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SOME ANALYTICAL RESULTS ON THE $\Delta-$FRACTIONAL DYNAMIC EQUATIONS

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Abstract. In this paper, we successfully solve some linear $\Delta-$fractional dynamic equations ($\Delta-$FDE) with Caputo $\Delta-$derivative analytically by solving an auxiliary linear $\Delta-$differential equation ($\Delta-$DE) with an integer order. The idea of the proposed method is based on transforming the given $\Delta-$FDE into a linear $\Delta-$DE with an integer order. This transformation removes certain terms of the solution of the considered $\Delta-$FDE, resulting in the remaining terms being a solution to the auxiliary equation. To demonstrate the ability and efficacy of this idea, several examples have been provided.

Keywords: Time Scale Calculus, Fractional time scale calculus, Caputo fractional $\Delta-$derivatives.

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1. Introduction

Time scale calculus was presented by Hilger [26, 27] to generalise and unify the study of theories of discrete and continuous differential equations, as well as to stretch these theories to other sorts of equations called dynamic equations, which have lately attracted a lot of attention. The two principal characteristics of time scale calculus are the unification and extension of discrete and continuous equations. There are numerous results concerning continuous dynamic equations that transfer over pretty readily to analogous results for discrete dynamic equations, whereas discrete dynamic equations' results may appear diametrically opposed to their continuous dual. On time scales, studying dynamic equations reveals these inconsistencies, allowing one to avoid having to repeat the proof of results twice for discrete and continuous dynamic equations. The beauty of time scale calculus is that it allows one to solve a given dynamic equation on any time scale set $\mathbb{T}$, and then this set will be selected later based on the type of dynamic system, such as $\mathbb{T} = \mathbb{R}$ when studying differential equations, and $\mathbb{T} = \mathbb{Z}$ when studying difference equations, and so on. This method yields results that are not only in connection with $\mathbb{Z}$ or $\mathbb{R}$, but also for any non-empty closed subset of $\mathbb{R}$. Since Hilger studied in his doctoral thesis on this topic,

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many scholars were interested in the extension of it and adapted many theories and issues in different fields [16, 3, 17, 2, 25, 39, 21, 34, 30, 45, 44, 24, 43, 40, 42, 41]. Many contributions and developments in time scale, applications of the theory and methods, have been made by many scholars in various disciplines, including: biology [15, 18], engineering [9, 19, 29, 36, 31], physics [12], economy [5, 7, 6], neural networks [20], and cellular neural networks [22]. Indeed, the delta derivative plays a fundamental role in time scale theory like the classical derivative (forward difference) of time for continuous (discrete) time.

The fractional and time scales calculus have been mixed by Bastos's Ph.D. thesis [10], to introduce fractional calculus on time scales. Recently, several results have been obtained, which includes fractional time scale calculus theory [14, 35], chaotic systems [37, 38, 1], applications of fractional time scales operators to dynamic equations [32, 11], recurrent neural networks [28], optimal control [8], existence and uniqueness of solutions to dynamic equations with fractional time scales [4, 13, 33, 50, 46, 49, 48, 47, 51].

Analytical solutions to $\Delta-$FDEs on time scale are important for describing physical phenomena. The analytical solutions make it easy to study the solutions, which are usually difficult to obtain through numerical calculations. However, most $\Delta-$FDEs fail to have exact analytical solutions due to the singular kernel appearing in the $\Delta-$fractional derivative. Although many research results have been given about the existence and uniqueness of fractional differential equations of various types, there are very few studies about how to solve these problems analytically. The applications of the $\Delta-$Mittag-Leffler function for presenting the existence of the solution of initial value problems of linear Caputo $\Delta-$FDEs were studied in [23]. In fact, this solution was obtained by means of the $\Delta-$Mittag-Leffler function, it is very difficult to express the solution in a finite analytic form because the solution formula concerns the fractional $\Delta-$integral of the $\Delta-$Mittag-Leffler function.

The motivation of this paper is to show that the solutions of $\Delta-$FDEs with Caputo $\Delta-$derivative are associated with the solutions of auxiliary linear $\Delta-$DEs with integer order. This relationship provides an analytical method for solving $\Delta-$FDEs by solving the auxiliary linear $\Delta-$DEs.

2. Preliminaries

This section includes some essential notions associated with the Time Scale Calculus.

