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NONLINEAR LOVE WAVES IN A HOMOGENEOUS LAYER OVERLYING A HOMOGENEOUS HALF-SPACE

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Abstract. In this article, the focus is on the propagation of Love waves in a solid layer overlying a solid half-space, assuming that both layer and half-space consist of nonlinear, isotropic, homogeneous, hyper-elastic, and generalized neo-Hookean materials, in addition to having different mechanical properties. As done by Love, displacements and stresses are assumed to be continuous at the interface between the layer and half-space, and the upper surface is to be free from traction, in addition to holding the radiation condition in the half-space. The method of multiple scales in the self modulation of the problem is used. Then, it is shown that the self modulation of the problem can be given by a nonlinear Schrödinger equation as a result of a balance between dispersion and nonlinearity. By using the coefficients of the nonlinear Schrödinger equation, the existence of the solitary waves is studied. In addition, the effects of the parameters of linear and nonlinear mediums on the functions of the wave propagation are studied.

Keywords: Love waves, solitary waves, nonlinear waves, nonlinear Schrödinger equation, homogeneous materials

AMS Subject Classification: 35B20, 35Q55,35G30, 35Q86.

1. Introduction

Body waves, called pressure (or compressional) and shear waves, are well-known not only in seismology as seismic waves propagating to the interior of the Earth during an earthquake but also in the classical theory of elasticity as elastic waves. As opposed to pressure motion, in which particles move in the same direction of wave propagation, the movement of a particle in shear motion is perpendicular to the direction of the wave propagation. In addition to body waves, two types of elastic surface waves, named Rayleigh and Love waves after their discoverers, are also well-known in the literature. Rayleigh waves result from an elliptical motion characterized as a combination of pressure and vertically shear motions; whereas, Love waves result from only horizontally shear motion. Love waves have found many applications better than Rayleigh waves, in different branches of science such as biosensors in biology and medicine, chemosensors in chemistry, non-destructive testing materials in engineering, and so on, in addition to knowing their great importance

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in seismology, geology, and geophysics. Some applications such as chemosensors, non-destructive testing of materials and sensors of various physical quantities can be found in [1–5]. At this point, we refer the reader to see an extensive review in [6] and references therein with recent developments and applications, in addition to rich sources [7–10] for wave propagation, linear elasticity, wave guides, and waves.

A simple geophysical model of the Earth was suggested by Augustus Edward Hough Love and awarded the Adams Prize in 1911 [11]. In this model, the propagation of Love waves on a solid half-space (or substrate) covered by a solid layer was considered. Both mediums were assumed to be linear, isotopic, and elastic, additionally having different mechanical properties. Besides, the upper surface to be free from traction, and displacements and stresses to be continuous at the interface were assumed, in addition to holding the radiation condition in the half-space. Under the condition that the shear wave velocity of the layer must be slower than the shear wave velocity of the half-space, the existence of a horizontally shear wave, having an amplitude decayed rapidly with the thickness in the half-space, was shown. Later, elastic waves in different contexts such as exact solutions [12], a comparative study [13], large-amplitude [14], heterogeneity [15–18], layered structures [19–23] were considered. Furthermore, some recent and relevant papers can be found in [24–29].

In this article, the focus will be on the propagation of Love waves in a solid half-space covered by a solid layer, assuming that both layer and half-space consist of nonlinear, isotropic, homogeneous, hyper-elastic, and generalized neo-Hookean materials, in addition to having different mechanical properties. As done by Love [11], displacements and stresses are assumed to be continuous at the interface between the layer and half-space, and the upper surface is to be free from traction, in addition to holding the radiation condition in the half-space. In other words, that is an improved version of Love's study since nonlinear materials are taken into consideration instead of linear materials. By adding nonlinearity to Love's study and considering the signs of nonlinear models, four different nonlinear models such as hardening-hardening (H-H), hardening-softening (H-S), softening-hardening (S-H), softening-softening (S-S) will be taken into account.

2. Formulation of the Problem

According to Cartesian system of axes, the spatial and material coordinates of a point are (x_1, x_2, x_3) and (X_1, X_2, X_3), respectively. A nonlinear solid layer of finite thickness $h > 0$ overlying a nonlinear solid half-space is considered. The layer is in the region between the planes $X_2 = -h$ and $X_2 = 0$, in addition to the half-space occupying the region $X_2 < -h$, please see Figure.1.

