

CENTRAL STAR DECOMPOSITION OF COMPLETE GRAPHS

M. CAAY^{1*}, R. MAZA², §

ABSTRACT. In this paper, we define the central star decomposition of graphs and show that the oriented graphs correspond to central star decomposition of a simple graph and vice versa.

Keywords: Decomposition, star decomposition, central star decomposition, orientation of graphs, Marriage theorem.

AMS Subject Classification: 05C51, 05C20

1. INTRODUCTION

Graph decomposition is one of the most researched concepts in graph theory for it offers a wide range of research ideas and applications. One of the studied notions of decomposition is the star decomposition. A star decomposition of graph G is defined as the partitioning into stars, and its first attempt of study was done by Ae, et.al. in [10] in the unpublished works entitled, “*Line-disjoint Decomposition of Complete Graph into Stars*”. Tarsi in [11] introduced the notion of decomposition of graphs into stars, and Zhao and Wu in [12] studied this notion and concluded that there exist infinitely many graphs that have certain star decompositions with given bounds with respect to a particular degree. Sebastian, et.al. in [8] studied the minimum number of vertices of stars in a star decomposition of graphs.

One of the classical notion in graph theory is the Marriage Theorem which was introduced by Hall [6] in 1935, and has paved a lot of applications to combinatorial mathematics and graph theory. In combinatorial formulation, it deals with a collection of finite sets and provides the necessary and sufficient condition for finding a matching that covers at least one side of the graph. Button, et.al. [7] studied that given a group G and a finite index subgroup H of G , the application of marriage theorem can be used to show that there is a set T such that T is a transversal for both the set of left cosets and right cosets of H in G . Hall in [9] showed the application of marriage theorem as a tool in the usual

¹ Polytechnic University of the Philippines, Department of Mathematics and Statistics, Sta. Mesa, Manila - Philippines.

e-mail: mlcaay@pup.edu.ph; orcid: 0000-0002-7477-2957.

² Mapua University, Department of Mathematics, Intramuros, Manila, Philippines.

e-mail: remaza@mapua.edu.ph; orcid: 0000-0002-3451-4734.

* corresponding author.

§ Manuscript received: April 28, 2024; accepted: September 27, 2024; TWMS Journal of Applied and Engineering Mathematics, Vol.15, No.6 © Işık University, Department of Mathematics, 2025; all rights reserved.

proofs of the fact that an $r \times n$ Latin rectangle can always be extended to an $(r + 1) \times n$ Latin rectangle when $r < n$, and so, ultimately to Latin square.

In 2022, Gunasekara in [5] applied Marriage Theorem by assigning vertices to the edges to study the star decomposition in graphs. In this paper, we use the Marriage Theorem by assigning edges to the vertices of graph to establish the star decomposition and thus introduces the notion of central star decomposition as a result of applying the Marriage Theorem.

2. PRELIMINARIES

Let G be a graph and $A \subseteq V(G)$. We denote by $E_G(A)$ the set of all the edges of G incident with some vertices in A . That is,

$$E_G(A) = \{uv \in E(G) : u \in A \text{ or } v \in A\}.$$

Remark 2.1. For any set A of vertices of a graph G ,

$$E_G(A) = \bigcup_{u \in A} E_G(u).$$

The papers in [1], [2] and [3] state that $D \subset V(G)$ is the *dominating set* of G if for every $v \in V(G) \setminus D$, there exists $u \in D$ such that $uv \in E(G)$. Thus, the collection of center of the stars will always form a dominating set, but a dominating set of a graph will not necessarily form a set of centers of some central star decomposition.

The decomposition of the complete graphs into stars is seemingly unrelated concept to tournaments (directed complete graphs). Recall that a *digraph* D (or *directed graph*) is a graph whose edges are directed. We call the edges of D as arcs. In this paper, we sometimes called digraphs as *oriented graphs*. For a given vertex v of D , we call **outdegree** of v , denoted by v_{out} , as the number of vertices of D to which v is adjacent, while we call **indegree** of v , denoted by v_{in} , as the number of vertices of D from which v is adjacent.

