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CERTAIN OPERATIONS ON m-POLAR PICTURE FUZZY GRAPHS

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ABSTRACT. Fuzzy graphs and their extensions, including m-polar fuzzy graphs and m-polar intuitionistic fuzzy graphs, find significant applications in various systems. The uncertainties in everyday life, stemming from inconsistent information, pose challenges for accurate modeling using existing techniques. To tackle this, researchers have introduced m-polar picture fuzzy graphs ($mPPFG$), a novel approach that addresses indeterminate and inconsistent information in real-valued problems. The paper presents several important operations like, Cartesian product, compositions, union, join, direct product, semi strong product and strong product of $mPPFG$s along with various properties and theorems. Also, product $mPPFG$, complete product $mPPFG$ are defined. The concepts of ring sum of two product $mPPFG$s are also studied.

Keyword: Cartesian product, composition, strong product, ring sum product.

AMS Subject Classification: 05C72;03E72.

1. INTRODUCTION

Graph theory serves as a powerful tool for modeling real-world systems, and fuzzy graphs (FGs) have emerged as an extension capable of handling imprecise and uncertain information. A recent advancement in this field involves m-polar picture fuzzy graphs ($mPPFG$s). These graphs offer a versatile framework for representing and analyzing complex systems, where nodes and edges possess degrees of membership with multiple polarities. By incorporating the advantages of FGs and m-polar picture fuzzy sets, $mPPFG$s provide a more comprehensive representation of uncertainty and imprecision. They capture diverse perspectives, facilitating a deeper understanding of relationships and interactions within the graph structure.

Utilizing fuzzy relations within fuzzy sets, Kauffman [1] pioneered the development of FG theory. Subsequently, Rosenfeld [2] introduced the notation for FGs in 1975. Notable contributions to the theoretical understanding of FG theory were made by various authors, including Mathew [3] and Mordeson [4]. Research conducted by Mordeson *et al.* [5]

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delved into FGs and fuzzy hypergraphs. Samanta *et al.* [6] introduced bipolar FGs, while Akram [7] explored the application of bipolar fuzzy digraph notation in social groups. A novel concept in FGs called fuzzy incident graphs was developed by Dinesh [8]. Additionally, Mathew *et al.* [9, 10] extensively studied fuzzy incidence graphs, investigating various properties. Fang *et al.* [11] made significant contributions to understanding the connectivity of fuzzy incidence graphs. Mahapatra *et al.* [12] explored fuzzy fractional coloring of FG along with its practical applications. Furthermore, Pal *et al.* [13] conducted research on fuzzy planar graphs, uncovering crucial properties in this domain. In 2016, Parvathi *et al.* [14] introduced the concept of intuitionistic fuzzy graphs (IFGs), laying the groundwork for intuitionistic fuzzy relations. Similarly, Shannon *et al.* [15] proposed various generalized forms of IFGs. Additionally, Sahoo *et al.* explored different products related to IFGs [16] and pioneered the development of intuitionistic fuzzy competition graphs [17]. The notion of picture fuzzy graphs (PFGs) was introduced in 2019 by Zuo *et al.* [18]. These PFGs have been approached diversely by different researchers. Das *et al.* introduced picture fuzzy tolerance graphs [19], and Amanathulla *et al.* presented blanched PFGs [20]. Furthermore, in 2021, Amanathulla *et al.* applied PFGs in the context of airline networks [21]. Additionally, Amanathulla *et al.* [22] explored various methods for decision-making involving multiple attributes within a picture fuzzy environment. Rashmanlu *et al.* [23] introduced the concept of cubic graphs with innovative applications. Amanathulla *et al.* provided insights into the application of discrete mathematics in real-world scenarios [24] and involvement of graph theory in reality [25]. Furthermore, Nazeer [26] introduced picture fuzzy incidence graphs, applying them to control illegal transportation in both India and America. The concept of m-polar fuzzy sets was initially introduced by Chen *et al.* in their work [27]. Subsequently, Ghorai *et al.* [28, 29] delved into the research of m-polar fuzzy graphs, exploring their characteristics and practical applications. They investigated the isomorphic properties of $mPFG$s with various applications [30] and examined the planarity of $mPFG$s [31]. Within the realm of m-polar fuzzy graphs, Mandal *et al.* [32] focused on different types of arcs, while Akram *et al.* [33] explored the relationship between m-polar fuzzy graphs and line graphs. Additionally, Akram *et al.* conducted studies on specific edge features and dominations in m-polar fuzzy graphs [34], along with applications [35]. The fuzzy coloring of $mPFG$s and its applications were explored by Mahapatra *et al.* in 2018 [36]. Mandal *et al.* [37] also investigated the genus values of $mPFG$s. For further exploration of new terminologies and applications in fuzzy graph theory, refer to [38]. In 2016, Wieslaw *et al.* [39] have studied Operation on level graph of bipolar fuzzy graphs. New concepts of vertex covering in cubic graphs with applications have studied by Huiqin *et al.* [40]. In 2020, Talebi *et al.* have studied new concepts of isrregular-intutionistic fuzzy graphs with application [41] and new concepts of m-polar interval valued intuitionistic fuzzy graph [42]. Khatun *et al.* have studied an application of neutrosophic graph in decision-making problem for alliances of companies [43], Picture fuzzy cubic graphs and their applications [44]. Banerjee *et al.* have introduced Optimization of disaster management using split domination in picture fuzzy graphs [45] and application of picture fuzzy bondage set to find least crowded passenger-friendly railway division in India [46].

1.1. **Motivation.** FG has been used in modeling and analyzing complex systems, providing flexibility in dealing with uncertainty. Concepts such as FGs, IFGs and PFGs have improved the representation of imprecise information. The study of m-polar FGs has further deepened our understanding of these structures. In practical applications, real-world problems often require a combination of different FG types to address uncertainty and incorporate visual representation. While existing fuzzy models offer some flexibility, $mPFS$ provide a more versatile approach. Intuitionistic fuzzy sets (IFSs) solely take into account membership and non-membership values, potentially falling short in encompassing all possible decision-making scenarios. However, in certain situations, individuals may want to remain neutral and not express a clear opinion. In such cases, picture fuzzy sets/graphs are used as a decision-making tool. PFS take into account three values: positive, neutral, and negative membership values. This facilitates a more inclusive portrayal of preferences or opinions. However, standard PFS are typically limited to being 1-polar, meaning they consider only one polar value. While this suffices for many problems, there are instances where the complexity of the problem demands a more nuanced approach. To address such situations, the concept of $mPPFG$ is introduced. This structure is more general and realistic compared to PFG, as it allows for the consideration of multiple polar values. By using $mPPFG$, a wider range of problems involving uncertainty can be effectively tackled, resulting in a more accurate and realistic representation of the underlying system. Consequently, the exploration of $mPPFG$s, which combine the features of m-polar FGs and PFGs, becomes a compelling avenue for constructing a powerful framework to model and analyze complex systems with both uncertain and visual aspects.

The remaining part of the article are organized as follows:
In section 2, some basic terminologies are given. Some operations and several properties and theorems of $mPPFG$s are presented in section 3. Finally, conclusion and future works are given in section 4.

Abbreviations: The following abbreviations are used in this study.

FG	Fuzzy graph.
IFG	Intuitionistic fuzzy graph.
PFG	Picture fuzzy graph.
$mPPFG$	m-polar picture fuzzy graph.

2. Preliminaries

In this section some basic terminologies are given.

Definition 2.1. [18] *A triple $G = (V_G, A, B)$ of the underling graph $G^* = (V_G, E_G)$, where $A = (\sigma_A, \mu_A, \eta_A)$ and $B = (\sigma_B, \mu_B, \eta_B)$ are picture fuzzy set on V_G and $V_G \times V_G$, respectively is called a PFG if*
(i) $\sigma_A : A \to [0,1]$, $\mu_A : A \to [0,1]$ and $\eta_A : A \to [0,1]$ are respectively the positive, neutral and negative membership functions, and they all meet the condition $0 \leq \sigma_A(t) + \mu_A(t) + \eta_A(t) \leq 1$ for all $t \in A$.
(ii) $\sigma_B : A \times A \to [0,1]$, $\mu_B : A \times A \to [0,1]$ and $\eta_B : A \times A \to [0,1]$, are respectively positive, neutral and negative membership functions of the edges $(s,t) \in A \times A$, and they all meet the conditions
$\sigma_B(s,t) \leq \sigma_A(s) \wedge \sigma_A(t)$, $\mu_B(s,t) \leq \mu_A(s) \wedge \mu_A(t)$, $\eta_B(s,t) \leq \eta_A(s) \vee \eta_A(t)$ and $0 \leq \sigma_B(s,t) + \mu_B(s,t) + \eta_B(s,t) \leq 1$ for each $(s,t) \in A \times A$.

Definition 2.2. [47] *A triplet $G = (V_G, A, B)$ of the underlying graph $G^* = (V_G, E_G)$, where $A = (\sigma_A, \mu_A, \eta_A)$ and $B = (\sigma_B, \mu_B, \eta_B)$ are picture fuzzy set on V_G and $V_G \times V_G$, respectively is called an m-polar PFG if for each $i = 1, 2, \ldots, m$.*
(i) $\sigma_A : A \to [0,1]^m$, $\mu_A : A \to [0,1]^m$ and $\eta_A : A \to [0,1]^m$ are respectively the positive, neutral and negative membership functions, and they all meet the condition $0 \leq p_i \circ \sigma_A(t) + p_\circ \mu_A(t) + p_i \circ \eta_A(t) \leq 1$ for all $t \in A$.
(ii) $\sigma_B : A \times A \to [0,1]^m$, $\mu_B : A \times A \to [0,1]^m$ and $\eta_B : A \times A \to [0,1]^m$, are respectively

positive, neutral and negative membership functions of the edges $(s,t) \in A \times A$, *and they all meet the conditions*
$p_i \circ \sigma_B(s,t) \leq p_i \circ \sigma_A(s) \wedge p_i \circ \sigma_A(t)$, $p_i \circ \mu_B(s,t) \leq p_i \circ \mu_A(s) \wedge p_i \circ \mu_A(t)$, $p_i \circ \eta_B(s,t) \leq p_i \circ \eta_A(s) \vee p_i \circ \eta_A(t)$ *and* $0 \leq p_i \circ \sigma_B(s,t) + p_i \circ \mu_B(s,t) + p_i \circ \eta_B(s,t) \leq 1$ *for each* $(s,t) \in E \subseteq V \times V$.

Note that $p_i \circ B(s,t) = 0$, for all $(s,t) \in V \times V - E$. Here, A is called the m-polar picture fuzzy vertex set of G and B is called the m-polar picture fuzzy edge set of G. An m-polar picture fuzzy relation B on V is called symmetric if for each $i = 1, 2, \ldots, m$, $p_i \circ B(s,t) = p_i \circ B(t,s)$ for all $s, t \in V$.

Definition 2.3. [47] *Let* $G = (V, A, B)$ *be an m-polar PFG, where* $A = (\sigma_A, \mu_A, \eta_A)$ *and* $B = (\sigma_B, \mu_B, \eta_B)$. *Then G is called a strong m-polar PFG if for each* $i = 1, 2, \ldots, m$, $p_i \circ \sigma_B(s,t) = p_i \circ \sigma_A(s) \wedge p_i \circ \sigma_A(t)$, $p_i \circ \mu_B(s,t) = p_i \circ \mu_A(s) \wedge p_i \circ \mu_A(t)$, $p_i \circ \eta_B(s,t) = p_i \circ \eta_A(s) \vee p_i \circ \eta_A(t)$ $\forall$ $(s,t) \in V_G^2$.

Definition 2.4. [47] *Suppose we have an m-polar PFG denoted by* $G = (V, A, B)$, *where* $A = (\sigma_A, \mu_A, \eta_A)$ *and* $B = (\sigma_B, \mu_B, \eta_B)$. *In this case, we refer to G as a complete m-polar PFG if for each* $i = 1, 2, \ldots, m$, $p_i \circ \sigma_B(s,t) = p_i \circ \sigma_A(s) \wedge p_i \circ \sigma_A(t)$, $p_i \circ \mu_B(s,t) = p_i \circ \mu_A(s) \wedge p_i \circ \mu_A(t)$, $p_i \circ \eta_B(s,t) = p_i \circ \eta_A(s) \vee p_i \circ \eta_A(t)$ $\forall$ $s, t \in A$.

3. Some operations on $mPFG$

In this section we defined some operations along with their properties and related theorems.

