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RESISTANCE DISTANCE IN k-COALESCENCE OF CERTAIN GRAPHS

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Abstract. Any graph can be considered as a network of resistors, each of which has a resistance of 1Ω. The resistance distance r_{ij} between a pair of vertices i and j in a graph is defined as the effective resistance between i and j. Graph operations have been widely used in the analysis of complex networks, with properties abstracted from the real world. This article deals with the resistance distance in the k-coalescence of any two graphs having a clique of order k and also gives results for the particular case of complete graphs. Furthermore, we find Kemeny's constant, Kirchhoff index, additive degree-Kirchhoff index, multiplicative degree-Kirchhoff index and mixed degree-Kirchhoff index of k-coalescence of two complete graphs. Moreover, we obtain the resistance distance in the k-coalescence of a complete graph with particular graphs. Additionally, we provide the resistance distance of certain graphs such as the vertex coalescence of a complete bipartite graph with a complete graph, a complete bipartite graph with a star graph, the windmill graph, dandelion graph, etc.

Keywords: Resistance distance, Laplacian matrix, Coalescence, Kirchhoff index, Kemeny's constant.

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1. Introduction

Let $G_n = (V(G_n), E(G_n))$ be a simple connected undirected graph, consisting of n vertices $\{v_1, v_2, \ldots, v_n\}$ and m edges $\{e_1, e_2, \ldots, e_m\}$. A block is a maximal connected subgraph of a given graph G_n that has no cut vertex. The *adjacency matrix* [20] $A(G_n) = (a_{ij})$ of G_n is defined such that $a_{ij} = 1$ if vertex v_i is adjacent to vertex v_j, and it is zero otherwise. Denote the all-one entry matrix by $J_{n\times m}$, and the identity matrix by I_n. The *complete graph*, *path*, *cycle* and *star graph* are denoted by K_n, P_n, C_n, and $K_{1,n}$ respectively, and K_{n_1,n_2} is said to be the complete bipartite graph. For any two matrices B and C, their Kronecker product is a block matrix $B \otimes C = (b_{ij}C)$. Let P be an $n \times n$ matrix and let $S, K \subseteq \{1, 2, \ldots, n\}$. We denote by $P(S|K)$, the matrix obtained by selecting the rows corresponds to vertices in S and columns corresponds to vertices in K. Let d_i denote the degree of a vertex v_i in G_n. Note that the *Laplacian matrix*

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[27],$L(G_n) = D(G_n) - A(G_n)$, where $D(G_n)$ is the diagonal matrix of vertex degrees. The Laplacian matrix is essential in graph theory for analyzing various aspects of a graph, see [19, 20].

The *normalized Laplacian matrix* [10] $\mathcal{L}(G_n)$ of a graph G_n is an $n \times n$ matrix defined as $\mathcal{L}(G_n) = I_n - D^{1/2}A(G_n)D^{1/2}$. The application of the normalized Laplacian matrix can be seen in [21, 30]. Denote the eigenvalues of $\mathcal{L}(G_n)$ by $0 = \theta_1 < \theta_2 \leq \cdots \leq \theta_n$. The concept of *resistance distance* was introduced by Klein and Randić in 1993 [17]. The authors presented a new point of view, if we assign fixed resistances to each edge of a connected graph, then the resulting effective resistance between pairs of vertices corresponds to a graphical distance. For an $m \times n$ matrix M, the matrix P of order $n \times m$ is said to be a $\{1\}$-inverse of M (denoted by $M^{(1)}$) if $MPM = M$. For any square matrix N, its group-inverse $N^{\#}$, refers to a distinct matrix X that satisfies three conditions: $NXN = N$, $XNX = X$, and $NX = XN$. Clearly, the group inverse of N is a $\{1\}$-inverse of N [5].

The standard method to compute the resistance distance r_{ij} [2] between two vertices v_i and v_j is by using the $\{1\}$-inverse and group inverse of the Laplacian matrix $L = (l_{ij})$ of the underlying graph G_n which is

$$r_{ij} = l_{ii}^{(1)} + l_{jj}^{(1)} - l_{ij}^{(1)} - l_{ji}^{(1)} = l_{ii}^{\#} + l_{jj}^{\#} - 2l_{ij}^{\#}.$$

The matrix $R(G_n) = (r_{ij})_{n \times n}$ is called the resistance distance matrix of G_n.

Resistance distance is significant in combinatorial matrix theory [2, 3] and spectral graph theory [1, 9, 4, 32, 8]. For a survey of methods for finding resistance distance in graphs see [12]. Resistance distance, beyond serving as a distance function on graphs and a key component of electrical circuit theory, has significant applications in chemistry. It is more effective than the shortest-path distance in describing fluid or wave-like communications in molecules. One can explore graph connectivity through expressions for equivalent conductance, the inverse of resistance. Additionally, calculating resistance distance in specific types of electrical circuits is of significant interest to electrical engineers. Moreover, due to the analogy between electric circuits and random walks on graphs, resistance distance offers valuable insights into various aspects of graph theory. A recent approach for calculating the resistance distance of a generic circuit was proposed in [16].

Random walks are a type of Markov chain where the vertices of a graph G_n represent the state space, and movement between adjacent vertices is modeled stochastically. *Kemeny's constant* for such a Markov chain, denoted $\kappa(G_n)$, measures how well-connected the vertices of the graph are by quantifying the average time it takes to travel from a randomly chosen starting vertex to a randomly chosen destination vertex. Large values of $\kappa(G_n)$ may suggest clustering within the graph, while smaller values indicate good expansion properties. Kemeny's constant can be calculated in terms of effective resistances using the formula [24],

$$\kappa(G_n) = \frac{1}{4m} \sum_{v_i, v_j \in V(G_n)} d_i d_j r_{ij}. \tag{1}$$

There exists an alternative formula for Kemeny's constant in terms of the eigenvalues of the normalized Laplacian matrix, which is defined as $\kappa(G_n) = \sum_{i=2}^{n} \frac{1}{\theta_i}$ [29, 31].

The *Kirchhoff index* of G_n, also known as the total resistance of a network, represented as $\mathcal{K}f(G_n)$ [17, 7], is defined as,

$$\mathcal{K}f(G_n) = \sum_{i<j} r_{ij}. \tag{2}$$

The Kirchhoff index, a molecular invariant, quantifies a chemical network's global connectivity, with higher values indicating greater connectivity. For studies on the Kirchhoff index in phenylene networks, see [27].

The following are three graph parameters which are in terms of vertex degrees and resistance distance of a graph G_n.

The *mixed degree-Kirchhoff index* of G_n [6] is

$$\hat{R}(G_n) = \sum_{i<j} \left(\frac{d_i}{d_j} + \frac{d_j}{d_i} \right) r_{ij}. \tag{3}$$

The *multiplicative degree-Kirchhoff index* [9] of G_n is

$$R^*(G_n) = \sum_{i<j} d_i d_j r_{ij}. \tag{4}$$

The *additive degree-Kirchhoff index* [13] of G_n is

$$R^+(G_n) = \sum_{i<j} (d_i + d_j) r_{ij}. \tag{5}$$

Suppose we have two graphs G_n and $G'_{n'}$ with $v \in V(G_n)$ and $v' \in V(G'_{n'})$, then the coalescence $G_n \circ_1 G'_{n'}$[11] of G_n and $G'_{n'}$ with respect to v and v' is formed by identifying v and v'. Sudhir et al. [14] introduced the concept of k-coalescence of two graphs in their work, and it is defined as follows:

Definition 1.1. [14] *Let G_n and $G'_{n'}$ be two connected graphs of orders n and n' and sizes m and m', respectively having an induced complete graph order k with $n, n' \geq k$. Then the k-coalescence $G_n \circ_k G'_{n'}$ of G_n and $G'_{n'}$ is the graph obtained by identifying k vertices on ${}^kC_2$ edges of induced K_k. The order and size of $G_n \circ_k G'_{n'}$ are $n + n' - k$ and $m + m' - {}^kC_2$, respectively.*

FIGURE 1. $K_4 \circ K_3$ and $K_4 \circ K_{1,2}$.

