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ON A CLASS OF CONSTACYCLIC CODES OVER THE RING $\frac{Z_4[u]}{< u^2 - 3 >}$

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Abstract. In this paper, λ-constacyclic codes and skew-λ-constacyclic codes over the ring $R = \frac{Z_4[u]}{< u^2 - 3 >}$ are studied for $\lambda = 3$ and $2 + 3u$. Introducing new Gray maps from R to the copies of Z_4, we observed that Gray images of λ-constacyclic codes over R are cyclic, quasi-cyclic and permutation equivalent to quasi-cyclic codes over Z_4. λ-constacyclic codes of odd length over R and generating polynomial of the Gray images are studied. Further, it is observed that the images of skew-λ- constacyclic codes over R are cyclic, quasi-cyclic and permutation equivalent to quasi-cyclic codes over Z_4.

Keywords: Cyclic codes, Constacyclic codes, Gray maps, Skew-constacyclic codes

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1. Introduction

In the early 1970s, codes over finite rings were studied for research in algebraic coding theory. Cyclic codes are pre-eminent among the class of linear codes and are researched on different rings. Skew-cyclic codes, constacyclic codes, skew-λ-constacyclic codes are some of the generalizations of cyclic codes. Among all finite rings, the ring Z_4 of integers modulo 4 has a special place in coding theory. Hammons [6] established connections between binary non-linear codes and Z_4-linear codes. Also, codes over Z_4 have a special place in coding theory due to their links to lattices, designs, cryptography, etc. Also, specific binary non-linear codes are obtained as Gray images of Z_4 and are related to lattices [12], combinatorial designs [5] and low correlation sequences [10].

Rings of order 16 are extensions of Z_4 and are of great interest. In 2015, Bandi and Bhaintwal [4] studied the ring Z_4+uZ_4, $u^2 = 0$ and discussed the Galois ring extensions and ideal structure of their extension. They studied cyclic codes of odd length and presented 1-generator cyclic codes over the ring in terms of n^{th} roots of unity. In 2023, ST Timothy *et al.*[11] worked on the ring $Z_4 + vZ_4 + v^2Z_4$ for $v^3 = 1$ and studied cyclic, λ-constacyclic codes and skew-λ-constacyclic codes for the unit $(1 + 2v)$ and introduced new Gray maps. The Gray images are cyclic, quasi-cyclic and permutation equivalent to quasi-cyclic codes

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over Z_4. In 2015, Ashraf and Mohammad [1] considered $Z_4 + uZ_4$ and studied $(1 + 2u)$-constacyclic codes over the ring of odd length for $u^2 = 0$ and proved that the Gray image of $(1 + 2u)$-constacyclic codes of length n over the ring are cyclic codes of length $2n$ over Z_4. In 2016, Ozen *et al.* [9], studied cyclic codes and constacyclic codes with shift constant $(2+u)$ over Z_4+uZ_4 with $u^2 = 1$. They determined the form of generators of cyclic codes and their spanning sets. They proved that the Z_4-images of a $(2 + u)$-constacyclic code of odd length is a cyclic code over Z_4. In 2018, Aydin *et al.* [2] studied λ-constacyclic codes over the ring $R = Z_4 + uZ_4$, $u^2 = 1$ for the units $(3 + 2u)$ and $(2 + 3u)$ and determined the Gray-images of λ-constacyclic codes. They also conducted a computer search and obtained codes with better parameters than currently best-known linear codes over Z_4. Recently, Bag *et al.* [3] in 2018, studied λ-constacyclic codes over the ring $R = Z_4 + uZ_4$ with $u^2 = 3$ for the units $(1 + 2u)$ and $(3 + 2u)$. New Gray maps from R to the copies of Z_4 are introduced and showed that the Gray images of λ-constacyclic codes over the ring are cyclic, quasi-cyclic and permutation equivalent to quasi-cyclic codes over Z_4. They extended and obtained the Gray-images of skew λ-constacyclic codes over R.

In this paper, we again consider the ring $\dfrac{Z_4[u]}{< u^2 - 3 >}$, an extension of Z_4 that is isomorphic to the ring $\dfrac{Z_4[u]}{< u^2 - 2(1 + u) >}$ using the map $u \to u + 1$, which was discussed in [3]. This ring can be expressed as $Z_4 + uZ_4$, $u^2 = 3$ and λ-constacyclic codes over R for $\lambda = 3,\ 2 + 3u$ are studied. We aim to find some relations among constacyclic codes over R and introduced new codes as the Gray images of λ-constacyclic codes over R which is cyclic, quasi-cyclic and permutation equivalent to quasi-cyclic codes over Z_4.Our results are in agreement with those of [3] even after the change of units and Gray maps.

This paper is organised as follows: Section 2 lists some definitions and basic ideas of the ring. Section 3 defines different Gray maps with respect to the units 3 and $2+3u$, and studies the Gray images of λ-constacyclic codes over R. In Section 4, constacyclic codes over R are studied along with Nechaev's permutation. In Section 5, skew λ-constacyclic codes over R are studied with their Gray images and we conclude this article in Section 6.

2. Preliminaries

In [3], Bag *et al.* studied the finite commutative ring $R = Z_4 + uZ_4$, $u^2 = 3$. It is a local ring with characteristic 4 and cardinality 16. Clearly $R \cong \dfrac{Z_4[u]}{< u^2 - 3 >}$. For an element $c \in R$, we write $c = a + ub, \forall\ a, b \in Z_4$. The unit elements are $\{1, 3, u, 3u, 2 + u, 1 + 2u, 3 + 2u, 2 + 3u\}$ and non-unit elements are $\{0, 2, 2u, 1 + u, 1 + 3u, 2 + 2u, 3 + u, 3 + 3u\}$. There are 4 ideals in this ring given by $\{< 0 >, < 2 >, < 2 + 2u >, < 1 + u >\}$, with the chain condition $< 0 > \subset < 2 + 2u > \subset < 2 > \subset < 1 + u >$ and the ideal generated by $< 1 + u >$ is the unique maximal ideal. In this paper, we study constacyclic codes over R for the units 3 and $2 + 3u$ and introduce new Gray images.