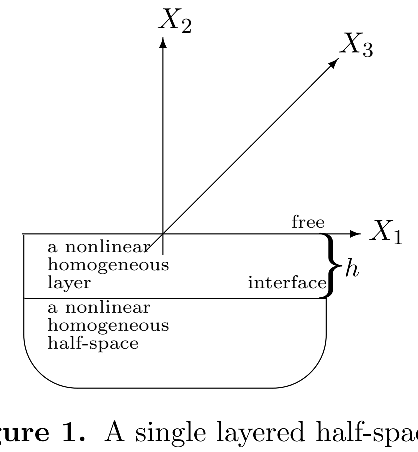

Figure 1. A single layered half-space

Waves of Love type, for which the motion takes place in the X_3-direction, are considered, so displacements in the X_1- and X_2-directions are taken equal to zero. It is assumed that the upper boundary $X_2 = 0$ is free from traction, and displacements and stresses at the interface $X_2 = -h$ are continuous, and also holding the radiation condition in the half-space. Then, a Love wave described by

$$x_1 = X_1, \quad x_2 = X_2, \quad x_3 = X_3 + \overset{\phi}{u}(X_1, X_2, t) \tag{1}$$

is assumed to propagate along X_1-axis, where $\overset{\phi}{u}$ are the displacements in the X_3-direction of the layer and half-space, using the superscripts 1 and 2 for ϕ, respectively. Moreover, t denotes the time.

By using (1), in the absent of body forces, governing equations and boundary conditions of the motion are approximately obtained [30]. Since the aim of this paper is to deal with small but finite amplitude wave motions, proceeding with the approximate equations, rather than the exact ones, will be more convenient. Then, the approximate governing equations and boundary conditions, of both mediums, involving terms not higher than the third degree in the deformation gradients are written as

$$\overset{1}{\mathcal{L}}(\overset{1}{u}) = \overset{1}{\mathcal{N}}(\overset{1}{u}), \tag{2}$$

$$\overset{2}{\mathcal{L}}(\overset{2}{u}) = \overset{2}{\mathcal{N}}(\overset{2}{u}), \tag{3}$$

$$\frac{\partial \overset{1}{u}}{\partial X_2} = -\eta_{0_1}\mathcal{K}(\overset{1}{u})\frac{\partial \overset{1}{u}}{\partial X_2} \quad \text{on} \quad X_2 = 0, \tag{4}$$

$$\frac{\partial \overset{1}{u}}{\partial X_2} - \mu_0\frac{\partial \overset{2}{u}}{\partial X_2} = -\eta_{0_1}\mathcal{K}(\overset{1}{u})\frac{\partial \overset{1}{u}}{\partial X_2} + \mu_0\eta_{0_2}\mathcal{K}(\overset{2}{u})\frac{\partial \overset{2}{u}}{\partial X_2} \quad \text{on} \quad X_2 = -h, \tag{5}$$

$$\overset{1}{u} - \overset{2}{u} = 0 \quad \text{on} \quad X_2 = -h, \tag{6}$$

$$\overset{2}{u} \to 0 \quad \text{as} \quad X_2 \to -\infty, \tag{7}$$

where

$$\overset{\phi}{\mathcal{L}}() = \frac{\partial^2()}{\partial t^2} - c_{0_\phi}^2\left(\frac{\partial^2()}{\partial X_1^2} + \frac{\partial^2()}{\partial X_2^2}\right), \tag{8}$$

$$\overset{\phi}{\mathcal{N}}() = n_{0_\phi}\left[\frac{\partial}{\partial X_1}\left(\frac{\partial()}{\partial X_1}\mathcal{K}()\right) + \frac{\partial}{\partial X_2}\left(\frac{\partial()}{\partial X_2}\mathcal{K}()\right)\right], \tag{9}$$

$$\mathcal{K}() = \left(\frac{\partial()}{\partial X_1}\right)^2 + \left(\frac{\partial()}{\partial X_2}\right)^2, \tag{10}$$

with the following relations and additional necessary non-dimensional parameters

$$\begin{aligned} &c_{0_\phi}^2 = \frac{\mu_{0_\phi}}{\rho_{0_\phi}} = \frac{2}{\rho_{0_\phi}}\overset{\phi}{\Sigma}{}'(3), \quad n_{0_\phi} = \frac{2}{\rho_{0_\phi}}\overset{\phi}{\Sigma}{}''(3), \\ &\mu_0 = \frac{\mu_{0_2}}{\mu_{0_1}}, \quad \rho_0 = \frac{\rho_{0_2}}{\rho_{0_1}}, \quad c_0 = \frac{c_{0_2}}{c_{0_1}}, \quad \eta_{0_\phi} = \frac{n_{0_\phi}}{c_{0_\phi}^2}. \end{aligned} \tag{11}$$

Note here that Lame constants, linear shear wave velocities, and nonlinear material constants are denoted by μ_{0_ϕ}, c_{0_ϕ} and n_{0_ϕ}, respectively, in addition to non-dimensional parameters of medium μ_0, ρ_0, c_0 and η_{0_ϕ}. Besides, n_{0_ϕ} is a real number. Therefore three possibilities exist, then the constituent materials of the layer and half-space soften in shear if $n_{0_\phi} < 0$ and harden but if $n_{0_\phi} > 0$, in addition to using the condition $n_{0_\phi} = 0$ in Eq. (2) or (3) for obtaining the well-known governing equation of SH waves in a linear and homogeneous medium [10]. Furthermore, the problem model includes a combination of the layer and half-space, and then nonlinear models such as hardening-hardening, hardening-softening, softening-hardening, and softening-softening exist, taking the signs of the models into account. The following sections will be based upon all discussions and assumptions above.