Graph operations have been widely used to analyse complex networks with properties abstracted from the real world. The formulas for resistance distance and Kirchhoff index pertaining to numerous graph classes and graph operations were presented in [8, 23, 26]. Refer to [18, 25] for computations of Kemeny's constant using the hitting time of vertices in a random walk and normalized Laplacian eigenvalues, respectively. In [28], the authors found the resistance distance of join, product, etc. by using the definition in terms of the orthonormalized eigenvectors of the Laplacian eigenvalues. Here we consider the definition of resistance distance in terms of entries in the $\{1\}$-inverse of the Laplacian matrix.

This work aims to give the resistance distance in the k-coalescence of any two graphs having a complete subgraph of order k and also to give a closed-form expression of resistance distance for k-coalescence of complete graphs. In addition, we provide parameters such as the Kemeny's constant, Kirchhoff index, additive degree-Kirchhoff index, multiplicative degree-Kirchhoff index and mixed degree-Kirchhoff index of k-coalescence of two complete graphs. Moreover, we obtain these results for some classes of graphs.

2. Preliminaries

Through this section we present some useful lemmas and theorems.

Lemma 2.1. [33] *Let* $C = \begin{bmatrix} C_0 & C_1 \\ C_2 & C_3 \end{bmatrix}$ *be a nonsingular matrix. If* C_0 *and* C_3 *are nonsingular, then*

$$C^{-1} = \begin{bmatrix} (C_0 - C_1 C_3^{-1} C_2)^{-1} & -C_0^{-1} C_1 P^{-1} \\ -P^{-1} C_2 C_0^{-1} & P^{-1} \end{bmatrix},$$

where $P = C_3 - C_2 C_0^{-1} C_1$.

Lemma 2.2. [8] *Let* $L = \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{bmatrix}$ *be the Laplacian matrix of a connected graph. If each column vector of* L_2^T *is* $-e$ *(the all-ones column vector) or a zero vector, then* $L^{(1)} = \begin{bmatrix} L_1^{-1} & 0 \\ 0 & S^{\#} \end{bmatrix}$*, where* $S = L_3 - L_2^T L_1^{-1} L_2$.

Lemma 2.3. [22] *Let* $L = \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{bmatrix}$ *be the Laplacian matrix of a connected graph* G*. If* L_3 *is non-singular, then*

$$L^{(1)} = \begin{bmatrix} H^{\#} & -H^{\#} L_2 L_3^{-1} \\ -L_3^{-1} L_2^T H^{\#} & L_3^{-1} + L_3^{-1} L_2^T H^{\#} L_2 L_3^{-1} \end{bmatrix},$$

where $H = L_1 - L_2 L_3^{-1} L_2^T$.

Lemma 2.4. [8] *Let* L *be the Laplacian matrix of a graph* G_n*. For any* $a > 0$*, we have* $(L + aI - \frac{a}{n} J_n)^{\#} = (L + aI)^{-1} - \frac{1}{an} J_n$.

Lemma 2.5. [15] *For any real numbers* $r, s > 0$,

$$(rI_n - sJ_n)^{-1} = \frac{1}{r} I_n + \frac{s}{r(r - ns)} J_n.$$

3. Main results

This section provides the resistance distance in k-coalescence of certain graphs and discusses some of its graph parameters. For a graph G_n, with $F \subseteq V(G_n)$, let $G_n \setminus F$ denote the graph obtained from G_n by removing the vertices in F.

Theorem 3.1. *Let* G_{n_1} *and* G_{n_2} *be two graphs of orders* n_1, n_2 *having a complete subgraph of order* $k < min\{n_1, n_2\}$*. Suppose that* K *denotes the set of vertices in the clique of order* k*,* N_i *denotes the diagonal matrix with diagonal entries corresponding to the number of edges in* G_{n_i} *which are incident to the vertices of* K*, and let* $S_i = V(G_{n_i} \setminus K)$ *for* $i = 1, 2$*. Then the resistance distance between vertices in* $G_{n_1} \circ_k G_{n_2}$ *is as follows:*

(i) *for any two vertices* v_i, $v_j \in V(G_{n_1} \setminus K)$,
$r_{ij} = H_{ii}^{\#} + H_{jj}^{\#} - 2H_{ij}^{\#}$,

(ii) *for* $v_i \in V(G_{n_1} \setminus K)$, $v_j \in K$,
$r_{ij} = H_{ii}^{\#} + M_{ii}^{-1} + (M^{-1} A_1(S_1|K)^T H^{\#} A_1(S_1|K) M^{-1})_{ii} - 2(H^{\#} A_1(S_1|K) M^{-1})_{ij}$,

(iii) *for* $v_i \in V(G_{n_1} \setminus K)$, $v_2 \in V(G_{n_2}) \setminus K$,
$r_{ij} = H_{ii}^{\#} + P_{jj}^{-1} + (T^T A_1^T(S_1|K) H^{\#} A_1(S_1|K) T)_{jj} - 2(H^{\#} A_1(S_1|K) T)_{ij}$,

(iv) *for* v_i, $v_j \in K$,
$r_{ij} = M_{ii}^{-1} + M_{jj}^{-1} + (M^{-1} A_1^T(S_1|K) H^{\#} A_1(S_1|K) M^{-1})_{ii} - 2M_{ij}^{-1}$
$+ (M^{-1} A_1^T(S_1|K) H^{\#} A_1(S_1|K) M^{-1})_{jj} - 2(M^{-1} A_1^T(S_1|K) H^{\#} A_1(S_1|K) M^{-1})_{ij}$,

(v) *for* $v_i \in K,\ v_j \in V(G_{n_2} \setminus K)$,
$$r_{ij} = M^{-1}_{ii} + (M^{-1}A_1^T(S_1|K)H^{\#}A_1(S_1|K)M^{-1})_{ii} + P^{-1}_{jj} - 2T_{ij} \\ + (T^TA_1^T(S_1|K)H^{\#}A_1(S_1|K)T_{jj} - 2(M^{-1}A_1^T(S_1|K)H^{\#}A_1(S_1|K)T)_{ij},$$

(vi) *for* $v_i,\ v_j \in V(G_{n_2} \setminus K)$,
$$r_{ij} = P^{-1}_{ii} + P^{-1}_{jj} + (T^TA_1^T(S_1|K)H^{\#}A_1(S_1|K)T)_{ii} + (T^TA_1^T(S_1|K)H^{\#}A_1(S_1|K)T)_{jj} \\ - 2P^{-1}_{ij} - 2(T^TA_1^T(S_1|K)H^{\#}A_1(S_1|K)T)_{ij},$$

where $H = L(G_{n_1} \setminus K) + N_1 - A(S_1|K)M^{-1}A_1^T(S_1|K),\ M = D' - J - A_2(S_2|K)(L(G_{n_2} \setminus K) + N_2)^{-1}A_2^T(S_2|K),\ P = L(G_{n_2} \setminus K) + N_2 - A_2^T(S_2|K)(D' - J)^{-1}A_2(S_2|K),$ *and* $T = (D' - J)^{-1}A_2(S_2|K)P^{-1}$.