A linear code $\mathcal{C}$ of length n over R is an R-submodule of R^n. Taking λ to be a unit in R, a linear code $\mathcal{C}$ over R is said to be λ-constacyclic code if and only if $\mathcal{C}$ is invariant under the constacyclic shift operator $\tau_\lambda : R^n \to R^n$ defined as $\tau_\lambda\big(c_0, c_1, ..., c_{n-1}\big) = \big(\lambda c_{n-1}, c_0, ..., c_{n-2}\big)$. For $\lambda = 1$, the constacyclic code is cyclic and negacyclic for $\lambda = -1$. Let us consider the cyclic shift operator defined as $\rho\big(c_0, c_1, ..., c_{n-1}\big) = \big(c_{n-1}, c_0, ..., c_{n-2}\big)$, then $\mathcal{C}$ is cyclic if $\rho\big(\mathcal{C}\big) = \mathcal{C}$.

For $a \in Z_4^{mn}$ with $\big(a^1|a^2| \cdots |a^m\big), a^i \in Z_4^n, \forall\ i = 1, 2, ..., m$, let η be a map from Z_4^{mn} to Z_4^{mn} defined by $\eta\,(a) = \big(\rho(a^1)|\rho(a^2)| \cdots |\rho(a^m)\big)$, where ρ is cyclic shift from Z_4^n to Z_4^n and "|" is the usual vector concatenation. A code of length mn over Z_4 is called quasi-cyclic code of index m is $\eta(\mathcal{C}) = \mathcal{C}$. A linear code $\mathcal{C}$ over the ring R is a s-quasi-cyclic if it is invariant under the cyclic shift ρ^s i.e. $\rho^s(\mathcal{C}) = \mathcal{C}$, where ρ is cyclic shift on R^n.

A linear code $\mathcal{C}$ of length n over R is λ-constacyclic code if and only if it is an ideal of $\dfrac{R[x]}{< x^n - \lambda >}$ under the R-module isomorphism $R^n \to \dfrac{R[x]}{< x^n - \lambda >}$ defined by

$$(c_0, c_1, ..., c_{n-1}) \to c_0 + c_1 x + \cdots + c_{n-1}x^{n-1} (mod < x^n - \lambda >).$$

3. Gray images of λ-Constacyclic Codes Over R

In this section, we introduce different Gray maps from R to copies of Z_4 and study λ-constacyclic codes over R for $\lambda = 3$ and $2 + 3u$.

3.1. λ-constacyclic codes over R.

For $\lambda = 3$, we establish three different Gray maps on the ring. Firstly, we define $\phi_1 : R \to Z_4^2$ by $\phi_1(a + ub) = (a + 2b, 3a + 2b)$. It is a Z_4-linear map and can be extended componentwise to a Gray map defined as, $\Phi_1 : R^n \to Z_4^{2n}$ by

$$\Phi_1(\bar{c}) = (a_0 + 2b_0, a_1 + 2b_1, ..., a_{n-1} + 2b_{n-1}, 3a_0 + 2b_0, ..., 3a_{n-1} + 2b_{n-1}),$$

where $\bar{c} = (c_0, c_1, ..., c_{n-1}) \in R^n$, $c_i = a_i + ub_i \ \forall \ a_i, b_i \in Z_4$ and $i = 0, 1, ..., n-1$.
Also, other two Gray maps are defined as

$$\Phi_2 : R^n \to Z_4^{2n} \quad \text{by}$$

$$\Phi_2(\bar{c}) = (2a_0, 2a_1, ..., 2a_{n-2}, 2a_{n-1}, 2b_0, 2b_1, ..., 2b_{n-2}, 2b_{n-1}),$$

and,

$$\Phi_3 : R^n \to Z_4^{3n} \quad \text{by}$$

$$\begin{aligned}\Phi_3(\bar{c}) =& (a_0 + 3b_0, a_1 + 3b_1, ..., a_{n-1} + 3b_{n-1}, 3a_0 + b_0, 3a_1 + b_1, \dots, 3a_{n-1} + b_{n-1},\\ & 2a_0, 2a_1, ..., 2a_{n-1}),\end{aligned}$$

where $\bar{c} = (c_0, c_1, ..., c_{n-1}) \in R^n$, $c_i = a_i + ub_i \ \forall \ a_i, b_i \in Z_4$ and $i = 0, 1, ..., n-1$.

With respect to $\lambda = 2 + 3u$, we define the Gray map as follows: $\Psi_1 : R^n \to Z_4^{3n}$ by

$$\begin{aligned}\Psi_1(\bar{c}) =& (2a_0, 2a_1, ..., 2a_{n-2}, 2a_{n-1}, 2b_0, 2b_1, ..., 2b_{n-2}, 2b_{n-1},\\ & 2a_0 + 2b_0, 2a_1 + 2b_1, \dots, 2a_{n-1} + 2b_{n-1}),\end{aligned}$$

where $\bar{c} = (c_0, c_1, ..., c_{n-1}) \in R^n$, $c_i = a_i + ub_i$ for $a_i, b_i \in Z_4$ and $i = 0, 1, ..., n-1$.
None of the Gray maps are bijective. By using the above defined Gray maps with cyclic shift, constacyclic shift and quasi-cyclic shift operations, we investigate the relation between them and obtain the following results.