We present the working definition of this study.

Definition 2.1. Given a graph G , we say G is centrally decomposable to the stars $S_{k_1}, S_{k_2}, \dots, S_{k_n}$ if

- (1) Every edge of G belongs exactly one of $S_{k_1}, S_{k_2}, \dots, S_{k_n}$.
- (2) The centers of the stars are distinct.

We shall occasionally call this decomposition a **central decomposition**.

Remark 2.2. The following are the notations used in this study:

- i) for simplicity, we denote by S_k the star S_k of size k ;
- ii) we denote $S_k(v)$ to be the star of size k whose center is v ; and
- iii) we let the trivial graph K_1 to be the star of size zero.

Following Definition 2.1, we have the following remarks:

Remark 2.3. Every graph has a central decomposition.

Remark 2.4. Every central star decomposition is a star decomposition, but converse is not necessarily true. In general, if a graph G with $V(G) = \{v_1, \dots, v_n\}$ is centrally decomposable to $S_{k_1}(v_1), \dots, S_{k_n}(v_n)$, then G is centrally decomposable to

$$S_{\deg(v_1)-k_1}(v_1), \dots, S_{\deg(v_n)-k_n}(v_n).$$

Definition 2.2. [4] A **tournament** is defined as a digraph such that for every pair u, v of distinct vertices, exactly one of (u, v) and (v, u) is an arc.

3. CENTRAL STAR DECOMPOSITION

We now state the two main results here:

Theorem 3.1. *Let G be a graph of order n and k_1, \dots, k_n be non-negative integers. A necessary and sufficient condition for G to be centrally decomposable to the stars S_{k_1}, \dots, S_{k_n} is that there exists a function ϕ from $V(G)$ onto $\{k_1, k_2, \dots, k_n\}$ and for every $A \subseteq V(G)$*

$$|E(G)| = \sum_{i=1}^n k_i \tag{1}$$

and

$$|E_G(A)| \geq \sum_{v \in A} \phi(v) \tag{2}$$

Proof. If a graph G is centrally decomposable to S_{k_1}, \dots, S_{k_n} , then there is an assignment of edges of G to the vertices of G and this assignment gives inequalities.

Conversely, suppose inequalities (2) and (1) holds then we can assign k_i edges to the set $E_G(v_i)$ and in turn assigns k_i edges to v_i . Thus, G is centrally decomposable to the stars $S_{k_1}, S_{k_2}, \dots, S_{k_n}$. □

The integers k_1, \dots, k_n in Theorem 3.1 are not necessarily distinct. However, if some of the k_i 's are equal then the function maps two distinct vertices, which are centers of some stars, to the same size. We are often using a much simpler version of theorem 3.1 in which we only need to consider positive values of k_i 's.

Theorem 3.2. *Let G be a graph with no isolated vertices and of order n , and k_1, \dots, k_m be positive integers with $m \leq n$. A necessary and sufficient for G to be centrally decomposable to the stars $S_{k_1}, S_{k_2}, \dots, S_{k_m}$ is that there exists a subset U of $V(G)$ with size m and a function ϕ from U onto $\{k_1, k_2, \dots, k_m\}$ such that for every $A \subseteq U$, we have*

$$|E(G)| = \sum_{i=1}^m k_i \tag{3}$$

and

$$|E_G(A)| \geq \sum_{v \in A} \phi(v). \tag{4}$$

Moreover, U forms a dominating set of G and consists of the centers of the stars in the decomposition.