Definition 2.1. [17] *The time scale $\mathbb{T}$ is defined as a non-empty arbitrary subset of $\mathbb{R}$ that is closed and non-empty.*

For examples, the complex numbers $\mathbb{C}$, the rational numbers $\mathbb{Q}$, and $[0,1)$, $(0,1)$, $(0,1]$, $(0,1]\cup\{2,6\}$ do not represent time scales. Whereas the integers numbers $\mathbb{Z}$, any closed interval $[a,b]\in\mathbb{R}$, the set $[0,1]\cup[4,5]$, the natural numbers $\mathbb{N}$, and the real numbers $\mathbb{R}$ represent time scales.

Definition 2.2. [16] *At $\vartheta\in\mathbb{T}$, the operator $\sigma:\mathbb{T}\to\mathbb{T}$ is referred to as follows:*

$$\sigma(\vartheta)=\inf\{r\in\mathbb{T}:r>\vartheta\},$$

it is called a forward jump operator. If $\sigma(\vartheta)=\vartheta$, then point ϑ is called right-dense.

Definition 2.3. [17] *At $\vartheta\in\mathbb{T}$, the operator $\rho:\mathbb{T}\to\mathbb{T}$ is referred to as follows:*

$$\rho(\vartheta)=\sup\{r\in\mathbb{T}:r<\vartheta\},$$

it is called a backward jump operator. If $\rho(\vartheta)=\vartheta$, and $\vartheta>\inf\mathbb{T}$, then point ϑ is called left-dense.

Definition 2.4. [16] *The function $\mu : \mathbb{T} \to [0, \infty)$ is known as a graininess function, and is represented by:*

$$\mu(\vartheta) = \sigma(\vartheta) - \vartheta, \quad \forall \vartheta \in \mathbb{T}.$$

Definition 2.5. [23] *The right dense continuous (rd−continuous) function is defined as a continuous function at right dense points in time scale, and its left-sided limits exist and are finite at left dense points in time scale. The set of rd−continuous functions $g : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. Also, $C^1_{rd}(\mathbb{T}, \mathbb{R})$ designates a set of $g : \mathbb{T} \to \mathbb{R}$ that can be differentiated using the rd-continuous derivative.*

Definition 2.6. [16] *The derived form of a time scale $\mathbb{T}$, referred to as $\mathbb{T}^\kappa$ is defined as:*

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T}\backslash(\rho(\sup\mathbb{T}), sup\mathbb{T}], & if\ sup(\mathbb{T}) \ < \infty, \\ \mathbb{T}, & if\ sup(\mathbb{T}) \ = \infty. \end{cases}$$

Definition 2.7. [17] *The Hilger or delta derivative of $g : \mathbb{T} \to \mathbb{R}$ at all $z \in \mathbb{T}^\kappa$ is denoted by $g^\Delta(z)$ as follows: $\forall \varepsilon > 0$, a neighborhood exists $\mathcal{V}_\mathbb{T}$ of z, $\mathcal{V}_\mathbb{T} = (z - \delta, z + \delta) \cap \mathbb{T}$ for some $\delta > 0$, we have*

$$\left| g\ (\sigma(z)) - g(\vartheta) - \ g^\Delta(z)\,(\sigma(z) \ - \ \vartheta) \right| \ \le \ \varepsilon\ |\sigma(z) \ - \ \vartheta|\,,$$

at $\vartheta \in \mathcal{V}_\mathbb{T}$, $\vartheta \neq \sigma(z)$.

Definition 2.8. [17]*The time scale monomials function $h_j(r, t_0) : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $j \in \mathbb{N}_0$ be defined by*

$$h_0(r, t_0) = 1 \quad \forall\, r, t_0 \in \mathbb{T},$$

and then recursively by

$$h_{j+1}(r, t_0) = \int_{t_0}^{r} h_\kappa(r, t_0)\Delta r, \ \ \forall r, t_0 \in \mathbb{T}.$$

As a result, the $\Delta-$derivative of h_j with respect to r satisfies for each fixed t_0.

$$h_j^\Delta(r, t_0) = h_{j-1}(r, t_0), \qquad r, t_0 \in \mathbb{T}, \ \ j \in \mathbb{N}.$$

Definition 2.9. [17, 23] *The time scale Laplace transform of a function $g : \mathbb{T} \to \mathbb{R}$ at all $t \in \mathbb{T}$, is defined by:*

$$\mathcal{L}\{g(t)\}(s) = G(s) := \int_0^\infty g(t) e^\sigma_{\ominus s}(t, 0)\Delta t,$$

for $s \in \mathcal{D}\{g\}$, such that $\mathcal{D}\{g\}$ involves all complex numbers $s \in \mathbb{C}$ that have an improper integral. While the time scale inverse Laplace transform is given by:

$$g(t) = \frac{1}{2\pi i} \int_\chi \mathcal{L}\{g(t)\}(s) \prod_{\ell=0}^{\eta-1} (1 + \mu(t_\ell)(s))ds, \eta \in \mathbb{N}_0,$$

where χ is any positively oriented closed curve.