Proposition 3.1. *Let τ_3 be the 3-constacyclic shift of R^n, then $\Phi_1\tau_3 = \rho\Phi_1$, where ρ and Φ_1 are introduced as above.*

Proof. Let $\bar{c} = (c_0, c_1, ..., c_{n-1}) \in R^n$, where $c_i = a_i + ub_i$, for $a_i, b_i \in Z_4$ and $i = 0, 1, 2, ..., n-1$.
Now,

$$\begin{aligned}\Phi_1\tau_3(\bar{c}) =& \Phi_1(3c_{n-1}, c_0, ..., c_{n-2})\\ =& (3a_{n-1} + 2b_{n-1}, a_0 + 2b_0, ..., a_{n-2} + 2b_{n-2}, a_{n-1} + 2b_{n-1}, 3a_0 + 2b_0,\\ & 3a_1 + 2b_1, ..., 3a_{n-2} + 2b_{n-2})\\ =& \rho\Phi_1(\bar{c}).\end{aligned}$$

Hence, $\Phi_1\tau_3 = \rho\Phi_1$.

Theorem 3.1. *The Φ_1-Gray image of 3-constacyclic code of length n over R is a cyclic code of length $2n$ over Z_4.*

Proof. Let $\mathcal{C}$ be a 3-constacyclic code of length n over R, then $\tau_3(\mathcal{C}) = \mathcal{C}$. Applying Φ_1 on both sides, we get $\Phi_1\tau_3(\mathcal{C}) = \Phi_1(\mathcal{C})$. By Proposition 3.1, we get $\rho\Phi_1(\mathcal{C}) = \Phi_1(\mathcal{C})$, which implies $\Phi_1(\mathcal{C})$ is a cyclic code of length $2n$ over Z_4.

Proposition 3.2. *Let τ_3 be the 3-constacyclic shift of R^n, then $\Phi_2\tau_3 = \eta\Phi_2$, where η and Φ_2 are introduced as above.*

Proof. Let $\bar{c} = (c_0, c_1, ..., c_{n-1}) \in R^n$, where $c_i = a_i + ub_i$, for $a_i, b_i \in Z_4$ and $i = 0, 1, 2, ..., n-1$. Now,

$$\begin{aligned}\Phi_2\tau_3(\bar{c}) =&\Phi_2(3c_{n-1}, c_0, ..., c_{n-2})\\ =&\big(2a_{n-1}, 2a_0, 2a_1, ..., 2a_{n-2}, 2b_{n-1}, 2b_0, 2b_1, ..., 2b_{n-2}\big)\\ =&\eta\Phi_2(\bar{c}).\end{aligned}$$

Hence, $\Phi_2\tau_3 = \eta\Phi_2$.

Theorem 3.2. *The Φ_2-image of 3-constacyclic code of length n over R is a quasi-cyclic code of index 2 over Z_4 of length $2n$.*

Proof. Let $\mathcal{C}$ be a 3-constacyclic code of length n over R, then $\tau_3(\mathcal{C}) = \mathcal{C}$. Applying Φ_2 on both sides, we get $\Phi_2\tau_3(\mathcal{C}) = \Phi_2(\mathcal{C})$. By Proposition 3.2, we get $\eta\Phi_2(\mathcal{C}) = \Phi_2(\mathcal{C})$. This implies that $\Phi_2(\mathcal{C})$ is a quasi-cyclic code of index 2 over Z_4 of length $2n$.

In the following results, we find the Gray-images of λ-constacyclic codes to be permutation equivalent to a quasi-cyclic codes over Z_4.

Proposition 3.3. *Let δ be a permutation of Z_4^{3n} defined by $\delta(x_1, x_2, ..., x_n, ..., x_{2n}, ..., x_{3n}) = \big(x_{\beta(1)}, ..., x_{\beta(n)}, .., x_{\beta(2n)}, ...x_{\beta(3n)}\big)$ with the permutation $\beta = (1, n+1)$ of $\{1, 2, ..., 3n\}$, then $\Phi_3\tau_3 = \delta\eta\Phi_3$, where τ_3, Φ_3 and η are introduced as above.*

Proof. Let $\bar{c} = (c_0, c_1, ..., c_{n-1}) \in R^n$, where $c_i = a_i + ub_i$, for $a_i, b_i \in Z_4$ and $i = 0, 1, 2, ..., n-1$. Now,

$$\begin{aligned}\Phi_3\tau_3(\bar{c}) =&\Phi_3(3c_{n-1}, c_0, ..., c_{n-2})\\ =&\big(3a_{n-1} + b_{n-1}, a_0 + 3b_0, a_1 + 3b_1, ..., a_{n-2} + 3b_{n-2}, a_{n-1} + 3b_{n-1}, 3a_0 + b_0,\\ &3a_1 + b_1, ..., 3a_{n-2} + b_{n-2}, 2a_{n-1}, 2a_0, 2a_1, ..., 2a_{n-2}\big).\end{aligned}$$

Also,

$$\begin{aligned}\eta\Phi_3(\bar{c}) =&\eta\big(a_0 + 3b_0, a_1 + 3b_1, ..., a_{n-1} + 3b_{n-1}, 3a_0 + b_0, 3a_1 + b_1, ..., 3a_{n-1} + b_{n-1},\\ &2a_0, 2a_1, ..., 2a_{n-1}\big)\\ =&\big(a_{n-1} + 3b_{n-1}, a_0 + 3b_0, ..., a_{n-2} + 3b_{n-2}, 3a_{n-1} + b_{n-1}, 3a_0 + b_0, ..., 3a_{n-2} + b_{n-2},\\ &2a_{n-1}, 2a_0, ..., 2a_{n-2}\big).\end{aligned}$$

Applying the permutation δ on $\eta\Phi_3(\bar{c})$, we get $\Phi_3\tau_3 = \delta\eta\Phi_3$.