Proof. If we assign zero edges to the rest of the vertices of G then the theorem 3.2 is exactly theorem 3.1, and the set U will consist of vertices mapped to a positive k_i . Hence, U will be the set of vertices where we assign the edges. In other words, U is a set of centers of the stars in the decomposition. □

Example 3.1. *Consider the figure below*

Lemma 3.1. *A complete graph K_n is centrally decomposable to p copies of the star S_m if and only if $p \leq 2(n - m) - 1$ and $pm = \frac{n(n-1)}{2}$.*

Proof. Let $G = K_n$. First note that for each $A \subseteq V(G)$, we have

$$|E_G(A)| = |A| \left(n - \frac{|A| + 1}{2} \right)$$

Indeed, every edge in $E_G(A)$ have either both end vertices are in A or exactly one end vertex in A . The number of edges with both the end vertices are in A is

$$\frac{|A|(|A| - 1)}{2}.$$

The number of edges incident to one vertex in A is $n - 1 - (|A| - 1) = n - |A|$. A total of $|A|(n - |A|)$ edges have one end vertex in A . Thus,

$$\begin{aligned} |E_G(A)| &= \frac{|A|(|A| - 1)}{2} + |A|(n - |A|) \\ &= |A| \left(n - \frac{|A| + 1}{2} \right). \end{aligned}$$

$$|E_G(S)| = |S| \left(n - \frac{|S| + 1}{2} \right)$$

If G is centrally decomposable to p copies of the star S_m , then there are p vertices in G that are centers of the stars S_i ($i \in \{1, 2, \dots, p\}$), and let Z be the set of the centers of the stars. All edges of K_n are incident to exactly one vertex in Z . Hence, $pm = \frac{n(n-1)}{2}$.

Now, we assign the edges of a star to the centers and assign no edge for any other vertices that are not centers (if any). Applying theorem 3.2 with p of the k_i 's are equal to m and all the other k_i are zero gives

$$\begin{aligned} |A| \left(n - \frac{|A| + 1}{2} \right) &= |E_G(A)| \\ &\geq |A|m \\ n - \frac{|A| + 1}{2} &\geq m \\ |A| &\leq 2(n - m) - 1 \end{aligned}$$

for $A \subseteq V(G)$ is non-empty. If Z is the set of centers, then

$$p = |Z| \leq 2(n - m) - 1$$

Conversely, suppose $p \leq 2(n - m) - 1$ and $pm = \frac{n(n-1)}{2}$, then let Z be any subset of $V(K_n)$ with cardinality p . Thus, for any $A \subseteq Z$,

$$|A| \leq 2(n - m) - 1$$

and so,

$$\begin{aligned} |E_G(A)| &= |A| \left(n - \frac{|A| + 1}{2} \right) \\ &\geq |A|m. \end{aligned}$$

This proves the claim. □

Theorem 3.3. *A complete graph K_n is centrally decomposable to p copies of the star S_m if and only if either i. $p = n$ and $m = \frac{n-1}{2}$ or ii. $p = n - 1$ and $m = \frac{n}{2}$.*

Proof. If K_n is centrally decomposable to p copies of the star S_m , then $p \leq 2(n - m) - 1$ and $pm = \frac{n(n-1)}{2}$ by lemma 3.1. From these two relations, we get the following.

$$\begin{aligned} \frac{n(n-1)}{2} &= pm \\ &\leq 2nm - 2m^2 - m \\ 2m^2 - m(2n-1) + \frac{n(n-1)}{2} &\leq 0 \end{aligned}$$

Thus,

$$\frac{n-1}{2} \leq m \leq \frac{n}{2}$$

and so the conclusion holds. The converse is obvious. □

In a star decomposition, we may think of a vertex as a center of multiple stars. Moreover, in a general decomposition into stars, a vertex which is a center of multiple stars can be considered to be a single center provided we think of the stars as one. For example, if the stars S_1, S_2, \dots, S_k all have the same center v , then we integrate all of them into one star with center v . In this manner, we can transform a problem of general decomposition to star into a problem of decompositions. We write this idea in the following remark.