Theorem 2.1. [17, 23] *Let $G(s)$ at $t \in \mathbb{T}$ be the time scale Laplace transform of $g(t) : \mathbb{T} \to \mathbb{R}$. Then,*

$$\mathcal{L}\{g^{\Delta^n}(t)\}(s) = s^n G(s) - \sum_{j=0}^{n-1} s^{n-j-1} g^{\Delta^j}(0).$$

Theorem 2.2. [16] *Let $1 + s\mu(a) \neq 0$ for all $s \in \mathbb{C}\backslash\{0\}$, and $j \in \mathbb{N}_0$, we have*

$$\mathcal{L}(h_j(a,0))(s) = \frac{1}{s^{j+1}}, \quad \forall a \in \mathbb{T}_0,$$

and

$$\lim_{a\to\infty} (h_j(a,0)e_{\ominus s}(a,0)) = 0.$$

Definition 2.10. [23] *On time scales $\mathbb{T}$, the generalized fractional $\Delta-$power function $h_\alpha(\vartheta, t_0)$ is*

$$h_\alpha(\vartheta, t_0) = \mathcal{L}^{-1}\left(\frac{1}{s^{\alpha+1}}\right)(\vartheta), \quad \forall \vartheta \geq t_0,$$

for all $s \in \mathbb{C}\backslash\{0\}$ is defined by

$$h_\alpha(\vartheta, r) = \widehat{h_\alpha(., t_0)}(\vartheta, r), \quad \vartheta, r \in \mathbb{T}, \quad \vartheta \geq r \geq t_0.$$

Example 2.1. [23] *Consider some elucidatory time scales*

(1) *Let $\mathbb{T} = \mathbb{R}$, $\sigma(\vartheta) = \vartheta$ at all $\vartheta \in \mathbb{T}$, we have*

$$h_k(\vartheta, t_0) = \frac{(r - t_0)^k}{k!}, \quad \forall \vartheta, t_0 \in \mathbb{T},\ k \in \mathbb{N}_0,$$

and define

$$h_\alpha(\vartheta, t_0) = \frac{(\vartheta - t_0)^\alpha}{\Gamma(\alpha + 1)}, \quad \vartheta \in \mathbb{T}, \quad \vartheta > t_0.$$

(2) *Let $\mathbb{T} = \mathbb{Z}$, $\sigma(\vartheta) = \vartheta + 1$ at all $\vartheta \in \mathbb{T}$, we have*

$$h_k(\vartheta, t_0) = \frac{(\vartheta - t_0)^k}{k!} = \binom{\vartheta - t_0}{k}, \quad k \in \mathbb{N}_0,$$

where $\vartheta^{(0)} = 1$, $\vartheta^{(k)} = \prod_{i=0}^{\mathrm{k}-1} (\vartheta - i)$, and define

$$h_\alpha(\vartheta, t_0) = \frac{(\vartheta - t_0)^\alpha}{\Gamma(\alpha + 1)}, \quad \vartheta \in \mathbb{T}, \quad \vartheta > t_0,$$

where $\vartheta^{(\alpha)} = \frac{\Gamma(\vartheta+1)}{\Gamma(\vartheta+1-\alpha)}$.