Definition 3.1. *Let* $G_1 = (V_{G_1}, A_1, B_1)$ *and* $G_2 = (V_{G_2}, A_2, B_2)$ *be two mPPFG of the underlying graphs* $G_1^* = (V_{G_1}, E_{G_1})$ *and* $G_2^* = (V_{G_2}, E_{G_2})$ *respectively. Then the Cartesian product of* G_1 *and* G_2 *is denoted by* $G_1 \times G_2$ *and is defined by* $G_1 \times G_2 = (V_{G_1} \times V_{G_2}, A_1 \times A_2, B_1 \times B_2)$, *where* $A_1 \times A_2 = (\sigma_{A_1} \times \sigma_{A_2}, \mu_{A_1} \times \mu_{A_2}, \eta_{A_1} \times \eta_{A_2})$ *and* $B_1 \times B_2 = (\sigma_{B_1} \times \sigma_{B_2}, \mu_{B_1} \times \mu_{B_2}, \eta_{B_1} \times \eta_{B_2})$ *such that for all* $i = 1, 2, \ldots, m$,
(i) For all $(u,v) \in V_{G_1} \times V_{G_2}$,
 (a) $p_i \circ (\sigma_{A_1} \times \sigma_{A_2})(u,v) = p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{A_2}(v)$;
 (b) $p_i \circ (\mu_{A_1} \times \mu_{A_2})(u,v) = p_i \circ \mu_{A_1}(u) \wedge p_i \circ \mu_{A_2}(v)$;
 (c) $p_i \circ (\eta_{A_1} \times \eta_{A_2})(u,v) = p_i \circ \eta_{A_1}(u) \vee p_i \circ \eta_{A_2}(v)$;
(ii) For all $s \in V_{G_1}$ *and* $uv \in E_{G_2}$,
 (a) $p_i \circ (\sigma_{B_1} \times \sigma_{B_2})((s,u)(s,v)) = p_i \circ \sigma_{A_1}(s) \wedge p_i \circ \sigma_{B_2}(uv)$;
 (b) $p_i \circ (\mu_{B_1} \times \mu_{B_2})((s,u)(s,v)) = p_i \circ \mu_{A_1}(s) \wedge p_i \circ \mu_{B_2}(uv)$;
 (c) $p_i \circ (\eta_{B_1} \times \eta_{B_2})((s,u)(s,v)) = p_i \circ \eta_{A_1}(s) \vee p_i \circ \eta_{B_2}(uv)$;
(iii) For all $s \in V_{G_2}$ *and* $uv \in E_{G_1}$,
 (a) $p_i \circ (\sigma_{B_1} \times \sigma_{B_2})((u,s)(v,s)) = p_i \circ \sigma_{B_1}(uv) \wedge p_i \circ \sigma_{A_2}(s)$;
 (b) $p_i \circ (\mu_{B_1} \times \mu_{B_2})((u,s)(v,s)) = p_i \circ \mu_{B_1}(uv) \wedge p_i \circ \mu_{A_2}(s)$;
 (c) $p_i \circ (\eta_{B_1} \times \eta_{B_2})((u,s)(v,s)) = p_i \circ \eta_{B_1}(uv) \vee p_i \circ \eta_{A_2}(s)$;
(iv) For all $(u,v)(s,t) \in V_{G_1} \times V_{G_2}^2 - E$,
 (a) $p_i \circ (\sigma_{B_1} \times \sigma_{B_2})((u,v)(s,t)) = 0$;
 (b) $p_i \circ (\mu_{B_1} \times \mu_{B_2})((u,v)(s,t)) = 0$;
 (c) $p_i \circ (\eta_{B_1} \times \eta_{B_2})((u,v)(s,t)) = 0$;

Where, $E = \{((u,s)(v,s)) : uv \in E_{G_1}, s \in V_{G_2}\} \cup \{((u,s)(u,t)) : u \in V_{G_1}, st \in E_{G_2}\}$.

Example 3.1. *Let* $G_1^* = (V_{G_1}, E_{G_1})$ *and* $G_2^* = (V_{G_2}, E_{G_2})$ *be two crisp graphs, where* $V_{G_1} = \{u, v\}$, $V_{G_2} = \{s, t\}$, $E_{G_1} = \{uv\}$ *and* $E_{G_2} = \{st\}$. *Let us consider two* $4PPFGs$

	(t,p)	(t,q)	(s,p)	(s,q)
$p_1 \circ (A_1 \times A_2)$	(.4,.1,.2)	(.3,.1,.1)	(.3,.1,.2)	(.3,.1,.2)
$p_2 \circ (A_1 \times A_2)$	(.2,.1,.1)	(.5,.1,.1)	(.2,.1,.1)	(.3,.1,.1)
$p_3 \circ (A_1 \times A_2)$	(.5,.1,.2)	(.4,.1,.1)	(.4,.1,.2)	(.4,.1,.1)
$p_4 \circ (A_1 \times A_2)$	(.3,.1,.1)	(.2,.1,.1)	(.4,.1,.1)	(.3,.1,.2)

TABLE 1. Membership values of the nodes of $G_1 \times G_2$

	$((t,p),(t,q))$	$((s,p),(s,q))$	$((s,p),(t,p))$	$((s,q),(t,q))$
$p_1 \circ (B_1 \times B_2)$	(.3,.1,.2)	(.3,.1,.3)	(.2,.1,.3)	(.2,.1,.3)
$p_2 \circ (B_1 \times B_2)$	(.2,.1,.1)	(.2,.1,.2)	(.3,.1,.2)	(.4,.1,.1)
$p_3 \circ (B_1 \times B_2)$	(.4,.1,.2)	(.4,.1,.2)	(.5,.1,.3)	(.5,.1,.1)
$p_4 \circ (B_1 \times B_2)$	(.3,.1,.1)	(.3,.1,.2)	(.3,.1,.1)	(.3,.1,.1)

TABLE 2. Membership values of the edges of $G_1 \times G_2$

$G_1 = (V_{G_1}, A_1, B_1)$ and $G_2 = (V_{G_2}, A_2, B_2)$ of the underlying graphs G_1^* and G_2^* respectively shown in Fig. 1. Then the Cartesian product $G_1 \times G_2 = (V_{G_1} \times V_{G_2}, A_1 \times A_2, B_1 \times B_2)$ of G_1 and G_2 are shown in Fig. 2. The membership values of nodes and edges of $G_1 \times G_2$ are shown in Table 1 and Table 2 respectively.

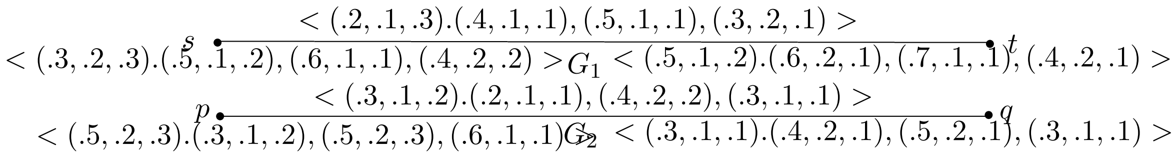

FIGURE 1. Two $4PPFG$s G_1 and G_2

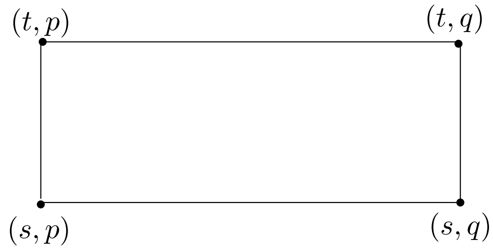

FIGURE 2. The Cartesian product $G_1 \times G_2$

Theorem 3.1. *The Cartesian product of two $mPPFGs$ is an $mPPFG$.*

Proof. Let $G_1 = (V_{G_1}, A_1, B_1)$ and $G_2 = (V_{G_2}, A_2, B_2)$ be two $mPPFG$ of the underlying graphs $G_1^* = (V_{G_1}, E_{G_1})$ and $G_2^* = (V_{G_2}, E_{G_2})$ respectively. Then $G_1 \times G_2 = (V_{G_1} \times V_{G_2}, A_1 \times A_2, B_1 \times B_2)$, where $A_1 \times A_2 = (\sigma_{A_1} \times \sigma_{A_2}, \mu_{A_1} \times \mu_{A_2}, \eta_{A_1} \times \eta_{A_2})$ and $B_1 \times B_2 = (\sigma_{B_1} \times \sigma_{B_2}, \mu_{B_1} \times \mu_{B_2}, \eta_{B_1} \times \eta_{B_2})$.
Since all condition of $mPPFG$ for $A_1 \times A_2$ are automatically satisfied. Therefore, verification needs only for $B_1 \times B_2$.

Let $s \in V_{G_1}$ and $uv \in E_{G_2}$. Then $\forall\ i = 1(1)m$,

$$\begin{aligned} p_i \circ (\sigma_{B_1} \times \sigma_{B_2})((s,u)(s,v)) &= p_i \circ \sigma_{A_1}(s) \wedge p_i \circ \sigma_{B_2}(uv) \\ &\leq p_i \circ \sigma_{A_1}(s) \wedge \{p_i \circ \sigma_{A_2}(u) \wedge p_i \circ \sigma_{A_2}(v)\} \\ &= \{p_i \circ \sigma_{A_1}(s) \wedge p_i \circ \sigma_{A_2}(u)\} \wedge \{p_i \circ \sigma_{A_1}(s) \wedge p_i \circ \sigma_{A_2}(v)\} \\ &= p_i \circ (\sigma_{A_1} \times \sigma_{A_2})(s,u) \wedge p_i \circ (\sigma_{A_1} \times \sigma_{A_2})(s,v) \end{aligned}$$

Therefore, $p_i \circ (\sigma_{B_1} \times \sigma_{B_2})((s,u)(s,v)) \leq p_i \circ (\sigma_{A_1} \times \sigma_{A_2})(s,u) \wedge p_i \circ (\sigma_{A_1} \times \sigma_{A_2})(s,v)$.

$$\begin{aligned} p_i \circ (\mu_{B_1} \times \mu_{B_2})((s,u)(s,v)) &= p_i \circ \mu_{A_1}(s) \wedge p_i \circ \mu_{B_2}(uv) \\ &\leq p_i \circ \mu_{A_1}(s) \wedge \{p_i \circ \mu_{A_2}(u) \wedge p_i \circ \mu_{A_2}(v)\} \\ &= \{p_i \circ \mu_{A_1}(s) \wedge p_i \circ \mu_{A_2}(u)\} \wedge \{p_i \circ \mu_{A_1}(s) \wedge p_i \circ \mu_{A_2}(v)\} \\ &= p_i \circ (\mu_{A_1} \times \mu_{A_2})(s,u) \wedge p_i \circ (\mu_{A_1} \times \mu_{A_2})(s,v) \end{aligned}$$

Therefore, $p_i \circ (\mu_{B_1} \times \mu_{B_2})((s,u)(s,v)) \leq p_i \circ (\mu_{A_1} \times \mu_{A_2})(s,u) \wedge p_i \circ (\mu_{A_1} \times \mu_{A_2})(s,v)$.

$$\begin{aligned} p_i \circ (\eta_{B_1} \times \eta_{B_2})((s,u)(s,v)) &= p_i \circ \eta_{A_1}(s) \vee p_i \circ \eta_{B_2}(uv) \\ &\leq p_i \circ \eta_{A_1}(s) \vee \{p_i \circ \eta_{A_2}(u) \vee p_i \circ \eta_{A_2}(v)\} \\ &= \{p_i \circ \eta_{A_1}(s) \vee p_i \circ \eta_{A_2}(u)\} \vee \{p_i \circ \eta_{A_1}(s) \vee p_i \circ \eta_{A_2}(v)\} \\ &= p_i \circ (\eta_{A_1} \times \eta_{A_2})(s,u) \vee p_i \circ (\eta_{A_1} \times \eta_{A_2})(s,v) \end{aligned}$$

Therefore, $p_i \circ (\eta_{B_1} \times \eta_{B_2})((s,u)(s,v)) \leq p_i \circ (\eta_{A_1} \times \eta_{A_2})(s,u) \vee p_i \circ (\eta_{A_1} \times \eta_{A_2})(s,v)$.