Remark 3.1. *If a graph G is decomposable to stars*

$$\begin{aligned} &S_{m_{1,1}}, S_{m_{1,2}}, \dots, S_{m_{1,k_1}} \\ &S_{m_{2,1}}, S_{m_{2,2}}, \dots, S_{m_{2,k_2}} \\ &\quad \vdots \\ &S_{m_{n,1}}, S_{m_{n,2}}, \dots, S_{m_{n,k_1}} \end{aligned}$$

such that there exists n vertices v_1, \dots, v_n in G for which v_i is a center of

$$S_{m_{i,1}}, S_{m_{i,2}}, \dots, S_{m_{i,k_1}}$$

for all $i \in 1, 2, \dots, n$, then G is centrally decomposable to the stars

$$\begin{aligned} &S_{m_{1,1}+m_{1,2}+\dots+m_{1,k_1}} \\ &S_{m_{2,1}+m_{2,2},\dots,+m_{2,k_2}} \\ &\quad \vdots \\ &S_{m_{n,1}+m_{n,2},\dots,+m_{n,k_1}} \end{aligned}$$

It is worth noting that there exists graph that can be decomposed into both n number of S_m and m number of S_n . Indeed, a complete bipartite graph $K_{n,m}$ of order $n \times m$ where n and m are any positive integers.

Theorem 3.4. *Let G be a graph and k be a positive integer. Then G is decomposable to stars of size k if and only if there exists a dominating set $U = \{u_1, \dots, u_j\}$ of G and positive integers m_1, \dots, m_j such that*

- i. $|E(G)| = k \sum_{i=1}^j m_i$; and
- ii. for all $A \subseteq U$, $|E_G(A)| \geq k \sum_{i=1}^j m_i \cdot \chi_A(u_i)$.

Proof. If G is decomposable into stars of size k , then some of these stars may share some center. Let $U = \{u_1, \dots, u_j\}$ be the set of all centers of star in the decomposition. Let m_i be the number of stars in the decomposition having u_i as the center. Thus, G is decomposable into stars of sizes

$$m_1 \cdot k, m_2 \cdot k, \dots, m_j \cdot k.$$

By Theorem 3.2 with ϕ defined as $\phi(u_i) = m_i \cdot k$, we have satisfied (i.) of the above Theorem and for all $A \subseteq U$, $|E_G(A)| \geq \sum_{u \in A} \phi(u) = k \sum_{i=1}^j m_i \cdot \chi_A(u_i)$. Moreover, by Theorem 3.2, U is a dominating set.

Conversely, suppose there exists a dominating set $U = \{u_1, \dots, u_j\}$ of $V(G)$ and let $m_1 \cdot k, \dots, m_j \cdot k$ be positive integers satisfying (i) and (ii) above. Define $\phi(u_i) = m_i \cdot k$. Then (ii) implies that for every $A \subseteq U$,

$$|E_G(A)| \geq k \cdot \sum_{i=1}^j m_i \cdot \chi_A(u_i) = \sum \phi(u).$$

Moreover, condition (i) implies that $|E(G)| = \sum_{i=1}^j m_i \cdot k$. □

4. TOURNAMENT

Let D be an orientation of a graph G , that is, G and D have the same set of vertices but D has an oriented or directed edges. Let U be the set of all vertices in D with positive outdegrees. Observe that for each vertex $v \in U$ with at least one arc directed away from v , we form the star $S_k(v)$ with edges in $E(G)$ whose corresponding arc in $E(D)$ is directed away from v where k is outdegree of v . Hence, from the directed graph D we have formed a central star decomposition of G . Reversing this process gives us a directed graph D from a given star decomposition of G . Thus, the number of central star decomposition of a graph is precisely $2^{|E(G)|}$.

Note that a tournament is an orientation of a complete graph. In this section, we present the following results.