(3) *Let $\mathbb{T} = \overline{q^{\mathbb{N}}}, q > 1$, $\sigma(\vartheta) = q\vartheta$ at all $\vartheta \in \mathbb{T}$, we have*

$$h_k(\vartheta, r) = \frac{(\vartheta - r)_q^k}{[k]!}, \quad k \in \mathbb{N},$$

where $[k]! = \prod_{n=1}^{k} [n]$, $[n] = \frac{q^n - 1}{q-1}$, $n \in \mathbb{R}$ and

$$h_\alpha(\vartheta, r) = \Gamma_q(\alpha)\,(\vartheta - r)_q^\alpha, \qquad \forall \vartheta, r \in \mathbb{T},$$

for the $q-$Gamma function $\Gamma_q : \mathbb{R}\backslash\mathbb{Z} \to \mathbb{R}$ is defined as follows:

$$\Gamma_q\left(\frac{1}{2}\right) = 1,\ \Gamma_q(\alpha - 1) = \Gamma_q(\alpha)\frac{q^\alpha - 1}{q - 1}, \alpha \in \mathbb{R}\backslash\mathbb{Z}.$$

Definition 2.11. [23] *On time scale $\mathbb{T}$, $\alpha > 0$, at all $r \in \mathbb{T}$, and $r > t_0$. For the function $g : \mathbb{T} \to \mathbb{R}$, the fractional $\Delta-$derivative of a type Riemann-Liouville for $g(t)$ is given by:*

$$I^0_{\Delta,t_0}\mathrm{g}(r) = g(r),$$

$$\begin{aligned}(I^{\alpha}_{\Delta,t_0}g)(r) &= (h_{\alpha-1}(\cdot,t_0)*g)(r)\\ &= \int_{t_0}^{r} \widehat{h_{\alpha-1}(\cdot,t_0)}(r,\sigma(u))g(u)\Delta u\\ &= \int_{t_0}^{r} h_{\alpha-1}(r,\sigma(u))g(u)\Delta u.\end{aligned}$$

Definition 2.12. [13] *On time scale $\mathbb{T}$, $\alpha \geq 0$, at all $r,t \in \mathbb{T}^{k^m}$, $r < t$. For the function $g : \mathbb{T} \to \mathbb{R}$, the fractional $\Delta-$derivative of a type Riemann-Liouville for $g(t)$ is given by:*

$$D^{\alpha}_{\Delta,r}g(t) = D^{m}_{\Delta}I^{m-\alpha}_{\Delta,r}g(t), \quad \forall t \in \mathbb{T},$$

where $m = -[-\alpha]$.

Definition 2.13. [23] *On time scale $\mathbb{T}$, $\alpha \geq 0$ at all $t \in \mathbb{T}$. For the function $g : \mathbb{T} \to \mathbb{R}$, the fractional $\Delta-$derivative of a type Caputo for $g(t)$ is given by:*

$${}^{C}D^{\alpha}_{\Delta,0}g(t) = D^{\alpha}_{\Delta,0}\left(g(t) - \sum_{k=0}^{m-1} h_k(t,0)g^{\Delta^k}(0)\right), \quad \forall t > 0,$$

where $m = [\alpha] + 1$.

Theorem 2.3. [23] *If $g(t) \in C^1_{rd}([0,\infty)_{\mathbb{T}},\mathbb{R})$ for all $t \in \mathbb{T}$, then*

$${}^{C}D^{\alpha_1}_{\Delta,0}\,{}^{C}D^{\alpha_2}_{\Delta,0}g(t) = {}^{C}D^{\alpha_2}_{\Delta,0}\,{}^{C}D^{\alpha_1}_{\Delta,0}g(t)) = {}^{C}D^{\alpha_1+\alpha_2}_{\Delta,0}g(t)$$

where $\alpha_1, \alpha_2 \in \mathbb{R}^+$ and $\alpha_1 + \alpha_2 \in (0,1]$.

Remark 2.1. *For any $\alpha, \beta > 0$, one can have*

$${}^{C}D^{\alpha}_{\Delta,0}h_{\beta}(t,0)\mid_{t=0} = \begin{cases} 1, & \alpha = \beta,\\ 0, & \beta > \alpha.\end{cases}$$

Theorem 2.4. [23] *Let $g(t) \in C^m_{rd}([0,\infty)_{\mathbb{T}},\mathbb{R})$ for all $t \in \mathbb{T}$, $m \in \mathbb{N}$, $m-1 < \alpha \leq m$ and $\alpha > 0$. Then*

$$\mathcal{L}\left({}^{C}D^{\alpha}_{\Delta,0}g(t)\right)(s) = s^{\alpha}\mathcal{L}(g(t))(s) - \sum_{j=0}^{m-1} s^{\alpha-j-1}g^{\Delta^j}(0),$$

at all $s \in \mathbb{C}$ for which

$$\lim_{t\to\infty}\left(g^{\Delta^j}(t)\,e_{\ominus s}(t,0)\right) = 0, \quad j \in \{0,\ldots,m-1\}.$$