3. Nonlinear Self-Modulation of Love Waves

The aim of this section is to solve the problem given in Eqs. (2)-(3) with the boundary conditions in Eqs. (4)-(5)-(6)-(7), using *the method of multiple scales* [31]. With this aim, new variables of the method are introduced as follows

$$x_j = \varepsilon^j X_1, \quad t_j = \varepsilon^j t, \quad y = X_2, \quad j = 0, 1, 2 \tag{12}$$

where x_1, x_2, t_1, t_2 are slow variables; x_0, y, t_0 are fast variables, and ε is a positive and small parameter which measures the weakness of the nonlinearity. Then, the functions $\overset{\phi}{u}$ can be expanded in the following asymptotic series in ε;

$$\overset{\phi}{u} = \sum_{j=1}^{\infty} \varepsilon^j \overset{\phi}{u}_j(x_0, x_1, x_2, y, t_0, t_1, t_2). \tag{13}$$

Using new variables in Eq. (12) and substituting the expansions in Eq. (13) into Eqs. (2)-(3) and Eqs. (4)-(5)-(6)-(7), and also equating the coefficients of the same powers of ε to get a hierarchy of equations and boundary conditions for determining $\overset{\phi}{u}_j$, successively.

Up to third order in ε these can be written as follows

$$\mathcal{O}(\varepsilon):\ \overset{1}{\mathcal{L}}_0(\overset{1}{u}_1)=0, \tag{14}$$

$$\overset{2}{\mathcal{L}}_0(\overset{2}{u}_1)=0, \tag{15}$$

$$\frac{\partial \overset{1}{u}_1}{\partial y}=0 \quad \text{on} \quad y=0, \tag{16}$$

$$\frac{\partial \overset{1}{u}_1}{\partial y}-\mu_0\frac{\partial \overset{2}{u}_1}{\partial y}=0 \quad \text{on} \quad y=-h, \tag{17}$$

$$\overset{1}{u}_1-\overset{2}{u}_1=0 \quad \text{on} \quad y=-h, \tag{18}$$

$$\overset{2}{u}_1\to 0 \quad \text{as} \quad y\to-\infty, \tag{19}$$

$$\mathcal{O}(\varepsilon^2):\ \overset{1}{\mathcal{L}}_0(\overset{1}{u}_2)+\overset{1}{\mathcal{L}}_1(\overset{1}{u}_1)=0, \tag{20}$$

$$\overset{2}{\mathcal{L}}_0(\overset{2}{u}_2)+\overset{2}{\mathcal{L}}_1(\overset{2}{u}_1)=0, \tag{21}$$

$$\frac{\partial \overset{1}{u}_2}{\partial y}=0 \quad \text{on} \quad y=0, \tag{22}$$

$$\frac{\partial \overset{1}{u}_2}{\partial y}-\mu_0\frac{\partial \overset{2}{u}_2}{\partial y}=0 \quad \text{on} \quad y=-h, \tag{23}$$

$$\overset{1}{u}_2-\overset{2}{u}_2=0 \quad \text{on} \quad y=-h, \tag{24}$$

$$\overset{2}{u}_2\to 0 \quad \text{as} \quad y\to-\infty, \tag{25}$$

$$\mathcal{O}(\varepsilon^3):\ \overset{1}{\mathcal{L}}_0(\overset{1}{u}_3)+\overset{1}{\mathcal{L}}_1(\overset{1}{u}_2)+\overset{1}{\mathcal{L}}_2(\overset{1}{u}_1)=\overset{1}{\mathcal{N}}_0(\overset{1}{u}_1), \tag{26}$$

$$\overset{2}{\mathcal{L}}_0(\overset{2}{u}_3)+\overset{2}{\mathcal{L}}_1(\overset{2}{u}_2)+\overset{2}{\mathcal{L}}_2(\overset{2}{u}_1)=\overset{2}{\mathcal{N}}_0(\overset{2}{u}_1), \tag{27}$$

$$\frac{\partial \overset{1}{u}_3}{\partial y}=-\eta_{0_1}\mathcal{K}_0(\overset{1}{u}_1)\frac{\partial \overset{1}{u}_1}{\partial y} \quad \text{on} \quad y=0, \tag{28}$$