Theorem 4.1. *Let n be a positive integer and $m_1 \leq m_2 \leq \dots \leq m_n$ be non-negative integers. There exists a tournament of order n with indegree sequence $m_1 \leq m_2 \leq \dots \leq m_n$ if and only if for every $k \in \{1, 2, \dots, n\}$*

$$\sum_{i=1}^n m_i = \frac{n(n-1)}{2} \tag{5}$$

and

$$\sum_{i=1}^k m_i \geq \frac{k(k-1)}{2}. \tag{6}$$

Proof. It only requires to prove the sufficient condition. Let G be a complete simple graph of order n with $V(G) = \{v_1, v_2, \dots, v_n\}$. Finding a tournament of order n with indegree sequence $m_1 \leq m_2 \leq \dots \leq m_n$ is equivalent to finding a tournament of order n with outdegree sequence

$$n-1-m_1 \geq n-1-m_2 \geq \dots \geq n-1-m_n.$$

This outdegree sequence corresponds to a central star decomposition of G of the form

$$S_{n-1-m_1}, S_{n-1-m_2}, \dots, S_{n-1-m_n}.$$

According to Theorem 3.1, in order to find such central star decomposition with $\phi(v_i) = n - 1 - m_i$, G must satisfy

$$|E(G)| = \sum_{i=1}^n (n - 1 - m_i) \tag{7}$$

and

$$|E_G(A)| \geq \sum_{v_i \in A} (n - 1 - m_i). \tag{8}$$

Note that the relation (5) implies (7).

To show (8), let $A \subseteq V(G)$. Condition (6) implies that

$$\sum_{v_i \in A} m_i \geq \frac{|A|(|A| - 1)}{2}$$

Suppose that

$$\begin{aligned} |E_G(A)| &< \sum_{v_i \in A} (n - 1 - m_i) \\ &= (n - 1)|A| - \sum_{v_i \in A} m_i \\ &\leq (n - 1)|A| - \frac{|A|(|A| - 1)}{2}. \end{aligned}$$

Then

$$|E_G(A)| < |A| \left(n - \frac{|A| + 1}{2} \right).$$

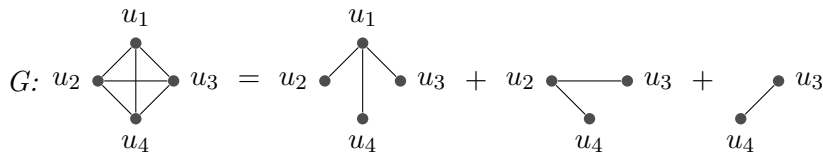
However, a similar computation to Lemma 3.1 implies that

$$|E_G(A)| = |A| \left(n - \frac{|A| + 1}{2} \right).$$

This shows a contradiction. □

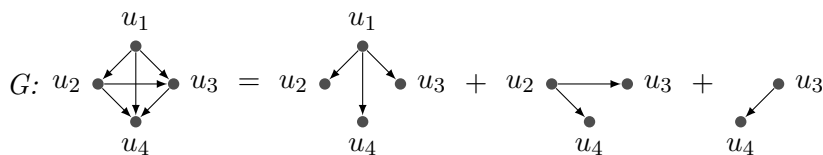
The following examples show the central star decomposition of K_4 and when it is viewed as a tournament.

Example 4.1. Consider the graph K_4 below and one of its central decompositions.



The centers of the stars in the central decomposition of K_4 in Example 4.1 are u_1 , u_2 and u_3 . Notice that we may choose u_4 to be the center of the third star but for the tournament below, we see that u_3 is the center for the case.

Example 4.2. The central decomposition of K_4 viewed as a tournament.



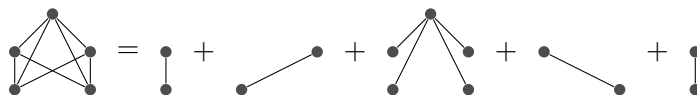
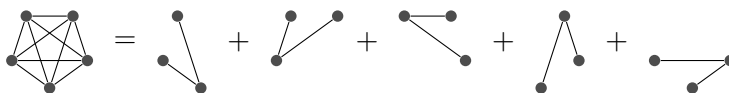


FIGURE 1. A non-trivial central star decomposition of a graph.

FIGURE 2. A central star decomposition of K_5 into 2-stars.