3. Main results

For any time scales $\mathbb{T}$, consider the linear $\Delta-$FDEs as follows:

$${}^{C}D^{\alpha}_{\Delta,0}y(t) = ay(t) + f(t), \tag{3.1}$$

$$y^{(\ell)}(0) = y_{\ell}, \quad \ell = 0,\ldots,m-1, \tag{3.2}$$

where $m = [\alpha] + 1$, $\alpha = \frac{r}{q} \in (m-1,m)$, $r\,and\,q \in \mathbb{N}$, $y(t) \in C^m_{rd}([0,\infty)_{\mathbb{T}},\mathbb{R})$, and $f(t) \in C^m_{rd}([0,\infty)_{\mathbb{T}},\mathbb{R})$.

In fact, the solution $y(t)$ of the problems (3.1)-(3.2) with $r=1$ has some terms accused of singularity. We plan to omit these terms in the first step of constructing the auxiliary linear DE with integer order by applying the following transform

$$w(t) = y(t) - \sum_{k=1}^{q} \gamma_k h_{\frac{k}{q}}(t,0), \tag{3.3}$$

where $\gamma_k, k=1,\dots,q-1$ and $w(t)$ is assumed to be a smooth function. Then, on both sides of Eq.(3.3), apply the operator ${}^{C}D^{\frac{1}{q}}_{\Delta,0}$ with Eq.(3.1) to get

$${}^{C}D^{\frac{1}{q}}_{\Delta,0} w(t) = ay(t) + f(t) - \sum_{k=1}^{q} \gamma_k {}^{C}D^{\frac{1}{q}}_{\Delta,0} h_{\frac{k}{q}}(t,0). \tag{3.4}$$

Now, at $t=0$, we force the RHS of Eq.(3.4) to zero and use the Remark 2.1, to get

$$\gamma_1 = ay(0) + f(0). \tag{3.5}$$

Using the ${}^{C}D^{\frac{1}{q}}_{\Delta,0}$ operator on both sides of Eq.(3.3) again, we get

$${}^{C}D^{\frac{1}{q}}_{\Delta,0} {}^{C}D^{\frac{1}{q}}_{\Delta,0} w(t) = a {}^{C}D^{\frac{1}{q}}_{\Delta,0} y(t) + {}^{C}D^{\frac{1}{q}}_{\Delta,0} f(t) - \sum_{k=2}^{q} \gamma_k {}^{C}D^{\frac{1}{q}}_{\Delta,0} {}^{C}D^{\frac{1}{q}}_{\Delta,0} h_{\frac{k}{q}}(t,0). \tag{3.6}$$

In order to simplify Eq.(3.6), Theorem 2.3 can be utilized with Eq.(3.1), resulting in

$${}^{C}D^{\frac{2}{q}}_{\Delta,0} w(t) = a^2 y(t) + af(t) + {}^{C}D^{\frac{1}{q}}_{\Delta,0} f(t) - \sum_{k=2}^{q} \gamma_k \; {}^{C}D^{\frac{2}{q}}_{\Delta,0} \mathrm{h}_{\frac{k}{q}}(t,0). \tag{3.7}$$

Forcing the RHS of Eq.(3.7) to zero at $t=0$ and using Remark 2.1, one can get

$$\gamma_2 = a^2 y(0) + af(0). \tag{3.8}$$

By following the same procedure $(q)^{th}-$times, one can arrive to

$$\begin{aligned} w^{\Delta}(t) &= a^q \; w(t) + \sum_{k=0}^{q-1} a^k \; {}^{C}D^{\frac{q-(k+1)}{q}}_{\Delta,0} f(t) + a^q \sum_{k=1}^{q} \gamma_k h_{\frac{k}{q}}(t,0) - \gamma_q, \\ w(0) &= \; y(0), \end{aligned} \tag{3.9}$$

where

$$\gamma_k = a^k y(0) + a^{k-1} f(0), \qquad k = 1,2,\dots,q. \tag{3.10}$$

The following theorem is stated and proved, to confirm the preceding result.