Let $s \in V_{G_2}$ and $uv \in E_{G_1}$. Then $\forall\ i = 1(1)m$,

$$\begin{aligned} p_i \circ (\sigma_{B_1} \times \sigma_{B_2})((u,s)(v,s)) &= p_i \circ \sigma_{B_1}(uv) \wedge p_i \circ \sigma_{A_2}(s) \\ &\leq \{p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{A_1}(v)\} \wedge\ p_i \circ \sigma_{A_2}(s) \\ &= \{p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{A_2}(s)\} \wedge \{p_i \circ \sigma_{A_1}(v) \wedge p_i \circ \sigma_{A_2}(s)\} \\ &= p_i \circ (\sigma_{A_1} \times \sigma_{A_2})(u,s) \wedge p_i \circ (\sigma_{A_1} \times \sigma_{A_2})(v,s) \end{aligned}$$

Therefore, $p_i \circ (\sigma_{B_1} \times \sigma_{B_2})((u,s)(v,s)) \leq p_i \circ (\sigma_{A_1} \times \sigma_{A_2})(u,s) \wedge p_i \circ (\sigma_{A_1} \times \sigma_{A_2})(v,s)$.

$$\begin{aligned} p_i \circ (\mu_{B_1} \times \mu_{B_2})((u,s)(v,s)) &= p_i \circ \mu_{B_1}(uv) \wedge p_i \circ \mu_{A_2}(s) \\ &\leq \{p_i \circ \mu_{A_1}(u) \wedge p_i \circ \mu_{A_1}(v)\} \wedge\ p_i \circ \mu_{A_2}(s) \\ &= \{p_i \circ \mu_{A_1}(u) \wedge p_i \circ \mu_{A_2}(s)\} \wedge \{p_i \circ \mu_{A_1}(v) \wedge p_i \circ \mu_{A_2}(s)\} \\ &= p_i \circ (\mu_{A_1} \times \mu_{A_2})(u,s) \wedge p_i \circ (\mu_{A_1} \times \mu_{A_2})(v,s) \end{aligned}$$

Therefore, $p_i \circ (\mu_{B_1} \times \mu_{B_2})((u,s)(v,s)) \leq p_i \circ (\mu_{A_1} \times \mu_{A_2})(u,s) \wedge p_i \circ (\mu_{A_1} \times \mu_{A_2})(v,s)$.

$$\begin{aligned} p_i \circ (\eta_{B_1} \times \eta_{B_2})((u,s)(v,s)) &= p_i \circ \eta_{B_1}(uv) \vee p_i \circ \eta_{A_2}(s) \\ &\leq \{p_i \circ \eta_{A_1}(u) \vee p_i \circ \eta_{A_1}(v)\} \vee\ p_i \circ \eta_{A_2}(s) \\ &= \{p_i \circ \eta_{A_1}(u) \vee p_i \circ \eta_{A_2}(s)\} \vee \{p_i \circ \eta_{A_1}(v) \vee p_i \circ \eta_{A_2}(s)\} \\ &= p_i \circ (\eta_{A_1} \times \eta_{A_2})(u,s) \vee p_i \circ (\eta_{A_1} \times \eta_{A_2})(v,s) \end{aligned}$$

Therefore, $p_i \circ (\eta_{B_1} \times \eta_{B_2})((u,s)(v,s)) \leq p_i \circ (\eta_{A_1} \times \eta_{A_2})(u,s) \vee p_i \circ (\eta_{A_1} \times \eta_{A_2})(v,s)$.

Let $(u,v)(s,t) \in V_{G_1} \times V_{G_2}^2 - E$. Then $\forall\ i = 1(1)m$,
$p_i \circ (\sigma_{B_1} \times \sigma_{B_2})((u,v)(s,t)) = 0 \leq p_i \circ (\sigma_{A_1} \times \sigma_{A_2})(u,v) \wedge p_i \circ (\sigma_{A_1} \times \sigma_{A_2})(s,t)$.
$p_i \circ (\mu_{B_1} \times \mu_{B_2})((u,v)(s,t)) = 0 \leq p_i \circ (\mu_{A_1} \times \mu_{A_2})(u,v) \wedge p_i \circ (\mu_{A_1} \times \mu_{A_2})(s,t)$.

	$((t,p),(t,q))$	$((s,p),(s,q))$	$((s,p),(t,p))$	$((s,q),(t,q))$	$((s,p),(t,q))$	$((s,q),(t,p))$
$p_1 \circ (B_1 \times B_2)$	(.3,.1,.2)	(.3,.1,.3)	(.2,.1,.3)	(.2,.1,.3)	(.2,.1,.3)	(.3,.1,.3)
$p_2 \circ (B_1 \times B_2)$	(.2,.1,.1)	(.2,.1,.2)	(.3,.1,.2)	(.4,.1,.1)	(.3,.1,.2)	(.2,.1,.2)
$p_3 \circ (B_1 \times B_2)$	(.4,.1,.2)	(.4,.1,.2)	(.5,.1,.3)	(.5,.1,.1)	(.5,.1,.3)	(.4,.1,.2)
$p_4 \circ (B_1 \times B_2)$	(.3,.1,.1)	(.3,.1,.2)	(.3,.1,.1)	(.3,.1,.1)	(.3,.1,.1)	(.3,.1,.2)

TABLE 3. Membership values of the edges of $G_1 \bullet G_2$

$p_i \circ (\eta_{B_1} \times \eta_{B_2})((u,v)(s,t)) = 0 \leq p_i \circ (\eta_{A_1} \times \eta_{A_2})(u,v) \vee p_i \circ (\eta_{A_1} \times \eta_{A_2})(s,t)$.
Hence the theorem. □

Definition 3.2. *Let $G_1 = (V_{G_1}, A_1, B_1)$ and $G_2 = (V_{G_2}, A_2, B_2)$ be two $mPPFG$ of the underlying graphs $G_1^* = (V_{G_1}, E_{G_1})$ and $G_2^* = (V_{G_2}, E_{G_2})$ respectively. Then the composition of G_1 and G_2 is denoted by $G_1 \bullet G_2$ and is defined by $G_1 \bullet G_2 = (V_{G_1} \times V_{G_2}, A_1 \bullet A_2, B_1 \bullet B_2)$, where $A_1 \bullet A_2 = (\sigma_{A_1} \bullet \sigma_{A_2}, \mu_{A_1} \bullet \mu_{A_2}, \eta_{A_1} \bullet \eta_{A_2})$ and $B_1 \bullet B_2 = (\sigma_{B_1} \bullet \sigma_{B_2}, \mu_{B_1} \bullet \mu_{B_2}, \eta_{B_1} \bullet \eta_{B_2})$ such that for all $i = 1, 2, \ldots, m$,*
(i) For all $(u,v) \in V_{G_1} \times V_{G_2}$,
(a) $p_i \circ (\sigma_{A_1} \bullet \sigma_{A_2})(u,v) = p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{A_2}(v)$;
(b) $p_i \circ (\mu_{A_1} \bullet \mu_{A_2})(u,v) = p_i \circ \mu_{A_1}(u) \wedge p_i \circ \mu_{A_2}(v)$;
(c) $p_i \circ (\eta_{A_1} \bullet \eta_{A_2})(u,v) = p_i \circ \eta_{A_1}(u) \vee p_i \circ \eta_{A_2}(v)$;
(ii) For all $s \in V_{G_1}$ and $(u,v) \in E_{G_2}$,
(a) $p_i \circ (\sigma_{B_1} \bullet \sigma_{B_2})((s,u)(s,v)) = p_i \circ \sigma_{A_1}(s) \wedge p_i \circ \sigma_{B_2}(u,v)$;
(b) $p_i \circ (\mu_{B_1} \bullet \mu_{B_2})((s,u)(s,v)) = p_i \circ \mu_{A_1}(s) \wedge p_i \circ \mu_{B_2}(u,v)$;
(c) $p_i \circ (\eta_{B_1} \bullet \eta_{B_2})((s,u)(s,v)) = p_i \circ \eta_{A_1}(s) \vee p_i \circ \eta_{B_2}(u,v)$;
(iii) For all $s \in V_{G_2}$ and $uv \in E_{G_1}$,
(a) $p_i \circ (\sigma_{B_1} \bullet \sigma_{B_2})((u,s)(v,s)) = p_i \circ \sigma_{B_1}(uv) \wedge p_i \circ \sigma_{A_2}(s)$;
(b) $p_i \circ (\mu_{B_1} \bullet \mu_{B_2})((u,s)(v,s)) = p_i \circ \mu_{B_1}(uv) \wedge p_i \circ \mu_{A_2}(s)$;
(c) $p_i \circ (\eta_{B_1} \bullet \eta_{B_2})((u,s)(v,s)) = p_i \circ \eta_{B_1}(uv) \vee p_i \circ \eta_{A_2}(s)$;
(iv) For all $(u,v)(s,t) \in E_0 - E$,
(a) $p_i \circ (\sigma_{B_1} \bullet \sigma_{B_2})((u,v)(s,t)) = p_i \circ \sigma_{B_1}(us) \wedge p_i \circ \sigma_{A_2}(v) \wedge p_i \circ \sigma_{A_2}(t)$;
(b) $p_i \circ (\mu_{B_1} \bullet \mu_{B_2})((u,v)(s,t)) = p_i \circ \mu_{B_1}(us) \wedge p_i \circ \mu_{A_2}(v) \wedge p_i \circ \mu_{A_2}(t)$;
(c) $p_i \circ (\eta_{B_1} \bullet \eta_{B_2})((u,v)(s,t)) = p_i \circ \eta_{B_1}(us) \vee p_i \circ \eta_{A_2}(v) \vee p_i \circ \eta_{A_2}(t)$;
(v) For all $(u,v)(s,t) \in (\widetilde{V}_{G_1} \times \widetilde{V}_{G_2})^2 - E_0$,
(a) $p_i \circ (\sigma_{B_1} \bullet \sigma_{B_2})((u,v)(s,t)) = 0$;
(b) $p_i \circ (\mu_{B_1} \bullet \mu_{B_2})((u,v)(s,t)) = 0$;
(a) $p_i \circ (\eta_{B_1} \bullet \eta_{B_2})((u,v)(s,t)) = 0$;
where, $E = \{((u,s)(v,s)) : uv \in E_{G_1}, s \in V_{G_2}\} \cup \{((u,s)(u,t)) : u \in V_{G_1}, st \in E_{G_2}\}$ and $E_0 = E \cup \{((u,s)(v,t)) : us \in E_{G_1}, s \neq t \in V_{G_2}\}$.

Example 3.2. *Here, we consider two $4PPFGs$ G_1 and G_2 same as the Example 3.1, then compute the composition $G_1 \bullet G_2$ of G_1 and G_2. The nodes of $G_1 \bullet G_2$ and their membership values are the same as the Cartesian product $G_1 \times G_2$, given in Table 1. The membership values of the edges of $G_1 \bullet G_2$ are displayed in Table 3.*

Theorem 3.2. *The composition of two $mPPFGs$ is an $mPPFG$.*

Proof. Let $G_1 = (V_{G_1}, A_1, B_1)$ and $G_2 = (V_{G_2}, A_2, B_2)$ be two $mPPFG$ of the underlying graphs $G_1^* = (V_{G_1}, E_{G_1})$ and $G_2^* = (V_{G_2}, E_{G_2})$ respectively. Then $G_1 \bullet G_2 = (V_{G_1} \times V_{G_2}, A_1 \bullet A_2, B_1 \bullet B_2)$, where $A_1 \bullet A_2 = (\sigma_{A_1} \bullet \sigma_{A_2}, \mu_{A_1} \bullet \mu_{A_2}, \eta_{A_1} \bullet \eta_{A_2})$ and $B_1 \bullet B_2 = (\sigma_{B_1} \bullet \sigma_{B_2}, \mu_{B_1} \bullet \mu_{B_2}, \eta_{B_1} \bullet \eta_{B_2})$.
Since all condition of $mPPFG$ for $A_1 \bullet A_2$ are automatically satisfied. Therefore, verification needs only for $B_1 \bullet B_2$.