$$\frac{\partial \overset{1}{u}_3}{\partial y}-\mu_0\frac{\partial \overset{2}{u}_3}{\partial y}=-\eta_{0_1}\mathcal{K}_0(\overset{1}{u}_1)\frac{\partial \overset{1}{u}_1}{\partial y} + \mu_0\eta_{0_2}\mathcal{K}_0(\overset{2}{u}_1)\frac{\partial \overset{2}{u}_1}{\partial y} \quad \text{on} \quad y=-h, \tag{29}$$

$$\overset{1}{u}_3-\overset{2}{u}_3=0 \quad \text{on} \quad y=-h, \tag{30}$$

$$\overset{2}{u}_3\to 0 \quad \text{as} \quad y\to-\infty, \tag{31}$$

where the differential operators $\overset{\phi}{\mathcal{L}}_0, \overset{\phi}{\mathcal{L}}_1, \overset{\phi}{\mathcal{L}}_2, \overset{\phi}{\mathcal{N}}_0$ and $\mathcal{K}_0$ are defined by

$$\overset{\phi}{\mathcal{L}}_0() = \frac{\partial^2()}{\partial t_0^2} - c_{0_\psi}^2 \left(\frac{\partial^2()}{\partial x_0^2} + \frac{\partial^2()}{\partial y^2} \right), \tag{32}$$

$$\overset{\phi}{\mathcal{L}}_1() = 2\frac{\partial^2()}{\partial t_0 \partial t_1} - 2c_{0_\phi}^2 \frac{\partial^2()}{\partial x_0 \partial x_1}, \tag{33}$$

$$\overset{\phi}{\mathcal{L}}_2() = \frac{\partial^2()}{\partial t_1^2} + 2\frac{\partial^2()}{\partial t_0 \partial t_2} - c_{0_\phi}^2 \left(\frac{\partial^2()}{\partial x_1^2} + 2\frac{\partial^2()}{\partial x_0 \partial x_2} \right), \tag{34}$$

$$\overset{\phi}{\mathcal{N}}_0() = n_{0_\phi} \left[\frac{\partial}{\partial x_0} \left(\mathcal{K}_0() \frac{\partial()}{\partial x_0} \right) + \frac{\partial}{\partial y} \left(\mathcal{K}_0() \frac{\partial()}{\partial y} \right) \right], \tag{35}$$

$$\mathcal{K}_0() = \left(\frac{\partial()}{\partial x_0} \right)^2 + \left(\frac{\partial()}{\partial y} \right)^2. \tag{36}$$

Note that the order problems mentioned above are linear at each step, and the solution of the first-order problem is identical with the solution of the linear problem. Only the difference between them is that, in the linear problem, it is explicitly possible to find the solution; but, in the first-order problem, for finding the explicit solution, it is needed to solve the second and third-order problems considering the dependence of the amplitude on the slow variables.

In order to build the first-order uniformly valid solution, *the method of separation of variables* can, firstly, be applied to Eqs. (14)-(15) and then the forms of the solutions to these equations for the layer and half-space can be written, respectively, as

$$\overset{1}{u}_1 = \sum_{a=1}^{\infty} \left[\overset{a}{A}_1(x_1, x_2, t_1, t_2) e^{ikap_{0_1} y} + \overset{a}{B}_1(x_1, x_2, t_1, t_2) e^{-ikap_{0_1} y} \right] e^{ia\psi} + \text{c.c.}, \tag{37}$$

$$\overset{2}{u}_1 = \sum_{a=1}^{\infty} \left[\overset{a}{C}_1(x_1, x_2, t_1, t_2) e^{kap_{0_2} y} \right] e^{ia\psi} + \text{c.c.}, \tag{38}$$

where

$$p_{0_1} = \left(\frac{v_p^2}{c_{0_1}^2} - 1 \right)^{1/2}, \ p_{0_2} = \left(1 - \frac{v_p^2}{c_{0_2}^2} \right)^{1/2}, \tag{39}$$

k is the wave number, ω is the angular frequency, $v_p = \omega/k$ is the phase velocity, a is a positive integer, c.c. is the complex conjugate to the preceding terms, and $\overset{a}{A}_1, \overset{a}{B}_1$ and $\overset{a}{C}_1$ are the first-order amplitude functions of wave propagation depending on the slow variables x_1, x_2, t_1, t_2. Note that $\psi = kx_0 - \omega t_0$. Note also that this analysis is only valid for the case $c_{0_1} < v_p < c_{0_2}$ [6, 9–11]. Otherwise, a Love wave doesn't propagate in such a medium. Additionally, the radiation condition in Eq. (19) is already applied to Eq. (38). By using the solutions in Eqs. (37)-(38) into the boundary conditions in Eqs. (16)-(17)-(18), the following system

$$\overset{a}{\mathbf{W}} \overset{a}{\mathbf{U}}_1 = \overset{a}{\mathbf{b}}_1 \tag{40}$$