Theorem 3.1. *For $r=1$, the linear $\Delta-$FDEs (3.1)-(3.2) have a solution as follows*

$$y(t) = w(t) + \sum_{k=1}^{q} \gamma_k h_{\frac{k}{q}}(t,0), \tag{3.11}$$

where $w(t)$ is the solution to an auxiliary linear $\Delta-$DE that follows

$$\begin{aligned} w^{\Delta}(t) &= a^q w(t) + \varphi(t), \\ w(0) &= \; y(0), \end{aligned} \tag{3.12}$$

such that

$$\varphi(t) = \sum_{k=0}^{q-1} a^k \; {}^{C}D^{\frac{q-(k+1)}{q}}_{\Delta,0} f(t) + a^q \sum_{k=1}^{q} \gamma_k \; h_{\frac{k}{q}}(t,0) - \gamma_q, \tag{3.13}$$

and γ_k, $k = 1, 2, \dots, q$ is provided in Eq.(3.10).

proof: Using the Laplace transform to Eq.(3.1), yield

$$(s^{\frac{1}{q}} - a)Y(s) = s^{\frac{1-q}{q}} y(0) + F(s). \tag{3.14}$$

Also, Eq.(3.14) is multiplied by $\sum_{k=0}^{q-1} a^k s^{\frac{q-1-k}{q}}$, yielding

$$(s - a^q)y(s) = \sum_{k=0}^{q-1} a^k \ s^{\frac{-k}{q}} y(0) + \sum_{k=0}^{q-1} a^k \ s^{\frac{q-1-k}{q}} F(s). \tag{3.15}$$

Applying the Laplace transform to Eq.(3.11), we get

$$y(s) = W(s) + \sum_{k=1}^{q-1} \gamma_k s^{\frac{-k-q}{q}}. \tag{3.16}$$

Then, by substituting Eq.(3.16) into Eq.(3.15), one could have

$$(s - a^q)W(s) = -(s - a^q) \sum_{k=1}^{q} \gamma_k s^{\frac{-k-q}{q}} + \sum_{k=0}^{q-1} a^q s^{-\frac{k}{q}} w(0) + \sum_{k=0}^{q-1} a^k s^{\frac{q-1-k}{q}} F(s). \tag{3.17}$$

By simplifying Eq.(3.17) and taking Eq.(3.10) into consideration, we have

$$sW(s) - w(0) = a^q W(s) + a^q \sum_{k=1}^{q} \gamma_k s^{\frac{-k-q}{q}} - \sum_{k=0}^{q-1} a^k f(0) s^{\frac{-k-1}{q}} + \sum_{k=0}^{q-1} a^k s^{\frac{q-1-k}{q}} F(s) - \gamma_q s^{-1}. \tag{3.18}$$

Applying the inverse Laplace transform to Eq.(3.18), yields

$$w^{\Delta}(t) = a^q w(t) + \sum_{k=0}^{q-1} a^k \ {}^{C}D_{\Delta,0}^{\frac{q-(k+1)}{q}} f(t) + a^q \sum_{k=1}^{q-1} \gamma_k h_{\frac{k}{q}}(t, 0) - \gamma_q. \tag{3.19}$$

This completes the proof.

The conclusions of Theorem 3.1 can be applicable to the following system of linear $\Delta-$FDEs on a time scales

$$\begin{aligned} {}^{C}D_{\Delta,0}^{\frac{1}{q}} Z(t) &= AZ(t) + G(t), \\ Z(0) &= Z_0, \end{aligned} \tag{3.20}$$

where $A \in \mathbb{R}^{p\times p}$ is a non-singular real constant matrix, $Z(t)$ is an unknown column vector and is of dimension p, and $G(t) = \begin{pmatrix} 0 & \cdots & g^T(t) \end{pmatrix}^T$ of dimension mp.

Now, using the same scheme of Theorem 3.1, the following theorem will be straightforwardly proven.

Theorem 3.2. *The solution to the system of linear $\Delta-$FDEs (3.20) is as follows*

$$Z(t) = \mathcal{V}(t) + \sum_{k=1}^{q} \xi_k \ h_{\frac{k}{q}}(t, 0), \tag{3.21}$$

where $\mathcal{V}(t)$ is a solution of the system of linear $\Delta-DEs$

$$\mathcal{V}^{\Delta}(t) = A^q\mathcal{V}(t) + \Phi(t), \qquad \mathcal{V}(0) = Z(0), \tag{3.22}$$