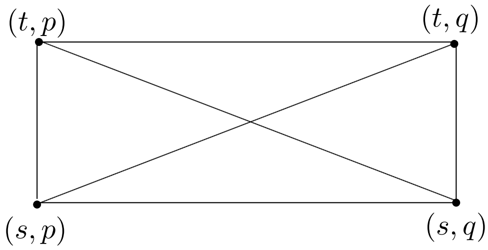

FIGURE 3. The composition $G_1 \bullet G_2$

Let $s \in V_{G_1}$ and $uv \in E_{G_2}$. Then $\forall\ i = 1(1)m$,

$$\begin{aligned} p_i \circ (\sigma_{B_1} \bullet \sigma_{B_2})((s,u)(s,v)) &= p_i \circ \sigma_{A_1}(s) \wedge p_i \circ \sigma_{B_2}(uv) \\ &\leq p_i \circ \sigma_{A_1}(s) \wedge \{p_i \circ \sigma_{A_2}(u) \wedge p_i \circ \sigma_{A_2}(v)\} \\ &= \{p_i \circ \sigma_{A_1}(s) \wedge p_i \circ \sigma_{A_2}(u)\} \wedge \{p_i \circ \sigma_{A_1}(s) \wedge p_i \circ \sigma_{A_2}(v)\} \\ &= p_i \circ (\sigma_{A_1} \bullet \sigma_{A_2})(s,u) \wedge p_i \circ (\sigma_{A_1} \bullet \sigma_{A_2})(s,v) \end{aligned}$$

Therefore, $p_i \circ (\sigma_{B_1} \bullet \sigma_{B_2})((s,u)(s,v)) \leq p_i \circ (\sigma_{A_1} \bullet \sigma_{A_2})(s,u) \wedge p_i \circ (\sigma_{A_1} \bullet \sigma_{A_2})(s,v)$.

$$\begin{aligned} p_i \circ (\mu_{B_1} \bullet \mu_{B_2})((s,u)(s,v)) &= p_i \circ \mu_{A_1}(s) \wedge p_i \circ \mu_{B_2}(uv) \\ &\leq p_i \circ \mu_{A_1}(s) \wedge \{p_i \circ \mu_{A_2}(u) \wedge p_i \circ \mu_{A_2}(v)\} \\ &= \{p_i \circ \mu_{A_1}(s) \wedge p_i \circ \mu_{A_2}(u)\} \wedge \{p_i \circ \mu_{A_1}(s) \wedge p_i \circ \mu_{A_2}(v)\} \\ &= p_i \circ (\mu_{A_1} \bullet \mu_{A_2})(s,u) \wedge p_i \circ (\mu_{A_1} \bullet \mu_{A_2})(s,v) \end{aligned}$$

Therefore, $p_i \circ (\mu_{B_1} \bullet \mu_{B_2})((s,u)(s,v)) \leq p_i \circ (\mu_{A_1} \bullet \mu_{A_2})(s,u) \wedge p_i \circ (\mu_{A_1} \bullet \mu_{A_2})(s,v)$.

$$\begin{aligned} p_i \circ (\eta_{B_1} \bullet \eta_{B_2})((s,u)(s,v)) &= p_i \circ \eta_{A_1}(s) \vee p_i \circ \eta_{B_2}(uv) \\ &\leq p_i \circ \eta_{A_1}(s) \vee \{p_i \circ \eta_{A_2}(u) \vee p_i \circ \eta_{A_2}(v)\} \\ &= \{p_i \circ \eta_{A_1}(s) \vee p_i \circ \eta_{A_2}(u)\} \vee \{p_i \circ \eta_{A_1}(s) \vee p_i \circ \eta_{A_2}(v)\} \\ &= p_i \circ (\eta_{A_1} \bullet \eta_{A_2})(s,u) \vee p_i \circ (\eta_{A_1} \bullet \eta_{A_2})(s,v) \end{aligned}$$

Therefore, $p_i \circ (\eta_{B_1} \bullet \eta_{B_2})((s,u)(s,v)) \leq p_i \circ (\eta_{A_1} \bullet \eta_{A_2})(s,u) \vee p_i \circ (\eta_{A_1} \bullet \eta_{A_2})(s,v)$.

Let $s \in V_{G_2}$ and $uv \in E_{G_1}$. Then $\forall\ i = 1(1)m$,

$$\begin{aligned} p_i \circ (\sigma_{B_1} \bullet \sigma_{B_2})((u,s)(v,s)) &= p_i \circ \sigma_{B_1}(uv) \wedge p_i \circ \sigma_{A_2}(s) \\ &\leq \{p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{A_1}(v)\} \wedge\ p_i \circ \sigma_{A_2}(s) \\ &= \{p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{A_2}(s)\} \wedge \{p_i \circ \sigma_{A_1}(v) \wedge p_i \circ \sigma_{A_2}(s)\} \\ &= p_i \circ (\sigma_{A_1} \bullet \sigma_{A_2})(u,s) \wedge p_i \circ (\sigma_{A_1} \bullet \sigma_{A_2})(v,s) \end{aligned}$$

Therefore, $p_i \circ (\sigma_{B_1} \bullet \sigma_{B_2})((u,s)(v,s)) \leq p_i \circ (\sigma_{A_1} \bullet \sigma_{A_2})(u,s) \wedge p_i \circ (\sigma_{A_1} \bullet \sigma_{A_2})(v,s)$.

$$\begin{aligned} p_i \circ (\mu_{B_1} \bullet \mu_{B_2})((u,s)(v,s)) &= p_i \circ \mu_{B_1}(uv) \wedge p_i \circ \mu_{A_2}(s) \\ &\leq \{p_i \circ \mu_{A_1}(u) \wedge p_i \circ \mu_{A_1}(v)\} \wedge\ p_i \circ \mu_{A_2}(s) \\ &= \{p_i \circ \mu_{A_1}(u) \wedge p_i \circ \mu_{A_2}(s)\} \wedge \{p_i \circ \mu_{A_1}(v) \wedge p_i \circ \mu_{A_2}(s)\} \\ &= p_i \circ (\mu_{A_1} \bullet \mu_{A_2})(u,s) \wedge p_i \circ (\mu_{A_1} \bullet \mu_{A_2})(v,s) \end{aligned}$$

Therefore, $p_i \circ (\mu_{B_1} \bullet \mu_{B_2})((u,s)(v,s)) \leq p_i \circ (\mu_{A_1} \bullet \mu_{A_2})(u,s) \wedge p_i \circ (\mu_{A_1} \bullet \mu_{A_2})(v,s)$.

$$\begin{aligned} p_i \circ (\eta_{B_1} \bullet \eta_{B_2})((u,s)(v,s)) &= p_i \circ \eta_{B_1}(uv) \vee p_i \circ \eta_{A_2}(s) \\ &\leq \{p_i \circ \eta_{A_1}(u) \vee p_i \circ \eta_{A_1}(v)\} \vee \ p_i \circ \eta_{A_2}(s) \\ &= \{p_i \circ \eta_{A_1}(u) \vee p_i \circ \eta_{A_2}(s)\} \vee \{p_i \circ \eta_{A_1}(v) \vee p_i \circ \eta_{A_2}(s)\} \\ &= p_i \circ (\eta_{A_1} \bullet \eta_{A_2})(u,s) \vee p_i \circ (\eta_{A_1} \bullet \eta_{A_2})(v,s) \end{aligned}$$

Therefore, $p_i \circ (\eta_{B_1} \bullet \eta_{B_2})((u,s)(v,s)) \leq p_i \circ (\eta_{A_1} \bullet \eta_{A_2})(u,s) \vee p_i \circ (\eta_{A_1} \bullet \eta_{A_2})(v,s)$.

Let $(u,v)(s,t) \in E_0 - E$, where,
$E = \{((u,s)(v,s)) : uv \in E_{G_1}, s \in V_{G_2}\} \cup \{((u,s)(u,t)) : u \in V_{G_1}, st \in E_{G_2}\}$ and $E_0 = E \cup \{((u,s)(v,t)) : us \in E_{G_1}, s \neq t \in V_{G_2}\}$. Then for $i = 1(1)m$,

$$\begin{aligned} p_i \circ (\sigma_{B_1} \bullet \sigma_{B_2})((u,v)(s,t)) &= p_i \circ \sigma_{B_1}(us) \wedge p_i \circ \sigma_{A_2}(v) \wedge p_i \circ \sigma_{A_2}(t) \\ &\leq \{p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{A_1}(s)\} \wedge p_i \circ \sigma_{A_2}(v) \wedge p_i \circ \sigma_{A_2}(t) \\ &= \{p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{A_2}(v)\} \wedge \{p_i \circ \sigma_{A_1}(s) \wedge p_i \circ \sigma_{A_2}(t)\} \\ &= p_i \circ (\sigma_{A_1} \bullet \sigma_{A_2})(u,v) \wedge p_i \circ (\sigma_{A_1} \bullet \sigma_{A_2})(s,t) \end{aligned}$$

Therefore, $p_i \circ (\sigma_{B_1} \bullet \sigma_{B_2})((u,v)(s,t)) \leq p_i \circ (\sigma_{A_1} \bullet \sigma_{A_2})(u,v) \wedge p_i \circ (\sigma_{A_1} \bullet \sigma_{A_2})(s,t)$.

$$\begin{aligned} p_i \circ (\mu_{B_1} \bullet \mu_{B_2})((u,v)(s,t)) &= p_i \circ \mu_{B_1}(us) \wedge p_i \circ \mu_{A_2}(v) \wedge p_i \circ \mu_{A_2}(t) \\ &\leq \{p_i \circ \mu_{A_1}(u) \wedge p_i \circ \mu_{A_1}(s)\} \wedge p_i \circ \mu_{A_2}(v) \wedge p_i \circ \mu_{A_2}(t) \\ &= \{p_i \circ \mu_{A_1}(u) \wedge p_i \circ \mu_{A_2}(v)\} \wedge \{p_i \circ \mu_{A_1}(s) \wedge p_i \circ \mu_{A_2}(t)\} \\ &= p_i \circ (\mu_{A_1} \bullet \mu_{A_2})(u,v) \wedge p_i \circ (\mu_{A_1} \bullet \mu_{A_2})(s,t) \end{aligned}$$

Therefore, $p_i \circ (\mu_{B_1} \bullet \mu_{B_2})((u,v)(s,t)) \leq p_i \circ (\mu_{A_1} \bullet \mu_{A_2})(u,v) \wedge p_i \circ (\mu_{A_1} \bullet \mu_{A_2})(s,t)$.

$$\begin{aligned} p_i \circ (\eta_{B_1} \bullet \eta_{B_2})((u,v)(s,t)) &= p_i \circ \eta_{B_1}(us) \vee p_i \circ \eta_{A_2}(v) \vee p_i \circ \eta_{A_2}(t) \\ &\leq \{p_i \circ \eta_{A_1}(u) \vee p_i \circ \eta_{A_1}(s)\} \vee p_i \circ \eta_{A_2}(v) \vee p_i \circ \eta_{A_2}(t) \\ &= \{p_i \circ \eta_{A_1}(u) \vee p_i \circ \eta_{A_2}(v)\} \vee \{p_i \circ \eta_{A_1}(s) \vee p_i \circ \eta_{A_2}(t)\} \\ &= p_i \circ (\eta_{A_1} \bullet \eta_{A_2})(u,v) \vee p_i \circ (\eta_{A_1} \bullet \eta_{A_2})(s,t) \end{aligned}$$

Therefore, $p_i \circ (\eta_{B_1} \bullet \eta_{B_2})((u,v)(s,t)) \leq p_i \circ (\eta_{A_1} \bullet \eta_{A_2})(u,v) \vee p_i \circ (\eta_{A_1} \bullet \eta_{A_2})(s,t)$.

Let $(u,v)(s,t) \in (V_{G_1} \times V_{G_2})^2 - E$. Then $\forall\ i = 1(1)m$,
$p_i \circ (\sigma_{B_1} \bullet \sigma_{B_2})((u,v)(s,t)) = 0 \leq p_i \circ (\sigma_{A_1} \bullet \sigma_{A_2})(u,v) \wedge p_i \circ (\sigma_{A_1} \bullet \sigma_{A_2})(s,t)$.
$p_i \circ (\mu_{B_1} \bullet \mu_{B_2})((u,v)(s,t)) = 0 \leq p_i \circ (\mu_{A_1} \bullet \mu_{A_2})(u,v) \wedge p_i \circ (\mu_{A_1} \bullet \mu_{A_2})(s,t)$.
$p_i \circ (\eta_{B_1} \bullet \eta_{B_2})((u,v)(s,t)) = 0 \leq p_i \circ (\eta_{A_1} \bullet \eta_{A_2})(u,v) \vee p_i \circ (\eta_{A_1} \bullet \eta_{A_2})(s,t)$.
Hence, the theorem. □

Definition 3.3. *Let $G_1 = (V_{G_1}, A_1, B_1)$ and $G_2 = (V_{G_2}, A_2, B_2)$ be two $mPPFG$ of the underlying graphs $G_1^* = (V_{G_1}, E_{G_1})$ and $G_2^* = (V_{G_2}, E_{G_2})$ respectively. Then the union of G_1 and G_2 is denoted by $G_1 \cup G_2$ and is defined by $G_1 \cup G_2 = (V_{G_1} \cup V_{G_2}, A_1 \cup A_2, B_1 \cup B_2)$, where $A_1 \cup A_2 = (\sigma_{A_1} \cup \sigma_{A_2}, \mu_{A_1} \cup \mu_{A_2}, \eta_{A_1} \cup \eta_{A_2})$ and $B_1 \cup B_2 = (\sigma_{B_1} \cup \sigma_{B_2}, \mu_{B_1} \cup \mu_{B_2}, \eta_{B_1} \cup \eta_{B_2})$ such that $\forall\ i = 1(1)m$,*
(i)

$$(a)\ \ p_i \circ (\sigma_{A_1} \cup \sigma_{A_2})(u) = \begin{cases} p_i \circ \sigma_{A_1}(u) & \text{if } u \in V_{G_1} \text{ and } u \notin V_{G_2} \\ p_i \circ \sigma_{A_2}(u) & \text{if } u \notin V_{G_1} \text{ and } u \in V_{G_2} \\ p_i \circ \sigma_{A_1}(u) \vee p_i \circ \sigma_{A_2}(u) & \text{if } u \in V_{G_1} \cap V_{G_2}. \end{cases}$$

$$(b)\ \ p_i \circ (\mu_{A_1} \cup \mu_{A_2})(u) = \begin{cases} p_i \circ \mu_{A_1}(u) & \text{if } u \in V_{G_1} \text{ and } u \notin V_{G_2} \\ p_i \circ \mu_{A_2}(u) & \text{if } u \notin V_{G_1} \text{ and } u \in V_{G_2} \\ p_i \circ \mu_{A_1}(u) \vee p_i \circ \mu_{A_2}(u) & \text{if } u \in V_{G_1} \cap V_{G_2}. \end{cases}$$

$$(c)\ \ p_i \circ (\eta_{A_1} \cup \eta_{A_2})(u) = \begin{cases} p_i \circ \eta_{A_1}(u) & \textit{if } u \in V_{G_1} \textit{ and } u \notin V_{G_2} \\ p_i \circ \eta_{A_2}(u) & \textit{if } u \notin V_{G_1} \textit{ and } u \in V_{G_2} \\ p_i \circ \eta_{A_1}(u) \wedge p_i \circ \eta_{A_2}(u) & \textit{if } u \in V_{G_1} \cap V_{G_2}. \end{cases}$$