Theorem 3.3. *The Φ_3-image of 3-constacyclic code of length n over R is permutation equivalent to a quasi-cyclic code of index 3 over Z_4 of length $3n$.*

Proof. Let $\mathcal{C}$ be a 3-constacyclic code of length n over R, so $\tau_3(\mathcal{C}) = \mathcal{C}$. Applying Φ_3 on both sides, we get $\Phi_3\tau_3(\mathcal{C}) = \Phi_3(\mathcal{C})$. By Proposition 3.3, we get $\delta\eta\Phi_3(\mathcal{C}) = \Phi_3(\mathcal{C})$. So, $\Phi_3(\mathcal{C})$ is permutation equivalent to a quasi-cyclic code of index 3 over Z_4 of length $3n$.

Proposition 3.4. *Let $\tau_{(2+3u)}$, Ψ_1 and η be the maps as introduced above, then $\Psi_1\tau_{(2+3u)} = \delta\eta\Psi_1$, where δ is a permutation of Z_4^{3n} defined in Proposition 3.1.5.*

Proof. Let $\bar{c} = (c_0, c_1, ..., c_{n-1}) \in R^n$, where $c_i = a_i + ub_i$, for $a_i, b_i \in Z_4$ and $i = 0, 1, 2, ..., n-1$. Now,

$$\begin{aligned}\Psi_1\tau_{(2+3u)}(\bar{c}) =& \Psi_1\left((2+3u)c_{n-1}, c_0, ..., c_{n-2}\right)\\ =& \Psi_1\left((2a_{n-1}+b_{n-1}) + u\left(3a_{n-1}+2b_{n-1}\right), a_0+ub_0, ..., a_{n-2}+ub_{n-2}\right)\\ =& \big(2b_{n-1}, 2a_0, 2a_1, ..., 2a_{n-2}, 2a_{n-1}, 2b_0, 2b_1, ..., 2b_{n-1}, 2a_{n-1}+2b_{n-1}, 2a_0+\\ & 2b_0, ..., 2a_{n-2}+2b_{n-2}\big).\end{aligned}$$

Also,

$$\begin{aligned}\delta\eta\Psi_1\left(\bar{c}\right) =& \delta\eta\big(2a_0, 2a_1, ..., 2a_{n-1}, 2b_0, 2b_1, ..., 2b_{n-1}, 2a_0+2b_0, ..., 2a_{n-1}+2b_{n-1}\big)\\ =& \delta\big(2a_{n-1}, 2a_0, ..., 2a_{n-2}, 2b_{n-1}, 2b_0, ..., 2b_{n-2}, 2a_{n-1}+2b_{n-1}, 2a_0+2b_0, ...,\\ & 2a_{n-2}+2b_{n-2}\big)\\ =& \big(2b_{n-1}, 2a_0, ..., 2a_{n-2}, 2a_{n-1}, 2b_0, ..., 2b_{n-2}, 2a_{n-1}+2b_{n-1}, 2a_0+2b_0, ...,\\ & 2a_{n-2}+2b_{n-2}\big).\end{aligned}$$

Hence, $\Psi 1\tau_{(2+3u)} = \delta\eta\Psi_1$.

Theorem 3.4. *The Ψ_1-image of $(2+3u)$-constacyclic code of length n over R is permutation equivalent to a quasi-cyclic code of index 3 over Z_4 of length $3n$.*

Proof. Let $\mathcal{C}$ be a $(2+3u)$-constacyclic code of length n over R, so $\tau_{(2+3u)}\left(\mathcal{C}\right) = \mathcal{C}$. Applying Ψ_1 on both sides, we get $\Psi_1\tau_{(2+3u)}\left(\mathcal{C}\right) = \Psi_1\left(\mathcal{C}\right)$. By Proposition 3.4, we get $\delta\eta\Psi_1\left(\mathcal{C}\right) = \Psi_1\left(\mathcal{C}\right)$. So, $\Psi_1\left(\mathcal{C}\right)$ is permutation equivalent to a quasi-cyclic code of index 3 over Z_4 of length $3n$.

3.2. **Permutation version of Gray maps.**

We use the permutation version $\Phi_{\pi,1}$ of Φ_1 defined as

$$\begin{aligned}\Phi_{\pi,1}\left(\bar{c}\right) =& \big(a_0+2b_0, 3a_0+2b_0, a_1+2b_1, 3a_1+2b_1, a_2+2b_2, 3a_2+2b_2, ...,\\ & a_{n-1}+2b_{n-1}, 3a_{n-1}+2b_{n-1}\big),\end{aligned}$$

where $\bar{c} = (c_0, c_1, ..., c_{n-1}) \in R^n$, $c_i = a_i + ub_i$ $\forall$ $a_i, b_i \in Z_4$ and $i = 0, 1, ..., n-1$.

Also, we introduce the permutation version $\Phi_{\pi,2}$ of Φ_2 as follows

$$\Phi_{\pi,2}\left(\bar{c}\right) = \big(2a_0, 2b_0, 2a_1, 2b_1, 2a_2, 2b_2, ..., 2a_{n-2}, 2b_{n-2}, 2a_{n-1}, 2b_{n-1}\big),$$

where $\bar{c} = (c_0, c_1, ..., c_{n-1}) \in R^n$, $c_i = a_i + ub_i$ $\forall$ $a_i, b_i \in Z_4$ and $i = 0, 1, ..., n-1$.

Proposition 3.5. *For any $\bar{c} \in R^n$, $\Phi_{\pi,1}\rho\left(\bar{c}\right) = \rho^2\Phi_{\pi,1}\left(\bar{c}\right)$.*

Proof. Here,

$$\begin{aligned}\rho^2\Phi_{\pi,1}(\bar{c}) =&\rho^2\big(a_0+2b_0, 3a_0+2b_0, a_1+2b_1, 3a_1+2b_1, a_2+2b_2, 3a_2+2b_2, ..., a_{n-1}+2b_{n-1},\\ &3a_{n-1}+2b_{n-1}\big)\\ =&\rho\big(3a_{n-1}+2b_{n-1}, a_0+2b_0, 3a_0+2b_0, a_1+2b_1, ..., a_{n-2}+2b_{n-2}, 3a_{n-2}+2b_{n-2},\\ &a_{n-1}+2b_{n-1}\big)\\ =&\big(a_{n-1}+2b_{n-1}, 3a_{n-1}+2b_{n-1}, a_0+2b_0, 3a_0+2b_0, a_1+2b_1, ..., a_{n-2}+2b_{n-2},\\ &3a_{n-2}+2b_{n-2}\big)\\ =&\Phi_{\pi,1}(c_{n-1}, c_0, c_1, ..., c_{n-2})\\ =&\Phi_{\pi,1}\rho(\bar{c}).\end{aligned}$$

Hence, the result follows.