such that

$$\Phi(t) = \sum_{k=0}^{q-1} A^k \, {}^C D_{\Delta,0}^{\frac{q-(k+1)}{q}} \mathrm{G}(t) + A^q \sum_{k=1}^{q} \xi_k h_{\frac{k}{q}}(t,0) - \xi_q,$$

where

$$\xi_k = A^k\mathcal{V}(0) + A^{k-1}\mathrm{G}(0), \quad = 1,\dots,q.$$

Theorem 3.3. *If $r < q$, then the linear $\Delta-FDEs$ (3.1)-(3.2) is equivalent to the system of $\Delta-FDEs$ with a derivative order of $1/q$.*
proof: Suppose

$$y_1(t) = y(t); \quad {}^C D_{\Delta,0}^{\frac{1}{q}} y_\kappa(t) = y_{\kappa+1}(t), \quad \forall\kappa = 1,\dots,r. \tag{3.23}$$

We have the following system

$$\begin{aligned}
{}^C D_{\Delta,0}^{\frac{1}{q}} y_1(t) &= y_2(t),\\
{}^C D_{\Delta,0}^{\frac{1}{q}} y_2(t) &= y_3(t),\\
&\vdots\\
{}^C D_{\Delta,0}^{\frac{1}{q}} y_{r-1}(t) &= y_r(t),\\
{}^C D_{\Delta,0}^{\frac{1}{q}} y_r(t) &= a y_1(t) + f(t).
\end{aligned} \tag{3.24}$$

Taking the Laplace transform of the system (3.24) as follows:

$$\begin{aligned}
&s^{\frac{1}{q}} y_1(s) - s^{\frac{1-q}{q}} y_1(0) = y_2(s),\\
&s^{\frac{1}{q}} y_2(s) - s^{\frac{1-q}{q}} y_2(0) = y_3(s),\\
&\vdots\\
&s^{\frac{1}{q}} y_{r-1}(s) - s^{\frac{1-q}{q}} y_{r-1}(0) = y_r(s),\\
&s^{\frac{1}{q}} y_r(s) - s^{\frac{1-q}{q}} y_r(0) = y_1(s) + F(s),
\end{aligned} \tag{3.25}$$

with initial conditions are defined as follows:

$$y_k(0) = \begin{cases} y(0), & if\ k = 1,\\ 0, & else. \end{cases} \tag{3.26}$$

By using the initial conditions (3.26), the backward substitution of the system (3.25), and taking into account that $\mathrm{y}_1(t) = y(t)$ one can have

$$s^{\frac{r}{q}} y(s) - s^{\frac{r-q}{q}} y(0) = a y(s) + F(s). \tag{3.27}$$

Indeed, Eq.(3.27) is precisely the Laplace transform of Eq.(3.1)with initial conditions (3.2). As a result, the system (3.24) with initial conditions (3.26) is equivalent to Eq.(3.1) with initial conditions (3.2).

Theorem 3.4. *If $r > q$, then the linear $\Delta-$FDEs (3.1)-(3.2) is equivalent to the system of $\Delta-$FDEs with a derivative order of $1/q$.*

Proof: Suppose

$$y_1(t) = y(t); \quad {}^{C}D^{\frac{1}{q}}_{\Delta,0}y_k(t) = y_{k+1}(t), \quad \forall\ k = 1, \ldots, r. \tag{3.28}$$

We have the following system

$$\begin{aligned} {}^{C}D^{\frac{1}{q}}_{\Delta,0}y_1(t) &= y_2(t), \\ {}^{C}D^{\frac{1}{q}}_{\Delta,0}y_2(t) &= y_3(t), \\ &\vdots \\ {}^{C}D^{\frac{1}{q}}_{\Delta,0}y_{r-1}(t) &= y_r(t), \\ {}^{C}D^{\frac{1}{q}}_{\Delta,0}y_r(t) &= ay_1(t) + f(t). \end{aligned} \tag{3.29}$$

Taking the Laplace transform of the system (3.29) as follows:

$$\begin{aligned} &s^{\frac{1}{q}}y_1(s) - s^{\frac{1-q}{q}}y_1(0) = y_2(s), \\ &s^{\frac{1}{q}}y_2(s) - s^{\frac{1-q}{q}}y_2(0) = y_3(s), \\ &\vdots \\ &s^{\frac{1}{q}}y_{r-1}(s) - s^{\frac{1-q}{q}}y_{r-1}(0) = y_p(s), \\ &s^{\frac{1}{q}}y_r(s) - s^{\frac{1-q}{q}}y_r(0) = y_1(s) + F(s), \end{aligned} \tag{3.30}$$

with initial conditions are defined as follows:

$$y_k(0) = \begin{cases} y^{(\ell)}(0), & if\ k = q\ell + 1,\ \ell = 0, 1, 2, \ldots, m-1, \\ 0, & else. \end{cases} \tag{3.31}$$