(ii)

$$(a)\ \ p_i \circ (\sigma_{B_1} \cup \sigma_{B_2})(uv) = \begin{cases} p_i \circ \sigma_{B_1}(uv) & \textit{if } uv \in E_{G_1} \textit{ and } uv \notin E_{G_2} \\ p_i \circ \sigma_{B_2}(uv) & \textit{if } uv \notin E_{G_1} \textit{ and } uv \in E_{G_2} \\ p_i \circ \sigma_{B_1}(uv) \vee p_i \circ \sigma_{B_2}(u) & \textit{if } uv \in E_{G_1} \cap E_{G_2}. \end{cases}$$

$$(b)\ \ p_i \circ (\mu_{B_1} \cup \mu_{B_2})(uv) = \begin{cases} p_i \circ \mu_{B_1}(uv) & \textit{if } uv \in E_{G_1} \textit{ and } uv \notin E_{G_2} \\ p_i \circ \mu_{B_2}(uv) & \textit{if } uv \notin E_{G_1} \textit{ and } uv \in E_{G_2} \\ p_i \circ \mu_{B_1}(uv) \vee p_i \circ \mu_{B_2}(u) & \textit{if } uv \in E_{G_1} \cap E_{G_2}. \end{cases}$$

$$(c)\ \ p_i \circ (\eta_{B_1} \cup \eta_{B_2})(uv) = \begin{cases} p_i \circ \eta_{B_1}(uv) & \textit{if } uv \in E_{G_1} \textit{ and } uv \notin E_{G_2} \\ p_i \circ \eta_{B_2}(uv) & \textit{if } uv \notin E_{G_1} \textit{ and } uv \in E_{G_2} \\ p_i \circ \eta_{B_1}(uv) \wedge p_i \circ \eta_{B_2}(u) & \textit{if } uv \in E_{G_1} \cap E_{G_2}. \end{cases}$$

(v) For all $uv \in (\widetilde{V_{G_1}}^2 \cup \widetilde{V_{G_2}^2}) - E_{G_1} \cup E_{G_2}$,
(a) $p_i \circ (\sigma_{B_1} \cup \sigma_{B_2})(uv) = 0$;
(b) $p_i \circ (\mu_{B_1} \cup \mu_{B_2})(uv) = 0$;
(a) $p_i \circ (\eta_{B_1} \cup \eta_{B_2})(uv) = 0$;

Theorem 3.3. *The union of two mPPFGs is an mPPFG.*

Proof. The proof is similar to Theorem 3.1. □

Definition 3.4. *Let* $G_1 = (V_{G_1}, A_1, B_1)$ *and* $G_2 = (V_{G_2}, A_2, B_2)$ *be two mPPFG of the underlying graphs* $G_1^* = (V_{G_1}, E_{G_1})$ *and* $G_2^* = (V_{G_2}, E_{G_2})$ *respectively. Then the join of* G_1 *and* G_2 *is denoted by* $G_1 + G_2$ *and is defined by* $G_1 + G_2 = (V_{G_1} \cup V_{G_2}, A_1 + A_2, B_1 + B_2)$, *where* $A_1 + A_2 = (\sigma_{A_1} + \sigma_{A_2}, \mu_{A_1} + \mu_{A_2}, \eta_{A_1} + \eta_{A_2})$ *and* $B_1 + B_2 = (\sigma_{B_1} + \sigma_{B_2}, \mu_{B_1} + \mu_{B_2}, \eta_{B_1} + \eta_{B_2})$ *such that for all* $i = 1(1)m$,
(i) For all $u \in V_{G_1} \cup V_{G_2}$,
(a) $p_i \circ (\sigma_{A_1} + \sigma_{A_2})(u) = p_i \circ (\sigma_{A_1} \cup \sigma_{A_2})(u)$;
(b) $p_i \circ (\mu_{A_1} + \mu_{A_2})(u) = p_i \circ (\mu_{A_1} \cup \mu_{A_2})(u)$;
(c) $p_i \circ (\eta_{A_1} + \eta_{A_2})(u) = p_i \circ (\eta_{A_1} \cup \eta_{A_2})(u)$;

(ii) For all $uv \in E_{G_1} \cup E_{G_2}$,
(a) $p_i \circ (\sigma_{B_1} + \sigma_{B_2})(uv) = p_i \circ (\sigma_{B_1} \cup \sigma_{B_2})(uv)$;
(b) $p_i \circ (\mu_{B_1} + \mu_{B_2})(uv) = p_i \circ (\mu_{B_1} \cup \mu_{B_2})(uv)$;
(c) $p_i \circ (\eta_{B_1} + \eta_{B_2})(uv) = p_i \circ (\eta_{B_1} \cup \eta_{B_2})(uv)$;

(iii) Let E^a *=the collection of all edges adding the elements of* V_{G_1} *and* V_{G_2} *and assuming that* $V_{G_1} \cap V_{G_2} = \emptyset$ *then* $\forall\ uv \in E^a$, *For all* $uv \in E_{G_1} \cup E_{G_2}$,
(a) $p_i \circ (\sigma_{B_1} + \sigma_{B_2})(u, v) = p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{A_2}(v)$;
(b) $p_i \circ (\mu_{B_1} + \mu_{B_2})(u, v) = p_i \circ \mu_{A_1}(u) \wedge p_i \circ \mu_{A_2}(v)$;
(c) $p_i \circ (\eta_{B_1} + \eta_{B_2})(u, v) = p_i \circ \eta_{A_1}(u) \vee p_i \circ \eta_{A_2}(v)$;

(iv) For all $uv \in (\widetilde{V_{G_1}}^2 \cup \widetilde{V_{G_2}^2}) - E_{G_1} \cup E_{G_2} \cup E^a$,
(a) $p_i \circ (\sigma_{B_1} + \sigma_{B_2})(uv) = 0$;
(b) $p_i \circ (\mu_{B_1} + \mu_{B_2})(uv) = 0$;
(a) $p_i \circ (\eta_{B_1} + \eta_{B_2})(uv) = 0$;

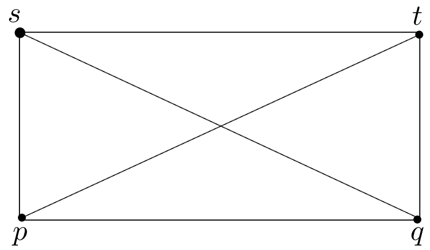

FIGURE 4. The join $G_1 + G_2$

	st	pq	sp	sq	tp	tq
$p_1 \circ (B_1 + B_2)$	(.2,.1,.3)	(.3,.1,.2)	(.3,.2,.3)	(.3,.1,.3)	(.5,.1,.3)	(.3,.1,.2)
$p_2 \circ (B_1 + B_2)$	(.4,.1,.1)	(.2,.1,.1)	(.3,.1,.2)	(.4,.1,.2)	(.3,.1,.2)	(.4,.2,.1)
$p_3 \circ (B_1 + B_2)$	(.5,.1,.1)	(.4,.2,.2)	(.5,.1,.3)	(.5,.1,.1)	(.5,.1,.3)	(.5,.1,.1)
$p_4 \circ (B_1 + B_2)$	(.3,.2,.1)	(.3,.1,.1)	(.4,.1,.2)	(.3,.1,.2)	(.4,.1,.1)	(.3,.1,.1)

TABLE 4. Membership values of the edges of $G_1 + G_2$

Example 3.3. *Here, we consider two $4PPFGs$ G_1 and G_2 same as the Example 3.1, then compute the join $G_1 + G_2$ of G_1 and G_2. The nodes of $G_1 + G_2$ are s, t, p and q and their membership values are shown in Fig. 4. The membership values of the edges of $G_1 + G_2$ are displayed in Table 4.*

Theorem 3.4. *The join of two $mPPFGs$ is an $mPPFG$.*

Proof. The proof is similar to Theorem 3.1. □

Theorem 3.5. *Let G_1 and G_2 be two strong $mPPFGs$ corresponding to the underlying graphs $G_1^* = (V_{G_1}, E_{G_1})$ and $G_2^* = (V_{G_2}, E_{G_2})$ respectively. Then $G_1 \times G_2$, $G_1 \bullet G_2$ and $G_1 + G_2$ are strong $mPPFG$.*

Proof. Let $G_1 = (V_{G_1}, A_1, B_1)$ and $G_2 = (V_{G_2}, A_2, B_2)$ be two strong $mPPFG$ of the underlying graphs $G_1^* = (V_{G_1}, E_{G_1})$ and $G_2^* = (V_{G_2}, E_{G_2})$ respectively.

Then for all $uv \in E_{G_1}$, $st \in E_{G_2}$ and for each $i = 1(1)m$, $p_i \circ \sigma_{B_1}(u, v) = p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{A_1}(v)$ and $p_i \circ \sigma_{B_2}(s, t) = p_i \circ \sigma_{A_2}(s) \wedge p_i \circ \sigma_{A_2}(t)$. Now, $\forall\ s \in V_{G_1}$ and $uv \in E_{G_2}$,

$$\begin{aligned} p_i \circ (\sigma_{B_1} \times \sigma_{B_2})((s,u)(s,v)) &= p_i \circ \sigma_{A_1}(s) \wedge p_i \circ \sigma_{B_2}(uv) \\ &= p_i \circ \sigma_{A_1}(s) \wedge p_i \circ \sigma_{A_2}(u) \wedge p_i \circ \sigma_{A_2}(v) \\ &= p_i \circ \sigma_{A_1}(s) \wedge p_i \circ \sigma_{A_2}(u) \wedge p_i \circ \sigma_{A_1}(s) \wedge p_i \circ \sigma_{A_2}(v) \\ &= p_i \circ (\sigma_{B_1} \times \sigma_{B_2})(s,u) \wedge p_i \circ (\sigma_{B_1} \times \sigma_{B_2})(u,v) \end{aligned}$$

$$\begin{aligned} p_i \circ (\mu_{B_1} \times \mu_{B_2})((s,u)(s,v)) &= p_i \circ \mu_{A_1}(s) \wedge p_i \circ \mu_{B_2}(uv) \\ &= p_i \circ \mu_{A_1}(s) \wedge p_i \circ \mu_{A_2}(u) \wedge p_i \circ \mu_{A_2}(v) \\ &= p_i \circ \mu_{A_1}(s) \wedge p_i \circ \mu_{A_2}(u) \wedge p_i \circ \mu_{A_1}(s) \wedge p_i \circ \mu_{A_2}(v) \\ &= p_i \circ (\mu_{B_1} \times \mu_{B_2})(s,u) \wedge p_i \circ (\mu_{B_1} \times \mu_{B_2})(u,v) \end{aligned}$$

$$\begin{aligned} p_i \circ (\eta_{B_1} \times \eta_{B_2})((s,u)(s,v)) &= p_i \circ \eta_{A_1}(s) \vee p_i \circ \eta_{B_2}(uv) \\ &= p_i \circ \eta_{A_1}(s) \vee p_i \circ \eta_{A_2}(u) \vee p_i \circ \eta_{A_2}(v) \\ &= p_i \circ \eta_{A_1}(s) \vee p_i \circ \eta_{A_2}(u) \vee p_i \circ \eta_{A_1}(s) \vee p_i \circ \eta_{A_2}(v) \\ &= p_i \circ (\eta_{B_1} \times \eta_{B_2})(s,u) \vee p_i \circ (\eta_{B_1} \times \eta_{B_2})(u,v) \end{aligned}$$