Proof. By a proper labelling of vertices in $G_{n_1} \circ_k G_{n_2}$, we get the Laplacian matrix of $G_{n_1} \circ_k G_{n_2}$ as

$$L(G_{n_1} \circ_k G_{n_2}) = \begin{bmatrix} L(G_{n_1} \setminus K) + N_1 & -A_1(S_1|K) & 0 \\ -A_1^T(S_1|K) & D' - J & -A_2(S_2|K) \\ 0 & -A_2^T(S_2|K) & L(G_{n_2} \setminus K) + N_2 \end{bmatrix}$$

where D' is a diagonal matrix with diagonal entries as $D'_{ii} = d_{G_{n_1}}(v_i) + d_{G_{n_2}}(v_i) - (k-2)$. If $L_1 = L(G_{n_1} \setminus K_k) + N_1$, $L_2 = \begin{bmatrix} -A_1(S_1|K) & 0 \end{bmatrix}$, and $L_3 = \begin{bmatrix} D' - J & -A_2(S_2|K) \\ -A_2^T(S_2|K) & L(G_{n_2} \setminus K) + N_2 \end{bmatrix}$, we can apply Lemma 2.3 to get $L^{(1)}(G_{n_1} \circ_k G_{n_2})$. First we compute L_3^{-1}, for that let $A_1 = D' - J$, $B_1 = -A_2(S_2|K)$, $C_1 = -A_2^T(S_2|K)$, and $D_1 = L(G_{n_2} \setminus K) + N_2$. By Lemma 2.1,

$$\begin{aligned} M^{-1} &= (A_1 - B_1D_1^{-1}C_1)^{-1} \\ &= \left(D' - J - A_2(S_2|K)(L(G_{n_2} \setminus K) + N_2)^{-1}A_2^T(S_2|K)\right)^{-1}, \end{aligned}$$

$$\begin{aligned} P^{-1} &= (D_1 - C_1A_1^{-1}B_1)^{-1} \\ &= \left(L(G_{n_2} \setminus K) + N_2 - A_2^T(S_2|K)(D' - J)^{-1}A_2(S_2|K)\right)^{-1}, \end{aligned}$$

$$-A_1^{-1}B_1P^{-1} = (D' - J)^{-1}A_2(S_2|K)P^{-1},$$

$$-P^{-1}C_1A_1^{-1} = P^{-1}A_2^T(S_2|K)(D' - J)^{-1},$$

Therefore, $L_3^{-1} = \begin{bmatrix} M^{-1} & T \\ T^T & P^{-1} \end{bmatrix}$, where $T = (D' - J)^{-1}A_2(S_2|K)P^{-1}$.

Now

$$\begin{aligned} H &= L_1 - L_2L_3^{-1}L_2^T \\ &= L(G_{n_1} \setminus K) + N_1 - A(S_1|K)M^{-1}A_1^T(S_1|K). \end{aligned}$$

$$-H^{\#}L_2L_3^{-1} = \begin{bmatrix} H^{\#}A_1(S_1|K)M^{-1} & H^{\#}A_1(S_1|K)T \end{bmatrix},$$

$$L_3^{-1} + L_3^{-1}L_2^TH^{\#}L_2L_3^{-1} = \begin{bmatrix} M^{-1} + M^{-1} + A_1^T(S_1|K)H^{\#}A_1(S_1|K)M^{-1} & T + M^{-1}A_1^T(S_1|K)H^{\#}A_1(S|K)T \\ T^T + T^TA_1^T(S_1|K)H^{\#}A_1(S_1|K)M^{-1} & P^{-1} + T^TA_1^T(S_1|K)H^{\#}A_1(S_1|K)T \end{bmatrix}.$$

Therefore,

$$L^{(1)} = \begin{bmatrix} H^{\#} & H^{\#}A_1(S_1|K)M^{-1} & H^{\#}A_1(S_1|K)T \\ M^{-1}A_1^T(S_1|K)H^{\#} & M^{-1}+M^{-1}A_1^T(S_1|K)H^{\#}A_1(S_1|K)M^{-1} & T+M^{-1}A_1^T(S_1|K)H^{\#}A_1(S|K)T \\ T^TA_1(S_1|K)H^{\#} & T^T+T^TA_1^T(S_1|K)H^{\#}A_1(S_1|K)M^{-1} & P^{-1}+T^TA_1^T(S_1|K)H^{\#}A_1(S_1|K)T \end{bmatrix}.$$

Then, by using the definition of resistance distance, we get the required result.

□

Example 3.1. *Consider $G \circ_3 G'$ as shown in Figure 2. From Theorem 3.1, we get the resistance matrix of $G \circ_3 G'$.*

$$R(G \circ_3 G') = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 0 & 0.63 & 0.63 & 1.18 & 1.18 & 1.63 & 1.63 & 1 \\ 0.63 & 0 & 0.54 & 0.72 & 0.9 & 1 & 1.54 & 1.63 \\ 0.63 & 0.54 & 0 & 0.9 & 0.72 & 1.54 & 1 & 1.63 \\ 1.18 & 0.72 & 0.9 & 0 & 0.72 & 1.72 & 1.9 & 2.18 \\ 1.18 & 0.9 & 0.72 & 0.72 & 0 & 1.9 & 1.72 & 2.18 \\ 1.63 & 1 & 1.54 & 1.72 & 1.9 & 0 & 2.54 & 2.63 \\ 1.63 & 1.54 & 1 & 1.9 & 1.72 & 2.54 & 0 & 2.63 \\ 1 & 1.63 & 1.63 & 2.18 & 2.18 & 2.63 & 2.63 & 0 \end{pmatrix} \end{array}$$

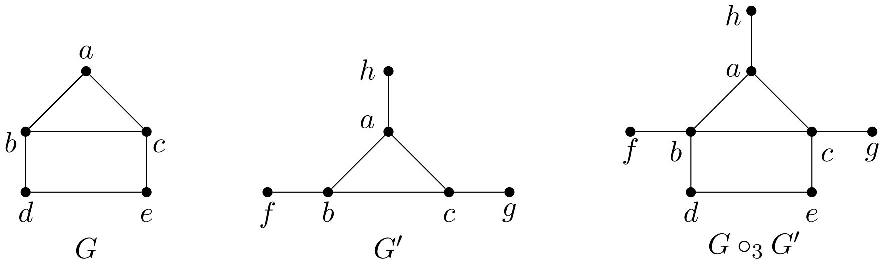

FIGURE 2. An example of 3-coalescence of two graphs

The graph $K_{p_1} \circ_k K_{p_2}$ consists of $t = p_1 + p_2 - k$ vertices. The next theorem gives the resistance distance in k-coalescence of two complete graphs.