By utilizing the initial conditions (3.31), the backward substitution of the system (3.30), and taking into account that $y_1(t) = y(t)$

$$s^{\frac{r}{q}}Y(s) - \sum_{k=1}^{m} s^{\frac{r-kq}{q}} y^{(k-1)}(0) = aY(s) + F(s). \tag{3.32}$$

Indeed, Eq.(3.32) is precisely the Laplace transform of Eq.(3.1) with initial conditions (3.2). As a result, the system (3.29) with initial conditions (3.31) is equivalent to Eq.(3.1) with initial conditions (3.2).

4. Illustrative Examples

To clarify and explain the proposed method, this section will discuss some examples.

Example 4.1. *For any time scale $\mathbb{T}$, consider the linear $\Delta-$FDE*

$$\begin{aligned} {}^{C}D^{\frac{1}{2}}_{\Delta,0}y(t) &= -y(t) + 2h_2(t,0) + 2h_{\frac{3}{2}}(t,0), \\ \mathrm{y}(0) &= 0. \end{aligned} \tag{4.1}$$

First, we construct the following auxiliary linear $\Delta-DE$

$$\begin{aligned} w^{\Delta}(t) &= w(t) + 2h_1(t,0) - 2h_2(t,0), \\ w(0) &= 0. \end{aligned} \tag{4.2}$$

It is clear that the solution to Eq.(4.2) is

$$w(t) = 2h_2(t,0). \tag{4.3}$$

According to Theorem 3.1, we use the exact solution in (4.3) to construct the exact solution to Eq.(4.1) as follows:

$$y(t) = 2\,h_2(t,0). \tag{4.4}$$

Example 4.2. *For any time scale* $\mathbb{T}$*, consider the linear* $\Delta-FDE$

$$\begin{aligned} {}^{C}D^{\frac{1}{2}}_{\Delta,0} y(t) &= y(t) + h_1(t,0), \\ \text{y}(0) &= -1. \end{aligned} \tag{4.5}$$

First, we construct the following auxiliary linear $\Delta-DE$

$$\begin{aligned} w^{\Delta}(t) &= w(t) + 1, \\ w(0) &= -1. \end{aligned} \tag{4.6}$$

It is clear that the solution to Eq.(4.6) is

$$w(t) = -1. \tag{4.7}$$

According to Theorem 3.1, we use the exact solution in (4.7) to construct the exact solution to Eq.(4.5) as follows:

$$y(t) = -h_1(t,0) - h_{\frac{1}{2}}(t,0) - 1. \tag{4.8}$$

Example 4.3. *For any time scale* $\mathbb{T}$*, consider the linear* $\Delta-FDE$

$$\begin{aligned} {}^{C}D^{\frac{1}{3}}_{\Delta,0} y(t) &= -y(t) + h_1(t,0) + 2, \\ \text{y}(0) &= 1. \end{aligned} \tag{4.9}$$

First, we construct the following auxiliary linear $\Delta-DE$

$$\begin{aligned} w^{\Delta}(t) &= -w(t) + h_1(t,0) + 2, \\ w(0) &= 1. \end{aligned} \tag{4.10}$$

It is clear that the solution to Eq.(4.10) is

$$w(t) = h_1(t,0) + 1. \tag{4.11}$$

According to Theorem 3.1, we use the exact solution in (4.11) to construct the exact solution to Eq.(4.9) as follows:

$$y(t) = h_1(t,0) + h_{\frac{1}{3}}(t,0) - h_{\frac{2}{3}}(t,0) + 1. \tag{4.12}$$

5. Conclusions

In this paper, we successfully solve the linear $\Delta-$FDE analytically via an auxiliary linear $\Delta-$DE. The advantage of the suggested method is that it enables us to find the analytic solution of the linear $\Delta-$FDEs throughout the corresponding linear $\Delta-$DE. Also, we believe that we can investigate the stability analysis of linear $\Delta-$FDEs by using the auxiliary linear $\Delta-$DE, and this will be the target of the next paper.

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