Again, $\forall\ s \in V_{G_2}$ and $uv \in E_{G_1}$,

$$\begin{aligned}
p_i \circ (\sigma_{B_1} \times \sigma_{B_2})((u,s)(v,s)) &= p_i \circ \sigma_{B_1}(uv) \wedge p_i \circ \sigma_{A_2}(s) \\
&= p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{A_1}(v) \wedge p_i \circ \sigma_{A_2}(s) \\
&= p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{A_2}(s) \wedge p_i \circ \sigma_{A_1}(v) \wedge p_i \circ \sigma_{A_2}(s) \\
&= p_i \circ (\sigma_{B_1} \times \sigma_{B_2})(u,s) \wedge p_i \circ (\sigma_{B_1} \times \sigma_{B_2})(v,s)
\end{aligned}$$

$$\begin{aligned}
p_i \circ (\mu_{B_1} \times \mu_{B_2})((u,s)(v,s)) &= p_i \circ \mu_{B_1}(uv) \wedge p_i \circ \mu_{A_2}(s) \\
&= p_i \circ \mu_{A_1}(u) \wedge p_i \circ \mu_{A_1}(v) \wedge p_i \circ \mu_{A_2}(s) \\
&= p_i \circ \mu_{A_1}(u) \wedge p_i \circ \mu_{A_2}(s) \wedge p_i \circ \mu_{A_1}(v) \wedge p_i \circ \mu_{A_2}(s) \\
&= p_i \circ (\mu_{B_1} \times \mu_{B_2})(u,s) \wedge p_i \circ (\mu_{B_1} \times \mu_{B_2})(v,s)
\end{aligned}$$

$$\begin{aligned}
p_i \circ (\eta_{B_1} \times \eta_{B_2})((u,s)(v,s)) &= p_i \circ \eta_{B_1}(uv) \vee p_i \circ \eta_{A_2}(s) \\
&= p_i \circ \eta_{A_1}(u) \vee p_i \circ \eta_{A_1}(v) \vee p_i \circ \eta_{A_2}(s) \\
&= p_i \circ \eta_{A_1}(u) \vee p_i \circ \eta_{A_2}(s) \vee p_i \circ \eta_{A_1}(v) \vee p_i \circ \eta_{A_2}(s) \\
&= p_i \circ (\eta_{B_1} \times \eta_{B_2})(u,s) \vee p_i \circ (\eta_{B_1} \times \eta_{B_2})(v,s)
\end{aligned}$$

Also, $\forall\ (u,v)(s,t) \in V_{G_1} \times V_{G_2}^2 - E$,
$p_i \circ (\sigma_{B_1} \times \sigma_{B_2})((u,v)(s,t)) = 0 = p_i \circ (\sigma_{B_1} \times \sigma_{B_2})(u,v) \wedge p_i \circ (\sigma_{B_1} \times \sigma_{B_2})(s,t)$,
$p_i \circ (\mu_{B_1} \times \mu_{B_2})((u,v)(s,t)) = 0 = p_i \circ (\mu_{B_1} \times \mu_{B_2})(u,v) \wedge p_i \circ (\mu_{B_1} \times \mu_{B_2})(s,t)$,
$p_i \circ (\eta_{B_1} \times \eta_{B_2})((u,v)(s,t)) = 0 = p_i \circ (\eta_{B_1} \times \eta_{B_2})(u,v) \vee p_i \circ (\eta_{B_1} \times \eta_{B_2})(s,t)$
Hence, $G_1 \times G_2$ is a strong $mPPFG$. Similarly, we can prove that $G_1 \bullet G_2$ and $G_1 + G_2$ are strong $mPPFG$. □

Theorem 3.6. *If $G_1 \times G_2$ is a strong $mPPFGs$ then at least one of G_1 and G_2 must be a strong $mPPFG$.*

Proof. Let $G_1 = (V_{G_1}, A_1, B_1)$ and $G_2 = (V_{G_2}, A_2, B_2)$ be two $mPPFG$ of the underlying graphs $G_1^* = (V_{G_1}, E_{G_1})$ and $G_2^* = (V_{G_2}, E_{G_2})$ respectively so that $G_1 \times G_2$ is a strong $mPPFG$. If possible let, both G_1 and G_2 are not strong $mPPFG$s. Then there exists at least one edge $uv \in E_{G_1}$ and at least one edge $st \in E_{G_2}$ such that $p_i \circ \sigma_{B_1}(u,v) < p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{A_1}(v)$ and $p_i \circ \sigma_{B_2}(s,t) < p_i \circ \sigma_{A_2}(s) \wedge p_i \circ \sigma_{A_2}(t)$. Without loss of generality, we assume that

$$\begin{aligned}
p_i \circ \sigma_{B_2}(s,t) &\leq p_i \circ \sigma_{B_1}(u,v) \\
&< p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{A_1}(v) \\
&\leq p_i \circ \sigma_{A_1}(u)
\end{aligned}$$

Let $\alpha \in V_{G_1}$ and $st \in E_{G_2}$. Then

$$\begin{aligned}
p_i \circ (\sigma_{B_1} \times \sigma_{B_2})((\alpha,s)(\alpha,t)) &= p_i \circ \sigma_{A_1}(\alpha) \wedge p_i \circ \sigma_{B_2}(st) \\
&< p_i \circ \sigma_{A_1}(\alpha) \wedge p_i \circ \sigma_{A_2}(s) \wedge p_i \circ \sigma_{A_2}(t)
\end{aligned}$$

Again, $p_i \circ (\sigma_{A_1} \times \sigma_{A_2})(\alpha,s) = p_i \circ \sigma_{A_1}(\alpha) \wedge p_i \circ \sigma_{A_2}(s)$ and $p_i \circ (\sigma_{A_1} \times \sigma_{A_2})(\alpha,t) = p_i \circ \sigma_{A_1}(\alpha) \wedge p_i \circ \sigma_{A_2}(t)$.
Therefore, $p_i \circ (\sigma_{A_1} \times \sigma_{A_2})(\alpha,s) \wedge p_i \circ (\sigma_{A_1} \times \sigma_{A_2})(\alpha,t) = p_i \circ \sigma_{A_1}(\alpha) \wedge p_i \circ \sigma_{A_2}(s) \wedge p_i \circ \sigma_{A_2}(t)$.
So,

$$\begin{aligned}
p_i \circ (\sigma_{B_1} \times \sigma_{B_2})((\alpha,s)(\alpha,t)) &= p_i \circ \sigma_{A_1}(\alpha) \wedge p_i \circ \sigma_{B_2}(st) \\
&< p_i \circ (\sigma_{A_1} \times \sigma_{A_2})(\alpha,s) \wedge p_i \circ (\sigma_{A_1} \times \sigma_{A_2})(\alpha,t)
\end{aligned}$$

This implies that $G_1 \times G_2$ is not a strong $mPPFG$, a contradiction. By contrapositively, at least one of G_1 or G_2 must be a strong $mPPFG$. □

Theorem 3.7. *If $G_1 \bullet G_2$ is a strong $mPPFGs$ then at least one of G_1 and G_2 must be a strong $mPPFG$.*

Proof. The proof is similar to Theorem 3.6. □

Definition 3.5. *Let $G_1 = (V_{G_1}, A_1, B_1)$ and $G_2 = (V_{G_2}, A_2, B_2)$ be two $mPPFG$ of the underlying graphs $G_1^* = (V_{G_1}, E_{G_1})$ and $G_2^* = (V_{G_2}, E_{G_2})$ respectively. Then the direct product $G_1 \sqcap G_2 = (V_{G_1} \times V_{G_2}, A_1 \sqcap A_2, B_1 \sqcap B_2)$ of the graph $G_1^* \sqcap G_2^* = (V_{G_1} \times V_{G_2}, E^2)$, where $A_1 \sqcap A_2 = (\sigma_{A_1} \sqcap \sigma_{A_2}, \mu_{A_1} \sqcap \mu_{A_2}, \eta_{A_1} \sqcap \eta_{A_2})$, $B_1 \sqcap B_2 = (\sigma_{B_1} \sqcap \sigma_{B_2}, \mu_{B_1} \sqcap \mu_{B_2}, \eta_{B_1} \sqcap \eta_{B_2})$, $E^2 = \{((u,s)(v,t)) : uv \in E_{G_1}, st \in E_{G_2}\}$ and $V_{G_1} \cap V_{G_2} = \emptyset$ such that for all $i = 1(1)m$,*
(i) For all $(u,v) \in V_{G_1} \times V_{G_2}$,
(a) $p_i \circ (\sigma_{A_1} \sqcap \sigma_{A_2})(u,v) = p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{A_2}(v)$;
(b) $p_i \circ (\mu_{A_1} \sqcap \mu_{A_2})(u,v) = p_i \circ \mu_{A_1}(u) \wedge p_i \circ \mu_{A_2}(v)$;
(c)$p_i \circ (\eta_{A_1} \sqcap \eta_{A_2})(u,v) = p_i \circ \eta_{A_1}(u) \vee p_i \circ \eta_{A_2}(v)$;

(ii) For all $((u,s)(v,t)) \in E^2$,
(a) $p_i \circ (\sigma_{B_1} \sqcap \sigma_{B_2})((u,s)(v,t)) = p_i \circ \sigma_{B_1}(uv) \wedge p_i \circ \sigma_{B_2}(st)$;
(b) $p_i \circ (\mu_{B_1} \sqcap \mu_{B_2})((u,s)(v,t)) = p_i \circ \mu_{B_1}(uv) \wedge p_i \circ \mu_{B_2}(st)$;
(c) $p_i \circ (\eta_{B_1} \sqcap \eta_{B_2})((u,s)(v,t)) = p_i \circ \eta_{B_1}(uv) \vee p_i \circ \eta_{B_2}(st)$;

Theorem 3.8. *The direct product of two $mPPFGs$ is also an $mPPFG$.*

Proof. The proof is similar to Theorem 3.1. □

Theorem 3.9. *If G_1 and G_2 are two strong $mPPFGs$ then $G_1 \sqcap G_2$ is also an $mPPFG$.*

Proof. Let $G_1 = (V_{G_1}, A_1, B_1)$ and $G_2 = (V_{G_2}, A_2, B_2)$ be two strong $mPPFG$ of the underlying graphs $G_1^* = (V_{G_1}, E_{G_1})$ and $G_2^* = (V_{G_2}, E_{G_2})$ respectively. Then for all $uv \in E_{G_1}$, for all $st \in E_{G_2}$ and for each $i = 1(1)m$,
(a) $p_i \circ \sigma_{B_1}(uv) = p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{A_1}(v)$ and $p_i \circ \sigma_{B_2}(st) = p_i \circ \sigma_{A_2}(s) \wedge p_i \circ \sigma_{A_2}(t)$.
(b) $p_i \circ \mu_{B_1}(uv) = p_i \circ \mu_{A_1}(u) \wedge p_i \circ \mu_{A_1}(v)$ and $p_i \circ \mu_{B_2}(st) = p_i \circ \mu_{A_2}(s) \wedge p_i \circ \mu_{A_2}(t)$.
(c) $p_i \circ \eta_{B_1}(uv) = p_i \circ \eta_{A_1}(u) \vee p_i \circ \eta_{A_1}(v)$ and $p_i \circ \eta_{B_2}(st) = p_i \circ \eta_{A_2}(s) \vee p_i \circ \eta_{A_2}(t)$.