Theorem 3.2. *For $p_1, p_2 \geq k$, let T be the collection of vertices in $K_{p_1} \circ_k K_{p_2}$, which are the identified vertices of some vertices in K_{p_1} and K_{p_2}. Also, let $t = p_1 + p_2 - k$, then,*

(i) *for any $v_i, v_j \in T$, $r_{ij} = \frac{2}{t}$.*
(ii) *for $v_i \in T, v_j \in V(K_{p_1} \setminus T)$, $r_{ij} = \frac{(k+1)(p_2-k)+2p_1k}{kp_1t}$.*
(iii) *for $v_i \in T, v_j \in V(K_{p_2} \setminus T)$, $r_{ij} = \frac{(k+1)(p_1-k)+2p_2k}{kp_2t}$.*
(iv) *for any $v_i, v_j \in V(K_{p_1} \setminus T), r_{ij} = \frac{2}{p_1}$.*
(v) *for $v_i \in V(K_{p_1} \setminus T), v_j \in V(K_{p_2} \setminus T), r_{ij} = \frac{(p_1+p_2)(k+1)}{kp_1p_2}$.*
(vi) *for any $v_i, v_j \in V(K_{p_2} \setminus T), r_{ij} = \frac{2}{p_2}$.*

Proof. The Laplacian matrix of $K_{p_1} \circ_k K_{p_2}$ is given by,

$$L(K_{p_1} \circ_k K_{p_2}) = \begin{bmatrix} tI_k - J_k & -J_{k \times p_1-k} & -J_{k \times p_2-k} \\ -J^T_{k \times p_1-k} & p_1I_{p_1-k} - J_{p_1-k} & 0 \\ -J^T_{k \times p_2-k} & 0 & p_2I_{p_2-k} - J_{p_2-k} \end{bmatrix}.$$

Let $L_1 = \begin{bmatrix} tI_k - J_k & -J_{k \times p_1-k} \\ -J^T_{k \times p_1-k} & p_1I_{p_1-k} - J_{p_1-k} \end{bmatrix}, L_2 = \begin{bmatrix} -J_{k \times p_2-k} \\ 0 \end{bmatrix}$, and $L_3 = p_2I_{p_2-k} - J_{p_2-k}$. Then by Lemmas 2.1 and 2.5 we get,

$$L_1^{-1} = \begin{bmatrix} \frac{1}{t}(I_k + \frac{p_1}{k(p_2-k)}J_k) & \frac{1}{k(p_2-k)}J_{k\times p_1-k} \\ \\ \frac{1}{k(p_2-k)}J_{k\times p_1-k}^T & \frac{1}{p_1}I_{p_1-k} + \frac{t}{p_1k(p_2-k)}J_{p_1-k} \end{bmatrix}.$$

Consider

$$\begin{aligned} S &= L_3 - L_2^TL_1^{-1}L_2 \\ &= p_2I_{p_2-k} - \frac{p_2}{p_2-k}J_{p_2-k} \end{aligned}$$

then, $S^{\#} = \frac{1}{p_2}(I_{p_2-k} - \frac{1}{p_2-k}J_{p_2-k})$.
From Lemma 2.2,

$$L^{(1)}(K_{p_1}\circ_k K_{p_2}) = \begin{bmatrix} \frac{1}{t}(I_k + \frac{p_1}{k(p_2-k)}J_k) & \frac{1}{k(p_2-k)}J_{k\times p_1-k} & 0 \\ \frac{1}{k(p_2-k)}J_{k\times p_1-k}^T & \frac{1}{p_1}\left(I_{p_1-k} + \frac{t}{k(p_2-k)}J_{p_1-k}\right) & 0 \\ 0 & 0 & \frac{1}{p_2}(I_{p_2-k} - \frac{1}{p_2-k}J_{p_2-k}) \end{bmatrix}.$$

For any $v_i, v_j \in T$,

$$\begin{aligned} r_{ij} &= \frac{2}{t}\left(1 + \frac{p_1}{k(p_2-k)}\right) - \frac{2p_1}{kt(p_2-k)} \\ &= \frac{2}{t}. \end{aligned}$$

For $v_i \in T, v_j \in V(K_{p_1} \setminus T)$,

$$\begin{aligned} r_{ij} &= \frac{1}{t}\left(1 + \frac{p_1}{k(p_2-k)}\right) + \frac{1}{p_1}\left(1 + \frac{t}{k(p_2-k)}\right) - \frac{2}{k(p_2-k)} \\ &= \frac{(k+1)(p_2-k) + 2p_1k}{kp_1t}. \end{aligned}$$

For $v_i \in T, v_j \in V(K_{p_2} \setminus T)$,

$$\begin{aligned} r_{ij} &= \frac{1}{t} + \frac{p_1}{k(p_2-k)t} + \frac{1}{p_2} - \frac{1}{p_2(p_2-k)} \\ &= \frac{(k+1)(p_1-k) + 2p_2k}{kp_2kt}. \end{aligned}$$

For any $v_i, v_j \in V(K_{p_1} \setminus T), r_{ij} = \frac{2}{p_1}$.

For $v_i \in V(K_{p_1} \setminus T), v_j \in V(K_{p_2} \setminus T)$,

$$\begin{aligned} r_{ij} &= \frac{1}{p_1}\left(1 + \frac{t}{k(p_2-k)}\right) + \frac{1}{p_2}\left(1 - \frac{1}{p_2-k}\right) \\ &= \frac{(p_1+p_2)(k+1)}{kp_1p_2}. \end{aligned}$$

For any $v_i,\ v_j \in V(K_{p_2} \setminus T), r_{ij} = \frac{2}{p_2}$.

Therefore, the resistance distance matrix of $K_{p_1} \circ_k K_{p_2}$ is

$$R(K_{p_1} \circ_k K_{p_2}) = \begin{bmatrix} \frac{2}{t}(J_k - I_k) & \frac{(k+1)(p_2-k)+2p_1k}{kp_1t} & \frac{(k+1)(p_1-k)+2p_2k}{p_2kt} J_{k\times p_2-k} \\ \frac{(k+1)(p_2-k)+2p_1k}{kp_1t} J^T_{k\times p_1-k} & \frac{2}{p_1}(J_{p_1-k} - I_{p_1-k}) & \frac{(p_1+p_2)(k+1)}{kp_1p_2} J_{p_1-k\times p_2-k} \\ \frac{(k+1)(p_1-k)+2kp_2}{kp_2t} J^T_{k\times p_2-k} & \frac{(p_1+p_2)(k+1)}{kp_1p_2} J^T_{p_1-k\times p_2-k} & \frac{2}{p_2}(J_{p_2-k} - I_{p_2-k}) \end{bmatrix}.$$

□

Example 3.2. *The kite graph $Kite_{p,2}$ is obtained by identifying a vertex in K_p to a pendant vertex of a path graph on 2 vertices, which can be viewed as $K_p \circ_1 K_2$. Let v^* be the identified vertex of a vertex v_1 in K_p and the vertex u_1 in K_2. Then by substituting $p_1 = p$, $p_2 = 2$, $k = 1$ and $T = \{v^*\}$ in Theorem 3.2 we have*

(a) *for $v_i = v^*$, $v_j \in V(K_p \setminus \{v^*\})$, $r_{ij} = \frac{2}{p}$,*
(b) *for $v_i, v_j \in V(K_p \setminus \{v^*\})$, $r_{ij} = \frac{2}{p}$,*
(c) *for $v_i = v^*$ and $v_j = u_2 \in V(K_2)$, $r_{ij} = 1$,*
(d) *for $v_i \in V(K_p \setminus \{v^*\})$ and $v_j = u_2 \in V(K_2)$, $r_{ij} = \frac{p+2}{p}$.*

The windmill graph W^t_{n+1} is the graph obtained by taking $t \geq 2$ copies of complete graph K_{n+1}, for $n \geq 1$, with a vertex in common. By the definition of coalescence of graphs one can easily write $W^t_{n+1} = K_{n+1} \underbrace{\circ_1 \cdots \circ_1}_{t-times} K_{n+1}$. Next proposition gives the resistance distance in W^t_{n+1}.