is obtained, for which

$$\overset{a}{\mathbf{W}} = \begin{bmatrix} ikap_{0_1} & -ikap_{0_1} & 0 \\ ikap_{0_1} e^{-ikap_{0_1} h} & -ikap_{0_1} e^{ikap_{0_1} h} & -kap_{0_2} \mu_0 e^{-kap_{0_2} h} \\ e^{-ikap_{0_1} h} & e^{ikap_{0_1} h} & -e^{-kap_{0_2} h} \end{bmatrix} \tag{41}$$

and

$$\overset{a}{\mathbf{U}}_1 = [\overset{a}{A}_1, \overset{a}{B}_1, \overset{a}{C}_1]^{\mathrm{T}}, \quad \overset{a}{\mathbf{b}}_1 = \mathbf{0}. \tag{42}$$

In order to find the nontrivial solution of the system in Eq. (40) for $a = 1$, the condition $\det\mathbf{W} = 0$, giving the dispersion relation, must be satisfied, where $\mathbf{W} = \overset{1}{\mathbf{W}}$. Nonlinear self-modulation of Love wave propagation is only taken in this analysis into consideration, and harmonic-resonance phenomenon is excluded, so it can be assumed that

$$\det\overset{a}{\mathbf{W}} \neq 0 \quad \text{for} \quad a \geq 2. \tag{43}$$

Then, the condition $\det\mathbf{W} = 0$ yields to the following classical Love wave dispersion relation [6, 9–11]

$$\tan(khp_{0_1}) = \frac{\mu_0 p_{0_2}}{p_{0_1}} \tag{44}$$

or, equally, to the dispersion branches

$$khp_{0_1} = \arctan\left(\frac{\mu_0 p_{0_2}}{p_{0_1}}\right) + n\pi, \quad n = 0, 1, 2, ... \tag{45}$$

The solution of the system in Eq. (40) can now be given as

$$\overset{a}{\mathbf{U}}_1 = \mathcal{A}_1\mathbf{R} \qquad \text{for} \quad a = 1, \tag{46}$$

$$\overset{a}{\mathbf{U}}_1 = \mathbf{0} \qquad \text{for} \quad a \geq 2, \tag{47}$$

where $\mathcal{A}_1 = \mathcal{A}_1(x_1, x_2, t_1, t_2)$ is a complex function of the first-order slowly varying amplitude of the wave modulation, and $\mathbf{R}$ is the right column vector satisfying

$$\mathbf{WR} = \mathbf{0}. \tag{48}$$

The vector $\mathbf{R}$ is obtained from Eq. (48), as

$$\mathbf{R} = [R_1, R_2, R_3]^{\mathrm{T}}, \tag{49}$$

where

$$R_1 = \frac{p_{0_1} - i\mu_0 p_{0_2}}{2p_{0_1}} e^{kh(ip_{0_1} - p_{0_2})}, \quad R_2 = \bar{R}_1, \quad R_3 = 1. \tag{50}$$

Note that $\bar{R}_1$ corresponds to the complex conjugate of R_1. Then, the first-order solutions for the layer and half-space are, respectively,

$$\overset{1}{u}_1 = \mathcal{A}_1(x_1, x_2, t_1, t_2)(R_1 e^{ikp_{0_1}y} + R_2 e^{-ikp_{0_1}y})e^{i\psi} + \text{c.c.}, \tag{51}$$

$$\overset{2}{u}_1 = \mathcal{A}_1(x_1, x_2, t_1, t_2)(R_3 e^{kp_{0_2}y})e^{i\psi} + \text{c.c.}. \tag{52}$$

The similar procedure as in the first-order problem for $\overset{1}{u}_2$, $\overset{2}{u}_2$, $\overset{1}{u}_3$, and $\overset{2}{u}_3$ can be followed. Then, from the compatibility condition of the second order problem, the first-order amplitude $\mathcal{A}_1 = \mathcal{A}_1(x_1 - v_g t_1, x_2, t_2)$ is partially determined. For the rest dependence of the amplitude on the slow variables, the compatibility condition of the third order problem must be considered. In the analysis, straight forward calculation shows that the compatibility condition of the third order problem reduces to a nonlinear Schrödinger equation.