The direct product of $G_1 = (V_{G_1}, A_1, B_1)$ and $G_2 = (V_{G_2}, A_2, B_2)$ is $G_1 \sqcap G_2 = (V_{G_1} \times V_{G_2}, A_1 \sqcap A_2, B_1 \sqcap B_2)$, where $A_1 \sqcap A_2 = (\sigma_{A_1} \sqcap \sigma_{A_2}, \mu_{A_1} \sqcap \mu_{A_2}, \eta_{A_1} \sqcap \eta_{A_2})$, $B_1 \sqcap B_2 = (\sigma_{B_1} \sqcap \sigma_{B_2}, \mu_{B_1} \sqcap \mu_{B_2}, \eta_{B_1} \sqcap \eta_{B_2})$. Now, for all $((\alpha,\beta)(\delta,\gamma)) \in E^2$ and for all $i = 1(1)m$,
(i)

$$\begin{aligned}
p_i \circ (\sigma_{B_1} \sqcap \sigma_{B_2})((\alpha,\beta)(\delta,\gamma)) &= p_i \circ \sigma_{B_1}(\alpha\delta) \wedge p_i \circ \sigma_{B_2}(\beta\gamma) \\
&= (p_i \circ \sigma_{A_1}(\alpha) \wedge p_i \circ \sigma_{A_1}(\delta)) \wedge (p_i \circ \sigma_{A_2}(\beta) \wedge p_i \circ \sigma_{A_2}(\gamma) \\
&= (p_i \circ \sigma_{A_1}(\alpha) \wedge p_i \circ \sigma_{A_2}(\beta)) \wedge (p_i \circ \sigma_{A_1}(\delta) \wedge p_i \circ \sigma_{A_2}(\gamma) \\
&= p_i \circ (\sigma_{A_1} \sqcap \sigma_{A_2})(\alpha,\beta) \wedge p_i \circ (\sigma_{A_1} \sqcap \sigma_{A_2})(\delta,\gamma)
\end{aligned}$$

(ii)

$$\begin{aligned}
p_i \circ (\mu_{B_1} \sqcap \mu_{B_2})((\alpha,\beta)(\delta,\gamma)) &= p_i \circ \mu_{B_1}(\alpha\delta) \wedge p_i \circ \mu_{B_2}(\beta\gamma) \\
&= (p_i \circ \mu_{A_1}(\alpha) \wedge p_i \circ \mu_{A_1}(\delta)) \wedge (p_i \circ \mu_{A_2}(\beta) \wedge p_i \circ \mu_{A_2}(\gamma) \\
&= (p_i \circ \mu_{A_1}(\alpha) \wedge p_i \circ \mu_{A_2}(\beta)) \wedge (p_i \circ \mu_{A_1}(\delta) \wedge p_i \circ \mu_{A_2}(\gamma) \\
&= p_i \circ (\mu_{A_1} \sqcap \mu_{A_2})(\alpha,\beta) \wedge p_i \circ (\mu_{A_1} \sqcap \mu_{A_2})(\delta,\gamma)
\end{aligned}$$

(iii)

$$\begin{aligned} p_i \circ (\eta_{B_1} \sqcap \eta_{B_2})((\alpha,\beta)(\delta,\gamma)) &= p_i \circ \eta_{B_1}(\alpha\delta) \vee p_i \circ \eta_{B_2}(\beta\gamma) \\ &= (p_i \circ \eta_{A_1}(\alpha) \vee p_i \circ \eta_{A_1}(\delta)) \vee (p_i \circ \eta_{A_2}(\beta) \vee p_i \circ \eta_{A_2}(\gamma) \\ &= (p_i \circ \eta_{A_1}(\alpha) \vee p_i \circ \eta_{A_2}(\beta)) \vee (p_i \circ \eta_{A_1}(\delta) \vee p_i \circ \eta_{A_2}(\gamma) \\ &= p_i \circ (\eta_{A_1} \sqcap \eta_{A_2})(\alpha,\beta) \vee p_i \circ (\eta_{A_1} \sqcap \eta_{A_2})(\delta,\gamma) \end{aligned}$$

□

Definition 3.6. *Let $G_1 = (V_{G_1}, A_1, B_1)$ and $G_2 = (V_{G_2}, A_2, B_2)$ be two $mPPFG$ of the underlying graphs $G_1^* = (V_{G_1}, E_{G_1})$ and $G_2^* = (V_{G_2}, E_{G_2})$ respectively. Then the semi strong product $G_1 * G_2 = (V_{G_1} \times V_{G_2}, A_1 * A_2, B_1 * B_2)$ of the graph $G_1^* * G_2^* = (V_{G_1} \times V_{G_2}, E^2)$, where $A_1 * A_2 = (\sigma_{A_1} * \sigma_{A_2}, \mu_{A_1} * \mu_{A_2}, \eta_{A_1} * \eta_{A_2})$, $B_1 * B_2 = (\sigma_{B_1} * \sigma_{B_2}, \mu_{B_1} * \mu_{B_2}, \eta_{B_1} * \eta_{B_2})$, $E^2 = ((V_{G_1} \times V_{G_2}) \times (V_{G_1} \times V_{G_2}))$, $E^3 = \{((\alpha,\beta)(\alpha,\gamma)) : \alpha \in E_{G_1}, \beta\gamma \in E_{G_2}\} \cup E^2$ and $V_{G_1} \cap V_{G_2} = \emptyset$ such that for all $i = 1(1)m$,*
(i) For all $(\alpha,\beta) \in V_{G_1} \times V_{G_2}$,
(a) $p_i \circ (\sigma_{A_1} * \sigma_{A_2})(\alpha,\beta) = p_i \circ \sigma_{A_1}(\alpha) \wedge p_i \circ \sigma_{A_2}(\beta)$;
(b) $p_i \circ (\mu_{A_1} * \mu_{A_2})(\alpha,\beta) = p_i \circ \mu_{A_1}(\alpha) \wedge p_i \circ \mu_{A_2}(\beta)$;
(c) $p_i \circ (\eta_{A_1} * \eta_{A_2})(\alpha,\beta) = p_i \circ \eta_{A_1}(\alpha) \vee p_i \circ \eta_{A_2}(\beta)$;

(ii) For all $((u,s)(v,t)) \in E^2$, for all $((\alpha,\beta)(\alpha,\gamma)) \in (E^3 - E^2)$,
(a) $p_i \circ (\sigma_{B_1} * \sigma_{B_2})((\alpha,\beta)(\alpha,\gamma)) = p_i \circ \sigma_{A_1}(\alpha) \wedge p_i \circ \sigma_{B_2}(\beta\gamma)$ and $p_i \circ (\sigma_{B_1} * \sigma_{B_2})((u,s)(v,t)) = p_i \circ \sigma_{B_1}(uv) \wedge p_i \circ \sigma_{B_2}(st)$
(b) $p_i \circ (\mu_{B_1} * \mu_{B_2})((\alpha,\beta)(\alpha,\gamma)) = p_i \circ \mu_{A_1}(\alpha) \wedge p_i \circ \mu_{B_2}(\beta\gamma)$ and $p_i \circ (\mu_{B_1} * \mu_{B_2})((u,s)(v,t)) = p_i \circ \mu_{B_1}(uv) \wedge p_i \circ \mu_{B_2}(st)$
(c) $p_i \circ (\eta_{B_1} * \eta_{B_2})((\alpha,\beta)(\alpha,\gamma)) = p_i \circ \eta_{B_1}(\alpha) \vee p_i \circ \eta_{B_2}(\beta\gamma)$ and $p_i \circ (\eta_{B_1} * \eta_{B_2})((u,s)(v,t)) = p_i \circ \eta_{B_1}(uv) \vee p_i \circ \eta_{B_2}(st)$

Theorem 3.10. *If G_1 and G_2 are two strong $mPPFGs$ then $G_1 * G_2$ is also a strong $mPPFG$.*

Proof. Let $G_1 = (V_{G_1}, A_1, B_1)$ and $G_2 = (V_{G_2}, A_2, B_2)$ be two strong $mPPFG$ of the underlying graphs $G_1^* = (V_{G_1}, E_{G_1})$ and $G_2^* = (V_{G_2}, E_{G_2})$ respectively. Then for all $uv \in E_{G_1}$, for all $st \in E_{G_2}$ and for each $i = 1(1)m$,
(a) $p_i \circ \sigma_{B_1}(uv) = p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{A_1}(v)$ and $p_i \circ \sigma_{B_2}(st) = p_i \circ \sigma_{A_2}(s) \wedge p_i \circ \sigma_{A_2}(t)$.
(b) $p_i \circ \mu_{B_1}(uv) = p_i \circ \mu_{A_1}(u) \wedge p_i \circ \mu_{A_1}(v)$ and $p_i \circ \mu_{B_2}(st) = p_i \circ \mu_{A_2}(s) \wedge p_i \circ \mu_{A_2}(t)$.
(c) $p_i \circ \eta_{B_1}(uv) = p_i \circ \eta_{A_1}(u) \vee p_i \circ \eta_{A_1}(v)$ and $p_i \circ \eta_{B_2}(st) = p_i \circ \eta_{A_2}(s) \vee p_i \circ \eta_{A_2}(t)$.

The semi strong product of $G_1 = (V_{G_1}, A_1, B_1)$ and $G_2 = (V_{G_2}, A_2, B_2)$ is $G_1 * G_2 = (V_{G_1} \times V_{G_2}, A_1 * A_2, B_1 * B_2)$, where $A_1 * A_2 = (\sigma_{A_1} * \sigma_{A_2}, \mu_{A_1} * \mu_{A_2}, \eta_{A_1} * \eta_{A_2})$, $B_1 * B_2 = (\sigma_{B_1} * \sigma_{B_2}, \mu_{B_1} * \mu_{B_2}, \eta_{B_1} * \eta_{B_2})$. Now, for all $((\alpha,\beta)(\delta,\gamma)) \in E^2$, for all $((u,v)(u,s)) \in (E^3 - E^2)$ and for all $i = 1(1)m$,
(i)

$$\begin{aligned} p_i \circ (\sigma_{B_1} * \sigma_{B_2})((u,v)(u,s)) &= p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{B_2}(vs) \\ &= p_i \circ \sigma_{A_1}(u) \wedge (p_i \circ \sigma_{A_2}(v) \wedge p_i \circ \sigma_{A_2}(s)) \\ &= (p_i \circ \sigma_{A_1}(u) \wedge p_i \circ \sigma_{A_2}(v)) \wedge (p_i \circ \sigma_{A_1}(u), p_i \circ \sigma_{A_2}(s)) \\ &= p_i \circ (\sigma_{A_1} * \sigma_{A_2})(u,v) \wedge \ p_i \circ (\sigma_{A_1} * \sigma_{A_2})(u,s) \end{aligned}$$

and

$$\begin{aligned} p_i \circ (\sigma_{B_1} * \sigma_{B_2})((\alpha,\beta)(\delta,\gamma)) &= p_i \circ \sigma_{B_1}(\alpha\delta) \wedge p_i \circ \sigma_{B_2}(\beta\gamma) \\ &= (p_i \circ \sigma_{A_1}(\alpha) \wedge p_i \circ \sigma_{A_1}(\delta)) \wedge (p_i \circ \sigma_{A_2}(\beta) \wedge p_i \circ \sigma_{A_2}(\gamma)) \\ &= (p_i \circ \sigma_{A_1}(\alpha) \wedge p_i \circ \sigma_{A_2}(\beta)) \wedge (p_i \circ \sigma_{A_1}(\delta) \wedge p_i \circ \sigma_{A_2}(\gamma)) \\ &= p_i \circ (\sigma_{A_1} * \sigma_{A_2})(\alpha,\beta) \wedge \ p_i \circ (\sigma_{A_1} * \sigma_{A_2})(\delta,\gamma) \end{aligned}$$

(ii)

$$\begin{aligned} p_i \circ (\mu_{B_1} * \mu_{B_2})((u,v)(u,s)) &= p_i \circ \mu_{A_1}(u) \wedge p_i \circ \mu_{B_2}(vs) \\ &= p_i \circ \mu_{A_1}(u) \wedge (p_i \circ \mu_{A_2}(v) \wedge p_i \circ \mu_{A_2}(s)) \\ &= (p_i \circ \mu_{A_1}(u) \wedge p_i \circ \mu_{A_2}(v)) \wedge (p_i \circ \mu_{A_1}(u), p_i \circ \mu_{A_2}(s)) \\ &= p_i \circ (\mu_{A_1} * \mu_{A_2})(u,v) \wedge \ p_i \circ (\mu_{A_1} * \mu_{A_2})(u,s) \end{aligned}$$

and

$$\begin{aligned} p_i \circ (\eta_{B_1} * \eta_{B_2})((\alpha,\beta)(\delta,\gamma)) &= p_i \circ \eta_{B_1}(\alpha\delta) \wedge p_i \circ \eta_{B_2}(\beta\gamma) \\ &= (p_i \circ \eta_{A_1}(\alpha) \wedge p_i \circ \eta_{A_1}(\delta)) \wedge (p_i \circ \eta_{A_2}(\beta) \wedge p_i \circ \eta_{A_2}(\gamma)) \\ &= (p_i \circ \eta_{A_1}(\alpha) \wedge p_i \circ \eta_{A_2}(\beta)) \wedge (p_i \circ \eta_{A_1}(\delta) \wedge p_i \circ \eta_{A_2}(\gamma)) \\ &= p_i \circ (\eta_{A_1} * \eta_{A_2})(\alpha,\beta) \wedge \ p_i \circ (\eta_{A_1} * \eta_{A_2})(\delta,\gamma) \end{aligned}$$