Proposition 3.1. *For $n > 1$, the resistance distance of vertices in W^t_{n+1} is given by,*

$$r_{ij} = \begin{cases} \frac{4}{n+1}, & \textit{if } v_i, v_j \textit{ are in different blocks,} \\ \frac{2}{n+1}, & \textit{otherwise.} \end{cases}$$

Proof. The Laplacian matrix of W^t_{n+1} is

$$L(W^t_{n+1}) = \begin{bmatrix} tnI_1 & -J_{1\times tn} \\ -J_{tn\times 1} & I_t \otimes (n+1)I_n - J_n \end{bmatrix}.$$

Then its $\{1\}$-inverse is

$$\begin{aligned} L^{(1)}(W^t_{n+1}) &= \begin{bmatrix} \frac{1}{tn}I_1 & 0 \\ 0 & (I_t \otimes (n+1)I_n - J_n - \frac{1}{tn}J_{tn})^{\#} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{tn}I_1 & 0 \\ 0 & I_t \otimes \frac{1}{n+1}(I_n + J_n) - \frac{1}{tn}J_{tn} \end{bmatrix}. \end{aligned}$$

Now by the definition of resistance distance we get the required result. □

From Proposition 3.1, we get the following corollaries.

Corollary 3.1. *The Kirchhoff index of W^t_{n+1} is*

$$Kf(W^t_{n+1}) = \frac{2n^2t^2 - n^2t + nt}{n+1}.$$

Proof. From (2), we have $Kf(G_n) = \sum_{i<j} r_{ij}$.

Then by applying Proposition 3.1, we get,

$$\begin{aligned} Kf(W^t_{n+1}) &= \left(\frac{2}{n+1}\right) nt + \left(\frac{2}{n+1}\right) \frac{n(n-1)t}{2} + \left(\frac{4}{n+1}\right) \frac{t(t-1)n^2}{2} \\ &= \frac{2n^2t^2 - n^2t + nt}{n+1}. \end{aligned}$$

□

Corollary 3.2. *The Kemeny's constant of* W_{n+1}^t *is*

$$\kappa(W_{n+1}^t) = \frac{n^2(2t-1)}{n+1}.$$

Proof. From (1), we have $\kappa(G_n) = \frac{1}{4m}\sum_{v_i,v_j\in V(G_n)} d_i d_j r_{ij}$.
Then by applying Proposition 3.1, we get,

$$\begin{aligned}\kappa(W_{n+1}^t) &= \frac{1}{tn(n+1)}\Bigg(\left(\frac{2}{n+1}\right)\frac{n(n-1)t}{2}(n^2) + \left(\frac{2}{n+1}\right)nt(n^2t)\\ &\quad + \left(\frac{4}{n+1}\right)\frac{n^2t(t-1)}{2}(n^2)\Bigg)\\ &= \frac{n^2(2t-1)}{n+1}.\end{aligned}$$

□

A 3-rose graph is a graph consisting of three cycles intersecting in a common vertex. Let $\mathcal{R}(r,s,t)$ denote the 3-rose graph on $n = r+s+t-2$ vertices, that is, the graph consisting of three cycles C_r, C_s and C_t intersecting in a common vertex.

Corollary 3.3. *The resistance distance for* $\mathcal{R}(3,3,3)$ *is given by,*

$$r_{ij} = \begin{cases} \frac{4}{3}, & \text{if } v_i, v_j \text{ are in different blocks,}\\ \frac{2}{3}, & \text{otherwise.}\end{cases}$$

Proof. By definition $\mathcal{R}(3,3,3)$ is a special case of W_{n+1}^t when $n=2$ and $t=3$. Then the proof directly follows from Proposition 3.1. □

For any two vertex-disjoint graphs G_{n_1} and G_{n_2}, their join $G_{n_1}\vee G_{n_2}$ is the graph in which each vertex of G_{n_1} is adjacent to every vertex of G_{n_2}.

Theorem 3.3. *Let* G_n *be a graph of order* n*. For* $p\geq k$*, let* T *be the collection of vertices in* $K_p \circ_k (G_n \vee K_k)$*, which are the identified vertices of* K_k *and some verices in* K_p*. Then,*

- (i) *for any* $v_i, v_j \in T$, $r_{ij} = \frac{2}{p+n}$.
- (ii) *for* $v_i \in T, v_j \in V(K_p\setminus T)$, $r_{ij} = \frac{k(2p+n)+n}{kp(p+n)}$.
- (iii) *for* $v_i \in T, v_j \in V(G_n)$, $r_{ij} = \frac{k-1}{k(p+n)} + (L(G_n)+kI_n)^{-1}_{jj}$.
- (iv) *for any* $v_i, v_j \in V(K_p\setminus T), r_{ij} = \frac{2}{p}$.
- (v) *for* $v_i \in V(K_p\setminus T), v_j\in V(G_n), r_{ij} = \frac{k+1}{kp} + (L(G_n)+kI_n)^{-1}_{jj}$.
- (vi) *for any* $v_i, v_j \in V(G_n), r_{ij} = (L(G_n)+kI_n)^{-1}_{ii} + (L(G_n)+kI_n)^{-1}_{jj} - 2(L(G_n)+kI_n)^{-1}_{ij}$.

Proof. The Laplacian matrix of $K_p \circ_k (G_n\vee K_k)$ is given by,

$$L(K_p\circ_k (G_n\vee K_k)) = \begin{bmatrix} (p+n)I_k - J_k & -J_{k\times p-k} & -J_{k\times n}\\ -J^T_{k\times p-k} & pI_{p-k} - J_{p-k} & 0\\ -J^T_{k\times n} & 0 & L(G_n)+kI_n\end{bmatrix}.$$

Let $L_1 = \begin{bmatrix}(p+n)I_k - J_k & -J_{k\times p-k}\\ -J_{p-k\times k} & pI_{p-k}-J_{p-k}\end{bmatrix}, L_2 = \begin{bmatrix}-J_{k\times n}\\ 0\end{bmatrix}$, and $L_3 = L(G_n)+kI_n$.
Then by Lemmas 2.1 and 2.5 we get,

$$L_1^{-1} = \begin{bmatrix} \frac{1}{p+n}(I_k + \frac{p}{nk}J_k) & \frac{1}{nk}J_{k\times p-k} \\ \frac{1}{nk}J_{p-k\times k} & \frac{1}{p}(I_{p-k} + \frac{p+n}{nk}J_{p-k}) \end{bmatrix}.$$

Now let $S = L_3 - L_2^T L_1^{-1} L_2$, then $S = L(G_n) + kI_n - \frac{k}{n}J_n$.
From Lemma 2.4, $S^{\#} = (L(G_n) + kI_n)^{-1} - \frac{k}{n}J_n$.

Therefore,

$$L^{(1)}(K_p \circ_k (G_n \vee K_k)) = \begin{bmatrix} \frac{1}{p+n}(I_k + \frac{p}{nk}J_k) & \frac{1}{nk}J_{k\times p-k} & 0 \\ \frac{1}{nk}J_{p-k\times k} & \frac{1}{p}(I_{p-k} + \frac{p+n}{nk}J_{p-k}) & 0 \\ 0 & 0 & (L(G_n) + kI_n)^{-1} - \frac{1}{kn}J_n \end{bmatrix}.$$

By applying the definition of resistance distance, we obtain the required result. □

Theorem 3.4. *If G_n is a graph of order n, then the resistance distance of the vertices in $K_{1,p-1} \circ_1 (G_n \vee K_1)$ is given by,*

(i) *for $v_i = u^*, v_j \in V(K_{1,p-1} \setminus \{u^*\})$, $r_{ij} = 1$,*
(ii) *for $v_i = u^*$, $v_j \in V(G_n)$, $r_{ij} = (L(G_n) + I_n)^{-1}_{jj}$,*
(iii) *for $v_i, v_j \in V(K_{1,p-1} \setminus \{u^*\}), r_{ij} = 2$,*
(iv) *for $v_i \in V(K_{1,p-1} \setminus \{u^*\}), v_j \in V(G_n)$, $r_{ij} = 1 + (L(G_n) + I_n)^{-1}_{jj}$,*
(v) *for $v_i, v_j \in V(G_n), r_{ij} = (L(G_n) + I_n)^{-1}_{ii} + (L(G_n) + I_n)^{-1}_{jj} - 2(L(G_n) + I_n)^{-1}_{ij}$,*

where u^* is the identified vertex of $K_{1,p-1}$ (center) and K_1.