After defining the following non-dimensional variables and constants

$$\tau = \frac{\varepsilon c_{0_1}}{h} t_1 = \frac{c_{0_1}}{h} t_2, \tag{53}$$

$$\xi = \frac{1}{h}(x_1 - v_g t_1) = \frac{\varepsilon^{-1}}{h}(x_2 - v_g t_2), \tag{54}$$

$$\mathcal{A} = \frac{\mathcal{A}_1}{h}, \quad \Gamma = \frac{\hat{\Gamma}}{h c_{0_1}}, \quad \Delta = \frac{h^3 \hat{\Delta}}{c_{0_1}} \tag{55}$$

the following nonlinear Schrödinger (NLS) equation is then obtained

$$i\frac{\partial \mathcal{A}}{\partial \tau} + \Gamma \frac{\partial^2 \mathcal{A}}{\partial \xi^2} + \Delta |\mathcal{A}|^2 \mathcal{A} = 0 \tag{56}$$

where

$$\begin{aligned}\hat{\Gamma} &= \frac{1}{2}\frac{dv_g}{dk} = \frac{1}{2}\frac{d^2\omega}{dk^2} \\ &= -\Big[\frac{1}{2}\mathbf{L}\left(v_g^2 \frac{\partial^2 \mathbf{W}}{\partial \omega^2} + 2v_g \frac{\partial^2 \mathbf{W}}{\partial \omega \partial k} + \frac{\partial^2 \mathbf{W}}{\partial k^2}\right)\mathbf{R} \\ &+ \mathbf{L}\left(\frac{\partial \mathbf{W}}{\partial k} + v_g \frac{\partial \mathbf{W}}{\partial \omega}\right)\left(\frac{\partial \mathbf{R}}{\partial k} + v_g \frac{\partial \mathbf{R}}{\partial \omega}\right)\Big] / \left(\mathbf{L}\frac{\partial \mathbf{W}}{\partial \omega}\mathbf{R}\right),\end{aligned} \tag{57}$$

$$\hat{\Delta} = -\mathbf{L}.\mathbf{G} / \left(\mathbf{L}\frac{\partial \mathbf{W}}{\partial \omega}\mathbf{R}\right). \tag{58}$$

and where the components of the vector $\mathbf{G} = [G_1, G_2, G_3]^T$ are given as

$$\begin{aligned} G_1 &= 0, \\ G_2 &= \frac{k^3 e^{-3hkp_{0_2}}}{32 p_{0_1}^4 p_{0_2}} \Big\{ 4\eta_{0_1} hk p_{0_1}^2 (9p_{0_1}^4 + 2p_{0_1}^2 + 9) p_{0_2} \\ &(\mu_0^2 p_{0_2}^2 + p_{0_1}^2) + \eta_{0_1}\mu_0^3(-33p_{0_1}^4 + 2p_{0_1}^2 + 27)p_{0_2}^4 + \mu_0 p_{0_1}^2 \\ &\Big[4\eta_{0_2} p_{0_1}^2(-3p_{0_2}^4 + 2p_{0_2}^2 + 9) - 3\eta_{0_1}(15p_{0_1}^4 + 2p_{0_1}^2 - 21)p_{0_2}^2\Big]\Big\}, \\ G_3 &= \frac{k^2 e^{-3hkp_{0_2}}}{32 p_{0_1}^4 p_{0_2}^2} \Big\{ \eta_{0_1} p_{0_2}^2 \Big[-4hk\mu_0 (9p_{0_1}^4 + 2p_{0_1}^2 + 9) p_{0_2} p_{0_1}^2 \\ &- 4hk\mu_0^3(9p_{0_1}^4 + 2p_{0_1}^2 + 9)p_{0_2}^3 + 3\mu_0^2(9p_{0_1}^4 - 2p_{0_1}^2 - 3)p_{0_2}^2 \\ &- 9p_{0_1}^6 + 2p_{0_1}^4 + 3p_{0_1}^2\Big] - 4\eta_{0_2} p_{0_1}^4 (9p_{0_2}^4 + 2p_{0_2}^2 - 3)\Big\}. \end{aligned}$$

the components of the vector $\mathbf{L} = [L_1, L_2, L_3]$ are given as

$$L_1 = 1, \quad L_2 = -\cos(khp_{0_1}), \quad L_3 = kp_{0_1}\sin(khp_{0_1}).$$

It is known that the sign of $\Gamma\Delta$ for the existence of solitary wave solutions to the NLS equation threats as a criterion, named as Zakharov-Shabat's criterion in the literature [32,33]. A qualitative review paper on the NLS equation can be found in [34]. In particular, if $\Gamma\Delta > 0$, then bright solitary Love waves will exist and propagate in such a medium; but if $\Gamma\Delta < 0$, then the dark solitary Love waves will exist and propagate in this medium [35]. More general discussion on nonlinear waves we refer the reader to [36–39]. Since the coefficients of the NLS equation are complicated in this discussion, a numerical approach will be given in the following section.