(iii)

$$\begin{aligned} p_i \circ (\sigma_{B_1} * \sigma_{B_2})((u,v)(u,s)) &= p_i \circ \sigma_{A_1}(u) \vee p_i \circ \sigma_{B_2}(vs) \\ &= p_i \circ \sigma_{A_1}(u) \vee (p_i \circ \sigma_{A_2}(v) \vee p_i \circ \sigma_{A_2}(s)) \\ &= (p_i \circ \sigma_{A_1}(u) \vee p_i \circ \sigma_{A_2}(v)) \vee (p_i \circ \sigma_{A_1}(u), p_i \circ \sigma_{A_2}(s)) \\ &= p_i \circ (\sigma_{A_1} * \sigma_{A_2})(u,v) \vee \ p_i \circ (\sigma_{A_1} * \sigma_{A_2})(u,s) \end{aligned}$$

and

$$\begin{aligned} p_i \circ (\sigma_{B_1} * \sigma_{B_2})((\alpha,\beta)(\delta,\gamma)) &= p_i \circ \sigma_{B_1}(\alpha\delta) \vee p_i \circ \sigma_{B_2}(\beta\gamma) \\ &= (p_i \circ \sigma_{A_1}(\alpha) \vee p_i \circ \sigma_{A_1}(\delta)) \vee (p_i \circ \sigma_{A_2}(\beta) \vee p_i \circ \sigma_{A_2}(\gamma)) \\ &= (p_i \circ \sigma_{A_1}(\alpha) \vee p_i \circ \sigma_{A_2}(\beta)) \vee (p_i \circ \sigma_{A_1}(\delta) \vee p_i \circ \sigma_{A_2}(\gamma)) \\ &= p_i \circ (\sigma_{A_1} * \sigma_{A_2})(\alpha,\beta) \vee \ p_i \circ (\sigma_{A_1} * \sigma_{A_2})(\delta,\gamma) \end{aligned}$$

Therefore, $G_1 * G_2$ is a strong $mPPFG$. □

Definition 3.7. *Let $G_1 = (V_{G_1}, A_1, B_1)$ and $G_2 = (V_{G_2}, A_2, B_2)$ be two $mPPFG$ of the underlying graphs $G_1^* = (V_{G_1}, E_{G_1})$ and $G_2^* = (V_{G_2}, E_{G_2})$ respectively. Then the strong product $G_1 \otimes G_2 = (V_{G_1} \times V_{G_2}, A_1 \otimes A_2, B_1 \otimes B_2)$ of the graph $G_1^* \otimes G_2^* = (V_{G_1} \times V_{G_2}, E)$, where $A_1 \otimes A_2 = (\sigma_{A_1} \otimes \sigma_{A_2}, \mu_{A_1} \otimes \mu_{A_2}, \eta_{A_1} \otimes \eta_{A_2})$, $B_1 \otimes B_2 = (\sigma_{B_1} \otimes \sigma_{B_2}, \mu_{B_1} \otimes \mu_{B_2}, \eta_{B_1} \otimes \eta_{B_2})$, $E^4 = E \cup E^2$ and $V_{G_1} \cap V_{G_2} = \emptyset$ is defined as follows:*
For all $i = 1(1)m$,
(i) For all $(\alpha, \beta) \in V_{G_1} \times V_{G_2}$,
(a) $p_i \circ (\sigma_{A_1} \otimes \sigma_{A_2})(\alpha, \beta) = p_i \circ \sigma_{A_1}(\alpha) \wedge p_i \circ \sigma_{A_2}(\beta)$;
(b) $p_i \circ (\mu_{A_1} \otimes \mu_{A_2})(\alpha, \beta) = p_i \circ \mu_{A_1}(\alpha) \wedge p_i \circ \mu_{A_2}(\beta)$;
(c) $p_i \circ (\eta_{A_1} \otimes \eta_{A_2})(\alpha, \beta) = p_i \circ \eta_{A_1}(\alpha) \vee p_i \circ \eta_{A_2}(\beta)$;

(ii) For all $((u,s)(v,t)) \in E^2$, for all $((\alpha,\beta)(\alpha,\gamma)) \in E$, $((u,v)(s,v)) \in E$,
(a) $p_i \circ (\sigma_{B_1} \otimes \sigma_{B_2})((\alpha,\beta)(\alpha,\gamma)) = p_i \circ \sigma_{A_1}(\alpha) \wedge p_i \circ \sigma_{B_2}(\beta\gamma)$, $p_i \circ (\sigma_{B_1} \otimes \sigma_{B_2})((u,v)(s,v)) = p_i \circ \sigma_{B_1}(us) \wedge p_i \circ \sigma_{A_2}(v)$ and $p_i \circ (\sigma_{B_1} \otimes \sigma_{B_2})((u,s)(v,t)) = p_i \circ \sigma_{B_1}(uv) \wedge p_i \circ \sigma_{B_2}(st)$
(b) $p_i \circ (\mu_{B_1} \otimes \mu_{B_2})((\alpha,\beta)(\alpha,\gamma)) = p_i \circ \mu_{A_1}(\alpha) \wedge p_i \circ \mu_{B_2}(\beta\gamma)$, $p_i \circ (\mu_{B_1} \otimes \mu_{B_2})((u,v)(s,v)) =$

$p_i \circ \mu_{B_1}(us) \wedge p_i \circ \mu_{A_2}(v)$ *and* $p_i \circ (\mu_{B_1} \otimes \mu_{B_2})((u,s)(v,t)) = p_i \circ \mu_{B_1}(uv) \wedge p_i \circ \mu_{B_2}(st)$

(c) $p_i \circ (\eta_{B_1} \otimes \eta_{B_2})((\alpha,\beta)(\alpha,\gamma)) = p_i \circ \eta_{A_1}(\alpha) \vee p_i \circ \eta_{B_2}(\beta\gamma)$, $p_i \circ (\eta_{B_1} \otimes \eta_{B_2})((u,v)(s,v)) = p_i \circ \eta_{B_1}(us) \vee p_i \circ \eta_{A_2}(v)$ *and* $p_i \circ (\eta_{B_1} \otimes \eta_{B_2})((u,s)(v,t)) = p_i \circ \eta_{B_1}(uv) \vee p_i \circ \eta_{B_2}(st)$.

Theorem 3.11. *If* G_1 *and* G_2 *are two complete* $mPPFGs$ *then* $G_1 \otimes G_2$ *is also a complete* $mPPFG$.

Proof. The proof is similar to Theorem 3.10. □

Definition 3.8. *Let* $G = (V_G, A, B)$ *be an* $mPPFG$ *of the underlying graph* $G^* = (V_G, E_G)$, *where* $A = (\sigma_A, \mu_A, \eta_A)$ *and* $B = (\sigma_B, \mu_B, \eta_B)$. *Then* G *is called product* $mPPFG$ *if* $\forall\ i = 1(1)m$, $p_i \circ \sigma_B(s,t) \leq p_i \circ \sigma_A(s) \times p_i \circ \sigma_A(t)$, $p_i \circ \mu_B(s,t) \leq p_i \circ \mu_A(s) \times p_i \circ \mu_A(t)$, $p_i \circ \eta_B(s,t) \leq p_i \circ \eta_A(s) \times p_i \circ \eta_A(t)\ \forall\ st \in \widetilde{V_G^2}$, *where,* $\times$ *denote the simple multiplication.*

Proposition 3.1. *Every product* $mPPFG$ *is an* $mPPFG$.

Definition 3.9. *Let* $G = (V_G, A, B)$ *be a product* $mPPFG$ *of the underlying graph* $G^* = (V_G, E_G)$, *where* $A = (\sigma_A, \mu_A, \eta_A)$ *and* $B = (\sigma_B, \mu_B, \eta_B)$. *Then* G *is called a complete product* $mPPFG$ *if* $\forall\ i = 1(1)m$, $p_i \circ \sigma_B(s,t) = p_i \circ \sigma_A(s) \times p_i \circ \sigma_A(t)$, $p_i \circ \mu_B(s,t) = p_i \circ \mu_A(s) \times p_i \circ \mu_A(t)$, $p_i \circ \eta_B(s,t) = p_i \circ \eta_A(s) \times p_i \circ \eta_A(t)\ \forall\ st \in \widetilde{V_G^2}$, *where,* $\times$ *denote the simple multiplication.*

Definition 3.10. *Let* $G_1 = (V_{G_1}, A_1, B_1)$ *and* $G_2 = (V_{G_2}, A_2, B_2)$ *be two product* $mPPFGs$ *of the underlying graphs* $G_1^* = (V_{G_1}, E_{G_1})$ *and* $G_2^* = (V_{G_2}, E_{G_2})$ *respectively. Then the ring sum* $G_1 \oplus G_2 = (V_{G_1} \cup V_{G_2}, A_1 \oplus A_2, B_1 \oplus B_2)$ *of the graph* $G_1^* \oplus G_2^* = (V_{G_1} \cup V_{G_2}, E_{G_1 \oplus G_2})$, *where* $A_1 \oplus A_2 = (\sigma_{A_1} \oplus \sigma_{A_2}, \mu_{A_1} \oplus \mu_{A_2}, \eta_{A_1} \oplus \eta_{A_2})$, $B_1 \oplus B_2 = (\sigma_{B_1} \oplus \sigma_{B_2}, \mu_{B_1} \oplus \mu_{B_2}, \eta_{B_1} \oplus \eta_{B_2})$ *is defined as follows:*
For all $i = 1(1)m$,
(i)

$$(a)\ \ p_i \circ (\sigma_{A_1} \oplus \sigma_{A_2})(\delta) = \begin{cases} p_i \circ \sigma_{A_1})(\delta) & \text{if } \delta \in V_{G_1} - V_{G_2} \\ p_i \circ \sigma_{A_2})(\delta) & \text{if } \delta \in V_{G_2} - V_{G_1} \\ p_i \circ \sigma_{A_1}(\delta) \vee p_i \circ \sigma_{A_2}(\delta) & \text{if } \delta \in V_{G_1} \cap V_{G_2} \end{cases}$$

$$(b)\ \ p_i \circ (\mu_{A_1} \oplus \mu_{A_2})(\delta) = \begin{cases} p_i \circ \mu_{A_1})(\delta) & \text{if } \delta \in V_{G_1} - V_{G_2} \\ p_i \circ \mu_{A_2})(\delta) & \text{if } \delta \in V_{G2} - V_{G_1} \\ p_i \circ \mu_{A_1}(\delta) \vee p_i \circ \mu_{A_2}(\delta) & \text{if } \delta \in V_{G_1} \cap V_{G_2} \end{cases}$$

$$(a)\ \ p_i \circ (\eta_{A_1} \oplus \eta_{A_2})(\delta) = \begin{cases} p_i \circ \eta_{A_1})(\delta) & \text{if } \delta \in V_{G_1} - V_{G_2} \\ p_i \circ \eta_{A_2})(\delta) & \text{if } \delta \in V_{G_2} - V_{G_1} \\ p_i \circ \eta_{A_1}(\delta) \wedge p_i \circ \eta_{A_2}(\delta) & \text{if } \delta \in V_{G_1} \cap V_{G_2} \end{cases}$$

(ii)

$$(a)\ \ p_i \circ (\sigma_{B_1} \oplus \sigma_{B_2})\delta\beta = \begin{cases} p_i \circ \sigma_{B_1})(\delta\beta) & \text{if } \delta\beta \in E_{G_1} - E_{G_2} \\ p_i \circ \sigma_{B_2})(\delta\beta) & \text{if } \delta\beta \in E_{G_2} - E_{G_1} \\ 0 & \text{otherwise} \end{cases}$$

$$(b)\ \ p_i \circ (\mu_{B_1} \oplus \mu_{B_2})\delta\beta = \begin{cases} p_i \circ \mu_{B_1})(\delta\beta) & \text{if } \delta\beta \in E_{G_1} - E_{G_2} \\ p_i \circ \mu_{B_2})(\delta\beta) & \text{if } \delta\beta \in E_{G_2} - E_{G_1} \\ 0 & \text{otherwise} \end{cases}$$

$$(a)\ p_i \circ (\eta_{B_1} \oplus \eta_{B_2})(\delta\beta) = \begin{cases} p_i \circ \eta_{B_1})(\delta\beta) & \text{if } \delta\beta \in E_{G_1} - E_{G_2} \\ p_i \circ \eta_{B_2})(\delta\beta) & \text{if } \delta\beta \in E_{G_2} - E_{G_1} \\ 0 & \text{otherwise} \end{cases}$$

Proposition 3.2. *The ring sum of two product $mPPFGs$ is a product $mPPFG$.*

4. Conclusion

The $mPPFG$ is a generalization of the m polar FG, m polar IFG and picture fuzzy graphs. The flexibility and comparability of $mPPFG$s are much higher than those of m polar FG, m polar IFG and picture fuzzy graphs. An $mPPFG$ can deal with uncertain problems, whereas an m polar FG, m polar IFG and picture fuzzy graphs may not be effective in such contexts. This article introduces new terminologies and explores the properties and operations of $mPPFG$s. It defines and discusses various types of operations of $mPPFG$s, including Cartesian product, compositions, union, join, direct product, semi strong product and strong product of $mPPFG$s. Several important proporties and theorems regarding different operations of $mPPFG$ are also provided. This article defined product $mPPFG$ and complete product $mPPFG$. The concepts of ring sum of two product $mPPFG$s are also studied in this article. In future we will study about different operations on interval-valued $mPPFG$s.

Compliance with ethical standards.

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