Proof. The Laplacian matrix of $K_{1,p-1} \circ_1 (G_n \vee K_1)$ is given by,

$$L(K_{1,p-1} \circ_1 (G_n \vee K_1)) = \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{bmatrix},$$

where $L_1 = \begin{bmatrix} (p+n-1)I_1 & -J_{1\times(p-1)} \\ -J_{(p-1)\times 1} & I_{p-1} \end{bmatrix}, L_2 = \begin{bmatrix} -J_{1\times n} \\ 0 \end{bmatrix}$, and $L_3 = L(G_n) + I_n$.
Then by Lemmas 2.1 and 2.5 we get,

$$L_1^{-1} = \begin{bmatrix} \frac{1}{n}I_1 & \frac{1}{n}J_{1\times(p-1)} \\ \frac{1}{n}J_{p-1\times 1} & I_p + \frac{1}{n}J_p \end{bmatrix}.$$

Now let $S = L_3 - L_2^T L_1^{-1} L_2$, then $S = L(G_n) + I_n - \frac{1}{n}J_n$.
From Lemma 2.4, $S^{\#} = (L(G_n) + I_n)^{-1} - \frac{1}{n}J_n$.
Therefore,

$$L^{(1)}(K_{1,p-1} \circ_1 (G_n \vee K_1)) = \begin{bmatrix} \frac{1}{n}I_1 & \frac{1}{n}J_{1\times(p-1)} & 0 \\ \frac{1}{n}J_{(p-1)\times 1} & I_{p-1} + \frac{1}{n}J_{p-1} & 0 \\ 0 & 0 & (L(G_n) + I_n)^{-1} - \frac{1}{n}J_n \end{bmatrix}.$$

Now by the definition of resistance distance we get the required result. □

Theorem 3.5. *The resistance distance matrix of $K_{p,q} \circ_1 K_{1,n}$ is given by,*

$$R(K_{p,q} \circ_1 K_{1,n}) = \begin{bmatrix} 0 & \frac{2}{q}J_{1\times(p-1)} & \frac{p+q-1}{pq}J_{1\times q} & J_{1\times n} \\ \frac{2}{q}J_{(p-1)\times 1} & \frac{2}{q}(J_{p-1} - I_{p-1}) & \frac{p+q-1}{pq}J_{(p-1)\times q} & \frac{q+2}{2}J_{(p-1)\times n} \\ \frac{p+q-1}{pq}J_{q\times 1} & \frac{p+q-1}{pq}J_{q\times(p-1)} & \frac{2}{p}(J_q - I_q) & \frac{q(p+1)+(p-1)}{pq}J_{q\times n} \\ J_{n\times 1} & \frac{q+2}{q}J_{n\times(p-1)} & \frac{q(p+1)+(p-1)}{pq}J_{n\times q} & 2(J_n - I_n) \end{bmatrix}.$$

Proof. The Laplacian matrix of $K_{p,q} \circ_1 K_{1,n}$ is given by,

$$L(K_{p,q} \circ_1 K_{1,n}) = \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{bmatrix},$$

where $L_1 = \begin{bmatrix} (q+n)I_1 & 0 & -J_{1\times q} \\ 0 & qI_{p-1} & -J_{(p-1)\times q} \\ -J_{q\times 1} & -J_{q\times(p-1)} & pI_q \end{bmatrix}$, $L_2 = \begin{bmatrix} -J_{1\times n} \\ 0 \\ 0 \end{bmatrix}$, and $L_3 = I_n$.

Then by Lemmas 2.1 and 2.5 we get,

$$L_1^{-1} = \begin{bmatrix} \frac{1}{n}I_1 & J_{1\times(p-1)} & \frac{1}{n}J_{1\times q} \\ \frac{1}{n}J_{(p-1)\times 1} & \frac{1}{q}I_{p-1} + \frac{n+q}{nq}J_{p-1} & \frac{q+n}{nq}J_{(p-1)\times q} \\ \frac{1}{n}J_{q\times 1} & \frac{q+n}{nq}J_{q\times(p-1)} & \frac{1}{p}I_q + \frac{p(q+n)-n}{npq}J_q \end{bmatrix}.$$

Consider

$$\begin{aligned} S &= L_3 - L_2^T L_1^{-1} L_2 \\ &= I_n - \frac{1}{n}J_n \end{aligned}$$

then, $S^{\#} = I_n - \frac{1}{n}J_n$.

Therefore,

$$L^{(1)}(K_{p,q} \circ_1 K_{1,n}) = \begin{bmatrix} \frac{1}{n}I_1 & J_{1\times(p-1)} & \frac{1}{n}J_{1\times q} & 0 \\ \frac{1}{n}J_{(p-1)\times 1} & \frac{1}{q}I_{p-1} + \frac{n+q}{nq}J_{p-1} & \frac{q+n}{nq}J_{(p-1)\times q} & 0 \\ \frac{1}{n}J_{q\times 1} & \frac{q+n}{nq}J_{q\times(p-1)} & \frac{1}{p}I_q + \frac{p(q+n)-n}{npq}J_q & 0 \\ 0 & 0 & 0 & I_n - \frac{1}{n}J_n \end{bmatrix}.$$

Now by the definition of resistance distance we get the required result. □

Theorem 3.6. *The resistance distance matrix of $K_{p,q} \circ_1 K_n$ is given by,*

$$R(K_{p,q} \circ_1 K_n) = \begin{bmatrix} 0 & \frac{2}{q}J_{1\times(p-1)} & \frac{p+q-1}{pq}J_{1\times q} & \frac{2}{n}J_{1\times n} \\ \frac{2}{q}J_{(p-1)\times 1} & \frac{2}{q}(J_{p-1} - I_{p-1}) & \frac{p+q-1}{pq}J_{(p-1)\times q} & \frac{2(q+n)}{qn}J_{(p-1)\times n} \\ \frac{p+q-1}{pq}J_{q\times 1} & \frac{p+q-1}{pq}J_{q\times(p-1)} & \frac{2}{p}(J_q - I_q) & \frac{q(n+2p)+n(p-1)}{npq}J_{q\times n} \\ \frac{2}{n}J_{n\times 1} & \frac{2(q+n)}{qn}J_{n\times(p-1)} & \frac{q(n+2p)+n(p-1)}{npq}J_{n\times q} & \frac{2}{n}(J_{n-1} - I_{n-1}) \end{bmatrix}.$$

Proof. The Laplacian matrix of $K_{p,q} \circ_1 K_n$ is given by,

$$L(K_{p,q} \circ_1 K_n) = \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3 \end{bmatrix},$$

where $L_1 = \begin{bmatrix} (q+n-1)I_1 & 0 & -J_{1\times q} \\ 0 & qI_{p-1} & -J_{(p-1)\times q} \\ -J_{q\times 1} & -J_{q\times(p-1)} & pI_q \end{bmatrix}$, $L_2 = \begin{bmatrix} -J_{1\times n-1} \\ 0 \\ 0 \end{bmatrix}$, and $L_3 = nI_{n-1} - J_{n-1}$.