Corollary 3.1. *Let $\mathcal{C}$ be a cyclic code of length n over R, then its Z_4 image $\Phi_{\pi,1}(\mathcal{C})$ is a 2-quasi-cyclic code of length $2n$ over Z_4.*

Proof. Here, $\mathcal{C}$ is cyclic code so, $\rho(\mathcal{C}) = \mathcal{C}$. Applying $\Phi_{\pi,1}$ on both sides, we get $\Phi_{\pi,1}\rho(\mathcal{C}) = \Phi_{\pi,1}(\mathcal{C})$. By Proposition 3.5, $\Phi_{\pi,1}\rho(\bar{c}) = \rho^2\Phi_{\pi,1}(\bar{c}), \forall \bar{c} \in R^n$, which means $\Phi_{\pi,1}(\mathcal{C}) = \rho^2\Phi_{\pi,1}(\mathcal{C})$. So, $\Phi_{\pi,1}(\mathcal{C})$ is a 2-quasi-cyclic code of length $2n$ over Z_4.

Proposition 3.6. *For any $\bar{c} \in R^n$, $\Phi_{\pi,2}\rho(\bar{c}) = \rho^2\Phi_{\pi,2}(\bar{c})$.*

Proof. Here,

$$\begin{aligned}\rho^2\Phi_{\pi,2}(\bar{c}) =&\rho^2\big(2a_0, 2b_0, 2a_1, 2b_1, 2a_2, 2b_2, ..., 2a_{n-2}, 2b_{n-2}, 2a_{n-1}, 2b_{n-1}\big)\\ =&\big(2a_{n-1}, 2b_{n-1}, 2a_0, 2b_0, 2a_1, 2b_1, 2a_2, 2b_2, ..., 2a_{n-2}, 2b_{n-2}\big)\\ =&\Phi_{\pi,2}(c_{n-1}, c_0, c_1, ..., c_{n-2})\\ =&\Phi_{\pi,2}\rho(\bar{c}).\end{aligned}$$

Hence, the result follows.

Corollary 3.2. *Let $\mathcal{C}$ be a cyclic code of length n over R, then its Z_4 image $\Phi_{\pi,2}(\mathcal{C})$ is a 2-quasi-cyclic code of length $2n$ over Z_4.*

Proof. Here $\mathcal{C}$ is cyclic code so, $\rho(\mathcal{C}) = \mathcal{C}$. Applying $\Phi_{\pi,2}$ on both sides, we get $\Phi_{\pi,2}\rho(\mathcal{C}) = \Phi_{\pi,2}(\mathcal{C})$. By Proposition 3.6, $\Phi_{\pi,2}\rho(\bar{c}) = \rho^2\Phi_{\pi,2}(\bar{c}), \forall\ \bar{c} \in R^n$ which gives $\Phi_{\pi,2}(\mathcal{C}) = \rho^2\Phi_{\pi,2}(\mathcal{C})$. So, $\Phi_{\pi,2}(\mathcal{C})$ is a 2-quasi-cyclic code of length $2n$ over Z_4.

4. 3-Constacyclic Codes of odd length with their generating polynomials

In this section, constacyclic codes of odd length are considered over the ring R. Here $\lambda^n = \lambda$, if n is odd and $\lambda = 3$ and we try to find the relation via the permutation map. Further, we also obtain the generator polynomial of the Gray images of the cyclic codes. Similar to the results in [2, 3, 7, 8, 13], the following results are stated without proofs.

Definition 4.1. *Let n be odd, the Nechaev's permutation π is defined as*

$$\pi(c_0, c_1, c_2, ..., c_{2n-1}) = \big(c_{\zeta(0)}, c_{\zeta(1)}, ..., c_{\zeta(2n-1)}\big)$$

with the permutation $\zeta = (1, n+1)(3, n+3)\cdots(2i+1, n+2i+1)\cdots(n-2, 2n-2)$ on $\{0, 1, 2, ..., 2n-1\}$.

Proposition 4.1. *Let n be an odd integer and $\lambda = 3$. Then the map $\Gamma : R_n \to R_{n,\lambda}$ defined by $\Gamma(c(x)) = c(\lambda x)$ is a ring isomorphism, where $R_n = \dfrac{R[x]}{< x^n - 1 >}$, $R_{n,\lambda} = \dfrac{R[x]}{< x^n - \lambda >}$.*

Corollary 4.1. *Let n be an odd integer. Then I is an ideal of R_n if and only if $\Gamma(I)$ is an ideal of $R_{n,\lambda}$.*

Corollary 4.2. *$\mathcal{C}$ is a cyclic code over R of length n if and only if $\mu(\mathcal{C})$ is a λ-constacyclic code of length n over R, where $\mu : R^n \to R^n$ is defined by $\mu(c_0, c_1, c_2, ..., c_{n-1}) = (c_0, \lambda c_1, \lambda^2 c_2, \ldots, \lambda^{n-1} c_{n-1})$.*

Theorem 4.1. *Let $\mathcal{C} =< a(x) + ub(x) >$ be a 3-constacyclic code of length n over R. Then $\Phi_1(\mathcal{C})$ is a cyclic code of length $2n$ over Z_4 generated by the polynomials $(a(x) + 2b(x)) + x^n(3a(x) + 2b(x))$ and $(2a(x) + 3b(x)) + x^n(2a(x) + b(x))$.*

Proof. For polynomials, we define the Gray map Φ_1 as

$$\Phi_1 : \frac{R[x]}{< x^n - \lambda >} \to \frac{Z_4[x]}{< x^n - 1 >} \times \frac{Z_4[x]}{< x^n - 1 >},$$

by $\Phi_1(a(x) + ub(x)) = (a(x) + 2b(x), 3a(x) + 2b(x)), \forall a(x), b(x) \in Z_4[x]$.