5. CONCLUSIONS AND RECOMMENDATIONS

In this paper, the authors have shown the existence of a central star decomposition of a complete graph, and concluded that an oriented graph is equivalent to a central star decomposition. Moreover, as a remark of the working definitions, it can easily be seen that a set of centers in a central star decomposition forms a dominating set of a graph, but the converse does not necessarily hold.

In view of the above conclusions, the authors would like to recommend interested researchers to answer the following:

- (i.) Naturally, what are the conditions that a dominating set of a graph coincides with the collection of centers of some central star decomposition?
- (ii.) Observe that a complete graph is a regular graph. What will happen if we consider any arbitrary regular graphs?

Acknowledgement. The authors would like to thank the anonymous reviewers of the TWMS Journal of Applied and Engineering Mathematics for the constructive feedback to improve the quality of the manuscript.

REFERENCES

- [1] Caay, M., Arugay, E., (2017), Perfect equitable domination of some graphs, *International Mathematical Forum*, 12(18), pp. 891-900.
- [2] Caay, M., Durog, M., (2021), On some independent equitable domination of some graphs, *Gulf Journal of Mathematics*, 11, pp. 57-64.
- [3] Caay, M., Palahang, S., (2022), On some results of perfect domination of some graphs, *TWMS Journal of Applied and Engineering Mathematics*, 12, pp. 600-607.
- [4] Chartrand, G., Zhang, P., (2012), *A first course in graph theory*, Dover Publications.
- [5] Gunasekara, A., (2022), *Completion and embedding problems for combinatorial designs* (PhD Thesis). Monash University.
- [6] Hall, P., (1935), On representatives of subsets, *Journal of the London Mathematical Society*, s1(10), pp. 26-30.
- [7] Buton, J., et. al., (2014), Coset intersection graphs for groups, *The American Monthly*, 121(10), pp. 922-926.
- [8] Sebastian, J., et. al., (2018), On star decomposition and star number of some graph classes, *Discrete Mathematics, Algorithms and Applications*, 5, pp. 81-85.
- [9] Hall, M. Jr., (1945), An existence theorem for Latin squares, *Bull. Amer. Math. Soc.*, 51, pp. 387-388.
- [10] Yamamoto, S., Ae, T., Yoshido, N., (to appear), Line-disjoint decomposition of complete graph into stars, *Journal of Combinatorial Theory, Series B*.
- [11] Tarsi, M., (1981), On the decomposition of a graph into stars, *Discrete Mathematics*, 36, pp. 299-304.
- [12] Zhao, Y., Wu, B., (2015), Star decomposition of graphs, *Discrete Mathematics, Algorithms and Applications*, 7.



Mark Lantaca Caay is a graduate of Bachelor of Science in Mathematics and Master of Science in Mathematics from the Mindanao State University - Iligan Institute of Technology (MSU-IIT) in Iligan City, Philippines in 2012 and 2020, respectively, where he focuses on research interests in Graph Theory, Combinatorics, Algebra and Mathematical Biology. He has also taken academic subjects in Doctor of Philosophy in Mathematics at the University of the Philippines - Diliman in Quezon City, Philippines and he is currently working as an Assistant Professor at the Polytechnic University of the Philippines in Manila, Philippines. Assistant Professor Caay has been handling and advising students in Graph Theory, Topology, Algebra and Mathematical Biology. Moreover, he has published several research papers in Graph Theory and Mathematical Biology, and one of his recent book publication is “The Calculus”.



Rodolfo Erodias Maza is a graduate of Bachelor of Science in Mathematics at the University of Mindanao in Davao City, Philippines in 2017, and a graduate of Master of Science in Mathematics at the Mindanao State University - Iligan Institute of Technology in Iligan City, Philippines, where he focuses on Analysis and Algebra. He is currently working on his Doctor of Philosophy in Mathematics at the University of the Philippines where he focuses on Analysis and Homogenization Theory. He is currently a faculty of mathematics at Mapua University in Manila Philippines. He has several publications in Analysis and Graph Theory. He has recent published book entitled “The Calculus” which is written together with Assistant Professor Caay in 2023.