Then by Lemmas 2.1 and 2.5 we get,

$$L_1^{-1} = \begin{bmatrix} \frac{1}{n-1}I_1 & \frac{1}{n-1}J_{1\times(p-1)} & \frac{1}{n-1}J_{1\times q} \\ \frac{1}{n-1} & \frac{1}{q}(I_{p-1} + \frac{q+n-1}{n-1}J_{p-1}) & \frac{q+n-1}{q(n-1)}J_{(p-1)\times q} \\ \frac{1}{n-1}J_{q\times 1} & \frac{q+n-1}{q(n-1)}J_{q\times(p-1)} & \frac{1}{p}(I_q + \frac{p(q+n-1)-(n-1)}{q(n-1)}J_q) \end{bmatrix}.$$

Consider

$$\begin{aligned} S &= L_3 - L_2^T L_1^{-1} L_2 \\ &= nI_{n-1} - \frac{n}{n-1}J_{n-1} \end{aligned}$$

then, $S^{\#} = \frac{1}{n}I_{n-1} - \frac{1}{n(n-1)}J_{n-1}$.
Therefore,

$$L^{(1)}(K_{p,q} \circ_1 K_n) = \begin{bmatrix} \frac{1}{n-1}I_1 & \frac{1}{n-1}J_{1\times(p-1)} & \frac{1}{n-1}J_{1\times q} & 0 \\ \frac{1}{n-1} & \frac{1}{q}(I_{p-1} + \frac{q+n-1}{n-1}J_{p-1}) & \frac{q+n-1}{q(n-1)}J_{(p-1)\times q} & 0 \\ \frac{1}{n-1}J_{q\times 1} & \frac{q+n-1}{q(n-1)}J_{q\times(p-1)} & \frac{1}{p}(I_q + \frac{p(q+n-1)-(n-1)}{q(n-1)}J_q) & 0 \\ 0 & 0 & 0 & \frac{1}{n}(I_{n-1} - \frac{1}{n-1}J_{n-1}) \end{bmatrix}.$$

Now by the definition of resistance distance we get the required result. ☐

The *pineapple graph* K_p^q is the coalescence of the complete graph K_p (at any vertex) with the star $K_{1,q}$ at the vertex of degree q. It has $n = p + q$ vertices and ${}^pC_2 + q$ edges.

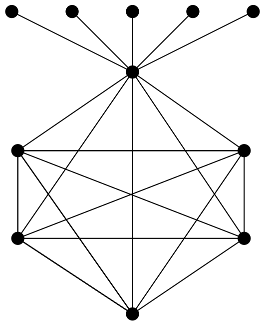

FIGURE 3. K_6^5.

Using Theorem 3.6, we get the following proposition.

Proposition 3.2. *The resistance distance in a pineapple graph K_p^q is given by*

(i) *for $v_i = v^{**}, v_j \in V(K_p \setminus \{v^{**}\})$, $r_{ij} = \frac{2}{p}$,*
(ii) *for $v_i = v^{**}$, $v_j \in V(K_{1,q} \setminus \{v^{**}\})$, $r_{ij} = 1$,*
(iii) *for $v_i, v_j \in V(K_p \setminus \{v^{**}\}), r_{ij} = \frac{2}{p}$,*
(iv) *for $v_i \in V(K_p \setminus \{v^{**}\}), v_j \in V(K_{1,q} \setminus \{v^{**}\})$, $r_{ij} = \frac{p+2}{p}$,*
(v) *for $v_i, v_j \in V(K_{1,q} \setminus \{v^{**}\}), r_{ij} = 2$,*

where v^{**} is the identified vertex K_p and $K_{1,q}$ (center).

From Proposition 3.2, we get the following corollaries.

Corollary 3.4. *The Kirchhoff index of K_p^q is*

$$Kf(K_p^q) = \frac{p(p+q)(q+1) - p - 2q}{p}.$$

Corollary 3.5. *The Kemeny's constant of K_p^q is*

$$\kappa(K_p^q) = \frac{p^4 - 3p^3 + p^3q + 3p^2q + 2pq^2 + 3p^2 - 9pq - p + 4q}{p^2(p-1) + 2pq}.$$

The *dandelion graph* $D(n, l)$ on n vertices is the coalescence of the star graph $K_{1,n-l}$ (at the center) with the path P_l at any pendant vertex.

The following theorem describes the resistance distance matrix of a dandelion graph $D(n, l)$.

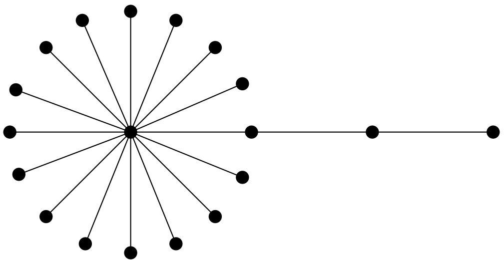

FIGURE 4. $D(19,4)$.

Theorem 3.7. *The resistance distance matrix of a dandelion graph $D(n,l)$ on n vertices is given by*

$$R(D(n,l)) = \begin{bmatrix} 0 & 1 & 2 & \cdots & l-1 & 1 & & \cdots & 1 \\ 1 & 0 & 1 & \cdots & l-2 & 2 & & \cdots & 2 \\ 2 & 1 & 0 & \cdots & l-3 & 3 & & \cdots & 3 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & & \vdots & \vdots \\ l-1 & l-2 & \cdots & 1 & 0 & l & & \cdots & l \\ 1 & 2 & & \cdots & l & 0 & 2 & \cdots & 2 \\ 1 & 2 & & \cdots & l & 2 & 0 & \cdots & 2 \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & & \cdots & l & 2 & 2 & \cdots & 0 \end{bmatrix}.$$

Proof. By a proper labelling of vertices in $D(n,l) = K_{1,n-l} \circ_1 P_l$, we can write its Laplacian matrix as

$$L(D(n,l)) = \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_3. \end{bmatrix},$$

where $L_1 = \begin{bmatrix} n+1-l & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}_{l \times l}$, $L_2 = \begin{bmatrix} -J_{1\times n-l} \\ 0_{l-1\times n-l} \end{bmatrix}$ and $L_3 = I_{n-l}$.

Then by applying Lemma 2.2 we get,

$$L^{(1)} = \begin{bmatrix} L_1^{-1} & 0 \\ 0 & I_{n-l} - J_{n-l} \end{bmatrix},$$

where $L_1^{-1} = \begin{bmatrix} \frac{1}{n-l} & \frac{1}{n-l} & \cdots & \frac{1}{n-l} \\ \frac{1}{n-l} & \frac{(n-l)+1}{n-l} & \cdots & \frac{(n-l)+1}{n-l} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n-l} & \frac{(n-l)+1}{n-l} & \cdots & \frac{(l-1)(n-l)+1}{n-l} \end{bmatrix}$.

Now by the definition of resistance distance we get the required result. □

From Theorem 3.7, we get the following corollaries.

Corollary 3.6. *The Kirchhoff index of $D(n,l)$ is*

$$Kf(D(n,l)) = \frac{l^2(3n-2l+2)+l(5-9n)+6(n^2-1)}{6}.$$

Corollary 3.7. *The Kemeny's constant of $D(n,l)$ is*

$$\kappa(D(n,l)) = \frac{(n+1)(2l^2-1)+2n(n-3l)}{2(n-1)} + \frac{l(5-2l^2)}{3(n-1)}.$$

The following corollaries describe various graph parameters of $K_{p_1} \circ_k K_{p_2}$.