For $p_1(x), p_2(x) \in Z_4[x]$, we obtain

$$\begin{aligned}\Phi_1[(p_1(x) + up_2(x))(a(x) + ub(x))] =& p_1(x)(a(x) + 2b(x), 3a(x) + 2b(x)) \\ &+ p_2(x)(2a(x) + 3b(x), 2a(x) + b(x)).\end{aligned}$$

For a vector $(a, b) \in \frac{Z_4[x]}{< x^n - 1 >} \times \frac{Z_4[x]}{< x^n - 1 >}$, it corresponds to the vector $a + bx^n \in \frac{Z_4[x]}{< x^{2n} - 1 >}$. Thus, $\Phi_1(\mathcal{C})$ is generated by the polynomials $[a(x) + 2b(x) + x^n(3a(x) + 2b(x))]$ and $[2a(x) + 3b(x) + x^n(2a(x) + b(x))]$.

Similar to this result, we obtain the following theorems.

Theorem 4.2. *Let $\mathcal{C} =< a(x)+ub(x) >$ be a 3-constacyclic code of length n over R. Then $\Phi_2(\mathcal{C})$ is a quasi-cyclic code of length $2n$ over Z_4 generated by the polynomials $2a(x) + 2b(x)x^n$ and $2b(x) + 2a(x)x^n$.*

Theorem 4.3. *Let $\mathcal{C} =< a(x)+ub(x) >$ be a 3-constacyclic code of length n over R. Then $\Phi_3(\mathcal{C})$ is permutation equivalent to a quasi-cyclic code of length $3n$ over Z_4 generated by the polynomials $(a(x)+3b(x))+x^n(3a(x)+2b(x))+x^{2n}(2a(x))$ and $(a(x)+3b(x))+x^n(a(x)+b(x)) + x^{2n}(2b(x))$.*

We study 3-constacyclic codes of odd length in relation with the Nechaev's permutation π over Z_4^{2n}.

Proposition 4.2. *Let μ be the map defined in Corollary 4.2, π be the Nechaev's permutation and n be an odd integer, then $\Phi_1\mu = \pi\Phi_1$.*

Proof. Here n is odd, so

$$\begin{aligned}\mu(\bar{c}) =& (c_0, 3c_1, c_2, 3c_3 ..., 3c_{n-2}, c_{n-1}) \\ =& \big(a_0 + ub_0, 3a_1 + 3ub_1, a_2 + ub_2, 3a_3 + 3ub_3, ..., 3a_{n-2} + 3ub_{n-2}, a_{n-1} + \\ & ub_{n-1}\big) \\ \Longrightarrow \Phi_1\mu(\bar{c}) =& \big(a_0 + 2b_0, 3a_1 + 2b_1, a_2 + 2b_2, 3a_3 + 2b_3, ..., 3a_{n-2} + 2b_{n-2}, a_{n-1} + 2b_{n-1}, \\ & 3a_0 + 2b_0, a_1 + 2b_1, 3a_2 + 2b_2, a_3 + 2b_3, ..., a_{n-2} + 2b_{n-2}, 3a_{n-1} + 2b_{n-1}\big).\end{aligned}$$

Also,

$$\begin{aligned}\Phi_1(\bar{c}) =&\Phi_1(c_0,c_1,c_2,...,c_{n-1})\\ =&\big(a_0+2b_0,a_1+2b_1,a_2+2b_2,...,a_{n-1}+2b_{n-1},3a_0+2b_0,3a_1+2b_1,3a_2+2b_2,...,\\ &3a_{n-1}+2b_{n-1}\big).\end{aligned}$$

Applying the permutation π on Φ_1 we get the result i.e. $\Phi_1\mu=\pi\Phi_1$.

Corollary 4.3. *Let π be the Nechaev's permutation and n be odd. If θ is the Φ_1 Gray image of a cyclic code $\mathcal{C}$ over R, then $\pi(\theta)$ is a cyclic code.*

Proof. Let $\mathcal{C}$ be a cyclic code over R and $\theta=\Phi_1(\mathcal{C})$. By Proposition 4.2, $\Phi_1\mu(\mathcal{C})=\pi\Phi_1(\mathcal{C})=\pi(\theta)$. Since $\mathcal{C}$ is cyclic, by Corollary 4.2 $\mu(\mathcal{C})$ is a 3-constacyclic code of length n over R. By Theorem 3.1, $\Phi_1\mu(\mathcal{C})$ is a cyclic code of length $2n$ over Z_4. So, $\pi(\theta)$ is cyclic code of length $2n$ over Z_4.