4. A Numerical Approach and Results

A numerical approach will be given on the basis of the non-dimensional functions and parameters of the wave propagation. Non-dimensional functions of the linear medium are $W = \omega h/c_{0_1}, V_P = v_p/c_{0_1}, V_G = v_g/c_{0_1}$, and $\Gamma = \hat{\Gamma}/hc_{0_1}$ depending on the non-dimensional parameters μ_0 and c_0. Non-dimensional functions of the nonlinear medium are Δ and $\Gamma\Delta$ depending on η_{0_1} and η_{0_2} in addition to parameters μ_0 and c_0. Non-dimensional parameters η_{0_1} and η_{0_2} are important to express nonlinear models such as hardening-hardening (H-H), hardening-softening (H-S), softening-hardening (S-H), softening-softening (S-S), that depend on the signs of the pair, $(\eta_{0_1}, \eta_{0_2}) = (\text{sign}(\eta_{0_1})\eta^\star_{0_1}, \text{sign}(\eta_{0_2})\eta^\star_{0_2})$, where $\eta^\star_{0_1}$ and $\eta^\star_{0_2}$ are positive. In the calculation of the complicated function Δ and $\Gamma\Delta$, they may, respectively, be chosen as $(+1.\eta_0, +1), (+1.\eta_0, -1)$, $(-1.\eta_0, +1)$, and $(-1.\eta_0, -1)$ where $\eta_0 = \eta^\star_{0_1}$ if the nonlinear medium is dominated by the layer, and as $(+1, +1.\eta_0), (+1, -1.\eta_0)$, $(-1, +1.\eta_0)$, and $(-1, -1.\eta_0)$ where $\eta_0 = \eta^\star_{0_2}$ if the nonlinear medium is dominated by the half-space, and also $(+1.\eta_0, +1.\eta_0), (+1.\eta_0, -1.\eta_0)$, $(-1.\eta_0, +1.\eta_0)$, and $(-1.\eta_0, -1.\eta_0)$ where $\eta_0 = \eta^\star_{0_1} = \eta^\star_{0_2}$ if the nonlinear medium is equally dominated by the layer and half-space. In addition, H-H and H-S models are, respectively, symmetric to S-S and S-H with respect to the K-axis. Therefore, it is enough to sketch H-H and H-S models.

Using the fixed parameter $c_0 = 1.297$ and unfixed parameter $\mu_0 = 1.250, \mu_0 = 2.159, \mu_0 = 3.068$, the effect of μ_0 on the respective variations of W, V_P, V_G and Γ, as functions of K, is plotted in Fig. 2 $(a), (b), (c)$, and (d) for the first branch of the dispersion relation. Using the fixed parameter $c_0 = 1.297$ and $\eta_0 = 1.500$ in the layer and half-space, and unfixed parameter $\mu_0 = 1.250, \mu_0 = 2.159, \mu_0 = 3.068$, the effect of μ_0 on the respective variations of Δ and $\Gamma\Delta$, as functions of K for respective nonlinear models H-H and H-S, is plotted in Fig. 3 $(a), (b)$, and (c), (d) for the first branch of the dispersion relation.

Using the fixed parameter $\mu_0 = 2.159$ and unfixed parameter $c_0 = 1.297, c_0 = 1.400, c_0 = 1.503$, the effect of c_0 on the respective variations of W, V_P, V_G and Γ, as functions of K, is plotted in Fig. 4 $(a), (b), (c)$, and (d) for the first branch of the dispersion relation. Using the fixed parameter $\mu_0 = 2.159$ and $\eta_0 = 1.500$ in the layer and half-space, and unfixed parameter $c_0 = 1.297, c_0 = 1.400, c_0 = 1.503$, the effect of c_0 on the respective variations of Δ and $\Gamma\Delta$, as functions of K for respective nonlinear models H-H and H-S, is plotted in Fig. 5 $(a), (b)$, and (c), (d) for the first branch of the dispersion relation.

Using the fixed parameter $\mu_0 = 2.159$ and $c_0 = 1.297$ and unfixed parameter $\eta_0 = 1.000, \eta_0 = 2.000, \eta_0 = 3.000$, the effect of η_0 in the layer on the respective variations of W, V_P, V_G and Γ, as functions of K, is plotted in Fig. 6 $(a), (b), (c)$, and (d) for the first branch of the dispersion relation. Using the fixed parameter $\mu_0 = 2.159$ and $c_0 = 1.297$ and unfixed parameter $\eta_0 = 1.000, \eta_0 = 2.000, \eta_0 = 3.000$, the effect of η_0 in the layer on the respective variations of Δ and $\Gamma\Delta$, as functions of K for respective nonlinear models H-H and H-S, is plotted in Fig. 7 $(a), (b)$, and (c), (d) for the first branch of the dispersion relation.