Corollary 3.8. *The Kirchhoff index of $K_{p_1} \circ_k K_{p_2}$ is given by*

$$\begin{aligned}\mathcal{K}f(K_{p_1} \circ_k K_{p_2}) = {} & \frac{1}{kp_1p_2t}\big((p_1-k)(p_2-k)(k+1)\left(p_1(k+t)+p_2+\frac{p_2k}{k+1}+\frac{p_2k(p_1k+p_1t-t)}{(p_2-k)(k+1)}\right) \\ & + kp_1(p_2-k)\left(p_2(t+2k)-t(k+1)+kp_2(k-1)\right)\big).\end{aligned}$$

Proof. By definition $\mathcal{K}f(G_n) = \sum_{i<j} r_{ij}(G_n)$. Then,

$$\begin{aligned}\mathcal{K}f(K_{p_1} \circ_k K_{p_2}) = {} & \sum_{v_i,v_j \in T} \frac{2}{t} + \sum_{v_i \in T, v_j \in V(K_{p_1} \backslash T)} \frac{k(p_1+t)+(p_2-k)}{p_1kt} \\ & + \sum_{v_i \in T, v_j \in V(K_{p_2} \backslash T)} \frac{(k+1)(p_1-k)+2p_2k}{p_2kt} + \sum_{v_i,v_j \in V(K_{p_1} \backslash T)} \frac{2}{p_1} \\ & + \sum_{v_i \in V(K_{p_1} \backslash T), v_j \in V(K_{p_2} \backslash T)} \frac{(p_1+p_2)(k+1)}{kp_1p_2} + \sum_{v_i,v_j \in V(K_{p_2} \backslash T)} \frac{2}{p_2}. \\ = {} & \frac{1}{kp_1p_2t}\Bigg((p_1-k)(p_2-k)(k+1)\Bigg(p_1(k+t)+p_2+\frac{p_2k}{k+1} \\ & + \frac{p_2k(p_1k+p_1t-t)}{(p_2-k)(k+1)}\Bigg) + kp_1(p_2-k)\big(p_2(t+2k)-tk-t)+kp_2(k-1)\big)\Bigg).\end{aligned}$$

□

Corollary 3.9. *The Kemeny's constant of $K_{p_1} \circ_k K_{p_2}$ is given by*

$$\begin{aligned}\kappa(K_{p_1} \circ_k K_{p_2}) = {} & \frac{1}{2mp_1p_2}\Bigg(\frac{t-1}{t}(p_1(t-1)(p_2k(k-1)+(p_2-1)(p_2-k)((k+1)(p_1-k)+2p_2k)) \\ & + p_2(p_1-1)(p_1-k)(2p_1k+(p_2-k)(k+1))) + p_1(p_2-k)(p_2-1)^2(p_2-k-1) \\ & + \frac{(p_1-k)(p_1-1)(p_2k(p_1-k-1)(p_1-1)+(p_2-k)(p_2-1)(p_1+p_2)(k+1))}{k}\Bigg).\end{aligned}$$

Corollary 3.10. *The additive degree-Kirchhoff index of $K_{p_1} \circ_k K_{p_2}$ is given by*

$$\begin{aligned}R^+(K_{p_1} \circ_k K_{p_2}) = {} & \frac{2p_1k(k-1)(t-1)+k(p_1-k)(2t-2)\left((2t)+p_2-k\right)}{p_1t} \\ & + \frac{(p_1-k)(p_2-k)(p_1+p_2-2)(p_1+p_2)(k+1)}{kp_1p_2} \\ & + \frac{(p_2-k)(p_1+2p_2-k-2)\left((k+1)(p_1-k)+2p_2k\right)}{p_2t} \\ & + \frac{2p_2(p_1-k)(p_1-k-1)(p_1-1)+2p_1(p_2-k)(p_2-k-1)(p_2-1)}{p_1p_2}.\end{aligned}$$

Corollary 3.11. *The multiplicative degree-Kirchhoff index of* $K_{p_1} \circ_k K_{p_2}$ *is given by*

$$\begin{aligned} R^*(K_{p_1} \circ_k K_{p_2}) =& \frac{1}{p_1p_2}\Bigg(\frac{t-1}{t}(p_1(t-1)(p_2k(k-1)+(p_2-1)(p_2-k)((k+1)(p_1-k)+2p_2k)) \\ &+p_2(p_1-1)(p_1-k)(2p_1k+(p_2-k)(k+1))) + p_1(p_2-k)(p_2-1)^2(p_2-k-1) \\ &+\frac{(p_1-k)(p_1-1)(p_2k(p_1-k-1)(p_1-1)+(p_2-k)(p_2-1)(p_1+p_2)(k+1))}{k}\Bigg). \end{aligned}$$

Corollary 3.12. *The mixed degree-Kirchhoff index of* $K_{p_1} \circ_k K_{p_2}$ *is given by*

$$\begin{aligned} \hat{R}(K_{p_1} \circ_k K_{p_2}) =& \frac{2k(k-1)}{t} + \frac{(p_1-k)((t-1)^2+(p_1-1)^2)(k(2t)+(p_2-k))}{p_1(p_1-1)t(t-1)} \\ &+ \frac{(p_2-k)((t-1)^2+(p_2-1)^2)((k+1)(p_1-k)+2p_2k)}{p_2(p_2-1)t(t-1)} \\ &+ \frac{2(p_1-k)(p_1-k-1)}{p_1} + \frac{2(p_2-k)(p_2-k-1)}{p_2} \\ &+ \frac{(p_1-k)(p_2-k)((p_1-1)^2+(p_2-1)^2)(p_1+p_2)(k+1)}{kp_1p_2(p_1-1)(p_2-1)}. \end{aligned}$$

In general, it is difficult to find the resistance energy of graphs. The following table provides the resistance energy of $K_{p_1} \circ_k K_{p_2}$, for different values of p_1, p_2 and k. From the table, we observe that the $RE(K_{p_1} \circ_k K_{p_2})$ depends on the values of p_1, p_2, and k.

No.	p_1	p_2	k	$RE(K_{p_1} \circ_k K_{p_2})$
1	3	2	1	6.21
2	4	2	1	6.92
3	5	2	1	7.79
4	6	2	1	7.91
5	2	3	1	6.23
6	3	3	1	7.29
7	4	3	1	7.91
8	5	3	1	8.37
9	6	3	1	8.76
10	2	4	1	6.92
11	3	4	1	7.9
12	4	4	1	8.44
13	5	4	1	8.82
14	6	4	1	9.14
15	5	5	1	9.13
16	6	5	1	9.4
17	7	5	1	9.6
18	2	2	2	4
19	3	2	2	4.36
20	4	2	2	4.45
21	5	2	2	4.48
22	6	2	2	4.48
23	2	3	2	4.4
24	3	3	2	4.92
25	4	3	2	5.63
26	5	3	2	5.42
27	6	3	2	5.59
28	2	4	2	4.56
29	3	4	2	5.23
30	4	4	2	5.64
31	5	4	2	5.92
32	6	4	2	6.17
33	5	5	2	6.27
34	6	5	2	6.52
35	7	5	2	6.73
36	4	3	3	8.14
37	5	3	3	8.18
38	6	3	3	8.19
39	4	4	3	8.25
40	5	4	3	8.3
41	6	4	3	8.43
42	5	5	3	8.36
43	6	5	3	8.39
44	7	5	3	8.41
45	6	5	4	2.92
46	7	5	4	2.77

4. Conclusion

This article explores the concept of resistance distance in any two graphs having K_k as a subgraph. A closed-form expression for the resistance distance in k-coalescence of two complete graphs is also provided. These results enable us to determine several graph parameters, including Kemeny's constant, Kirchhoff index, additive degree-Kirchhoff index, multiplicative degree-Kirchhoff index, and mixed degree-Kirchhoff index of k-coalescence of complete graphs. In addition, the resistance distance in the k-coalescence of a complete graph with particular graphs is obtained. Furthermore, the article applies the findings to determine the resistance distance of specific graphs like the vertex coalescence of a complete bipartite graph with a complete graph, a complete bipartite graph with a star graph, the windmill graphs, the pineapple graph, etc.

5. Declarations

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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