Proposition 4.3. *Let μ be the map defined above, then $\Phi_3\mu=\chi\Phi_3$, where the permutation χ of Z_4^{3n} is defined by $\chi\big(c_1,c_1,...,c_{3n}\big)=\big(c_{\delta(1)},c_{\delta(2)},...,c_{\delta(3n)}\big)$, with the permutation $\delta=\big(2,n+2\big),\big(4,n+4\big),...,\big(n-1,2n-1\big)$ of $\big\{1,2,...,3n\big\}$.*

Proof. We have,

$$\begin{aligned}&\mu(\bar{c})=\big(a_0+ub_0,3a_1+3ub_1,a_2+ub_2,...,3a_{n-2}+3ub_{n-2},a_{n-1}+ub_{n-1}\big)\\ \Longrightarrow\ &\Phi_3\mu\big(\bar{c}\big)=\big(a_0+3b_0,3a_1+b_1,a_2+3b_2,...,a_{n-1}+3b_{n-1},3a_0+b_0,a_1+3b_1,3a_2+b_2,...,\\ &\qquad 3a_{n-1}+b_{n-1},2a_0,2a_1,2a_2,...,2a_{n-2},2a_{n-1}\big).\end{aligned}$$

Also,

$$\begin{aligned}\Phi_3(\bar{c})=&\big(a_0+3b_0,a_1+3b_1,a_2+3b_2,...,a_{n-1}+3b_{n-1},3a_0+b_0,3a_1+b_1,3a_2+b_2,...,\\ &3a_{n-1}+b_{n-1},2a_0,2a_1,2a_2,...,2a_{n-1}\big).\end{aligned}$$

Applying the permutation χ on Φ_3 we get, $\Phi_3\mu=\chi\Phi_3$.

Theorem 4.4. *Let n be odd integer, if α is the Φ_3-Gray image of the cyclic code $\mathcal{C}$ over R,then α is permutation equivalent to a quasi-cyclic code over Z_4 of index 3 of length $3n$ via the permutation map χ.*

Proof. Let $\alpha=\Phi_3(\mathcal{C})$, where $\mathcal{C}$ is cyclic code over R. By Proposition 4.3, $\Phi_3\mu(\mathcal{C})=\chi\Phi_3(\mathcal{C})=\chi\alpha$. Since $\mathcal{C}$ is cyclic, by Corollary 4.2, $\mu(\mathcal{C})$ is a 3-constacyclic code of length n over R. By Theorem 3.3, $\Phi_3\mu(\mathcal{C})$ is permutation equivalent to quasi-cyclic code of index 3 over Z_4 of length $3n$. So, α is permutation equivalent to a quasi-cyclic code over Z_4 of index 3 of length $3n$.

5. Skew λ-Constacyclic Codes and their Gray images

In this section, skew λ-constacyclic codes of length n over $R=Z_4+uZ_4$ over R are studied with reference to Gray images. For this, we consider the automorphism given in [3], $\theta:R\to R$, defined as $\theta(0)=0,\theta(1)=1,\theta(u)=3u$ i.e. $\theta(a+ub)=a+3ub\ \forall a,b\in Z_4$. The order of the automorphism is 2. The set $R\big[x;\theta\big]=\big\{a_0+a_1x+a_2x^2+\cdots+a_{n-1}x^{n-1}\big\}$ forms a ring (skew polynomial ring) under the usual addition of polynomials and multiplication with respect to the condition $\big(ax^i\big)\big(bx^j\big)=a\theta^i\big(b\big)x^{i+j}$.

Definition 5.1. [7] *A subset $\mathcal{C}$ of R^n is called a skew λ-constacyclic code of length n if $\mathcal{C}$ satisfies the following conditions:*
(a) $\mathcal{C}$ is an R-submodule of R^n.
(b) If $\bar{c}=(c_0,c_1,c_2,...,c_{n-1})\in\mathcal{C}$, then $\sigma_{\theta,\lambda}(\bar{c})=(\theta(\lambda c_{n-1}),\theta(c_0),...,\theta(c_{n-2}))\in\mathcal{C}$.

Theorem 5.1. [7] *Let $\mathcal{C}$ be a linear code of length n over R. Then $\mathcal{C}$ is a skew λ-constacyclic code over R if and only if $\mathcal{C}$ is a left $R[x;\theta]$-submodule of* $\dfrac{R[x;\theta]}{< x^n - \lambda >}$.

Proposition 5.1. *Let $\sigma_{\theta,\lambda}$ be a skew λ-constacyclic shift on R^n, then $\rho\Phi_1 = \Phi_1\sigma_{\theta,\lambda}$, where Φ_1 and ρ are defined above.*

Proof. Here,

$$\begin{aligned}\sigma_{\theta,\lambda}(\bar{c}) =& \big(\theta(3c_{n-1}), \theta(c_0), \theta(c_1), ..., \theta(c_{n-2})\big)\\ =& \big(3a_{n-1} + ub_{n-1}, a_0 + 3ub_0, ..., a_{n-2} + 3ub_{n-2}\big)\\ \implies \Phi_1\sigma_{\theta,\lambda}(\bar{c}) =& \big(3a_{n-1} + 2b_{n-1}, a_0 + 2b_0, ..., a_{n-2} + b_{n-2}, a_{n-1} + 2b_{n-1}, 3a_0 + 2b_0,\\ & ..., 3a_{n-2} + 2b_{n-2}\big)\\ =& \rho\Phi_1(\bar{c}).\end{aligned}$$

Theorem 5.2. *The Φ_1-Gray image of a skew λ-constacyclic code over R of length n is a cyclic code over Z_4 with length $2n$.*

Proof. Let $\mathcal{C}$ be a 3-skew constacyclic code of length n over R, then $\sigma_{\theta,\lambda}(\mathcal{C}) = \mathcal{C}$. Applying Φ_1 on both sides, we get $\Phi_1\sigma_{\theta,\lambda}(\mathcal{C}) = \Phi_1(\mathcal{C})$. Using Proposition 5.1, we obtain $\rho\Phi_1(\mathcal{C}) = \Phi_1(\mathcal{C})$ which means $\Phi_1(\mathcal{C})$ is a cyclic code of length $2n$ over Z_4.

Proposition 5.2. *Let $\sigma_{\theta,\lambda}$ be a skew λ-constacyclic shift on R^n and Φ_2 the Gray map from R^n to Z_4^{2n}, then $\Phi_2\tau_{(3)} = \Phi_2\sigma_{\theta,\lambda}$.*

Proof. Here,

$$\begin{aligned}\sigma_{\theta,\lambda}(\bar{c}) =& \big(\theta(3c_{n-1}), \theta(c_0), \theta(c_1), ..., \theta(c_{n-2})\big)\\ =& \big(3a_{n-1} + ub_{n-1}, a_0 + 3ub_0, ..., a_{n-2} + 3ub_{n-2}\big)\\ \implies \Phi_2\sigma_{\theta,\lambda}(\bar{c}) =& \big(2a_{n-1}, 2a_0, 2a_1, ..., 2a_{n-3}, 2a_{n-2}, 2b_{n-1}, 2b_0, 2b_1, ..., 2b_{n-3}, 2b_{n-2}\big).\end{aligned}$$

Also,

$$\begin{aligned}\Phi_2\tau_3(\bar{c}) =& \Phi_2(3c_{n-1}, c_0, ..., c_{n-2}) = \Phi_2(3a_{n-1} + 3ub_{n-1}, a_0 + ub_0, ..., a_{n-2} + ub_{n-2})\\ =& \big(2a_{n-1}, 2a_0, 2a_1, ..., 2a_{n-3}, 2a_{n-2}, 2b_{n-1}, 2b_0, 2b_1, ..., 2b_{n-3}, 2b_{n-2}\big)\\ =& \Phi_2\sigma_{\theta,\lambda}(\bar{c}).\end{aligned}$$

Theorem 5.3. *The Φ_2-Gray image of a skew λ-constacyclic code over R of length n is a quasi-cyclic code of index 2 over Z_4 of length $2n$.*

Proof. Let $\mathcal{C}$ be a 3-skew constacyclic code of length n over R, then $\sigma_{\theta,\lambda}(\mathcal{C}) = \mathcal{C}$. Applying Φ_2 on both sides, we get $\Phi_2\sigma_{\theta,\lambda}(\mathcal{C}) = \Phi_2(\mathcal{C})$. By Proposition 5.2, we get $\Phi_2\tau_{(3)}(\mathcal{C}) = \Phi_2(\mathcal{C})$. Using Proposition 3.2, we get $\eta\Phi_2(\mathcal{C}) = \Phi_2(\mathcal{C})$ which means $\Phi_2(\mathcal{C})$ is a quasi-cyclic code of index 2 over Z_4 of length $2n$.

Proposition 5.3. *Let $\sigma_{\theta,\lambda}$ be a skew λ-constacyclic shift on R^n, then $\Psi_1\tau_{(2+3u)} = \Psi_1\sigma_{\theta,\lambda}$, where Ψ_1 is the Gray map from R^n to Z_4^{3n}.*

Proof. Here,

$$\begin{aligned}\sigma_{\theta,\lambda}(\bar{c}) =& \big(\theta((2+3u)c_{n-1}), \theta(c_0), \theta(c_1), ..., \theta(c_{n-2})\big)\\ =& \big((2a_{n-1} + b_{n-1}) + u(a_{n-1} + 2b_{n-1}), a_0 + 3ub_0, ..., a_{n-2} + 3ub_{n-2}\big)\\ \implies \Psi_1\sigma_{\theta,\lambda}(\bar{c}) =& \big(2b_{n-1}, 2a_0, ..., 2a_{n-2}, 2a_{n-1}, 2b_0, ..., 2b_{n-2}, 2a_{n-1} + 2b_{n-1}, 2a_0 + 2b_0,\\ & ..., 2a_{n-2} + 2b_{n-2}\big).\end{aligned}$$

Now,

$$\begin{aligned}\Psi_1\tau_{(2+3u)}(\bar{c}) =&\Psi_1\left((2+3u)\,c_{n-1},c_0,...,c_{n-2}\right)\\ =&\Psi_1\left((2a_{n-1}+b_{n-1})+u\left(3a_{n-1}+2b_{n-1}\right),a_0+ub_0,...,a_{n-2}+ub_{n-2}\right)\\ =&\big(2b_{n-1},2a_0,...,2a_{n-2},2a_{n-1},2b_0,...,2b_{n-2},2a_{n-1}+2b_{n-1},2a_0+2b_0,...,\\ &2a_{n-2}+2b_{n-2}\big)\\ =&\Psi_1\sigma_{\theta,\lambda}\left(\bar{c}\right).\end{aligned}$$

Theorem 5.4. *The Ψ_1-Gray image of a skew λ-constacyclic code over R of length n is permutation equivalent to a quasi-cyclic code over Z_4 of length $3n$.*

Proof. Let $\mathcal{C}$ be a $(2+3u)$- skew constacyclic code of length n over R, then $\sigma_{\theta,\lambda}(\mathcal{C})=\mathcal{C}$. Applying Ψ_1 on both sides, we get $\Psi_1\sigma_{\theta,\lambda}(\mathcal{C})=\Psi_1(\mathcal{C})$. By Proposition 5.3, we get $\Psi_1\tau_{(2+3u)}(\mathcal{C})=\Psi_1(\mathcal{C})$. By Proposition 3.4, $\delta\eta\Psi_1(\mathcal{C})=\Psi_1(\mathcal{C})$ which means $\Psi_1(\mathcal{C})$ is permutation equivalent to a quasi-cyclic code of index 3 over Z_4 of length $3n$.

6. Conclusions

We have studied λ-constacyclic codes over $R=Z_4+uZ_4, u^2=3$ for $\lambda=3,2+3u$. By defining different Gray maps from R to the copies of Z_4, the Gray images of λ- constacyclic codes over R are observed to be cyclic, quasi-cyclic, 2-quasi-cyclic and permutation equivalent to a quasi-cyclic codes over Z_4. λ-constacyclic codes of odd length over R as well as the generating polynomial of the Gray images are considered. Further, it is observed that the images of skew-λ-constacyclic codes over R are cyclic, quasi-cyclic and permutation equivalent to quasi-cyclic code over Z_4.

Conflicts of Interests

The authors declare that they have no competing interests.

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