Using the fixed parameter $\mu_0 = 2.159$ and $c_0 = 1.297$ and unfixed parameter $\eta_0 = 1.000, \eta_0 = 2.000, \eta_0 = 3.000$, the effect of η_0 in the half-space on the respective variations of W, V_P, V_G and Γ, as functions of K, is plotted in Fig. 8 $(a), (b), (c)$, and (d) for the first branch of the dispersion relation. Using the fixed parameter $\mu_0 = 2.159$ and $c_0 = 1.297$ and unfixed parameter $\eta_0 = 1.000, \eta_0 = 2.000, \eta_0 = 3.000$, the effect of η_0 in the half-space on the respective variations of Δ and $\Gamma\Delta$, as functions of K for respective nonlinear models H-H and H-S, is plotted in Fig. 9 (a), (b), and (c), (d) for the first branch of the dispersion relation.

In all figures of the functions $\Gamma\Delta$, it takes both positive and negative values. Therefore, bright and dark solitary Love waves can exist and propagate in this layered half-space.

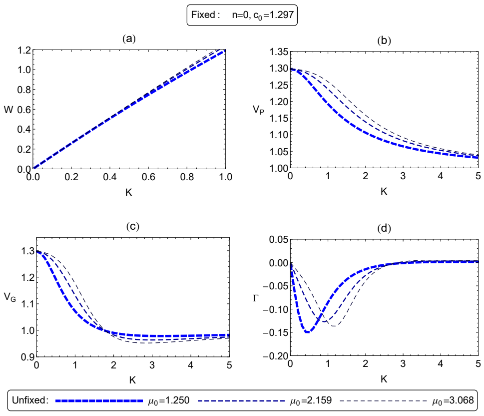

Figure 2. Effect of the parameter μ_0 on the variation of W, V_P, V_G, and Γ as functions of K for $n = 0$.

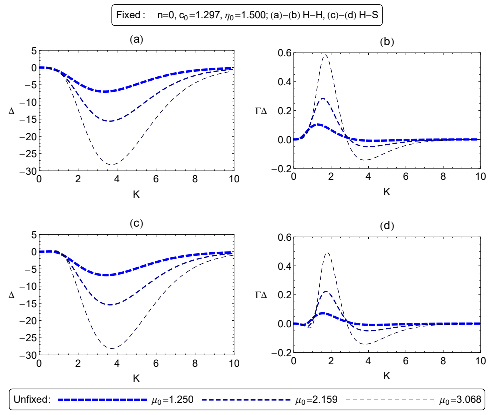

Figure 3. Effect of the parameter μ_0 on the variation of Δ and $\Gamma\Delta$ as functions of K for H-H and H-S models, and for $n = 0$.

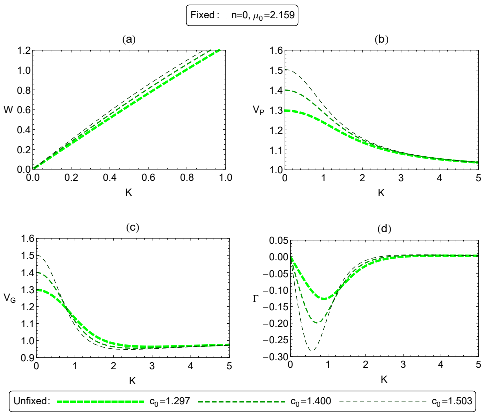

Figure 4. Effect of the parameter c_0 on the variation of W, V_P, V_G, and Γ as functions of K for $n = 0$.

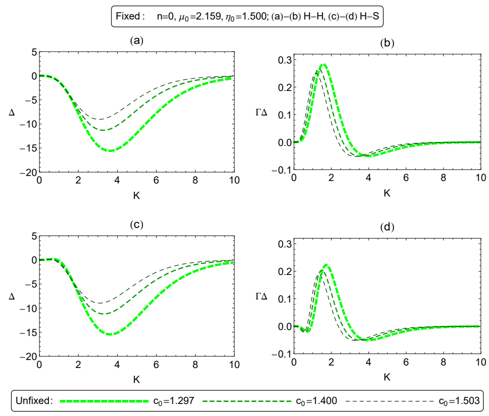

Figure 5. Effect of the parameter c_0 on the variation of Δ and $\Gamma\Delta$ as functions of K for H-H and H-S models, and for $n = 0$.

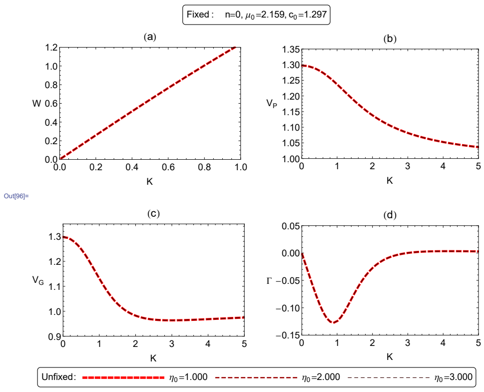

Figure 6. Effect of the parameter η_0 in the layer on the variation of W, V_P, V_G, and Γ as functions of K for $n = 0$.

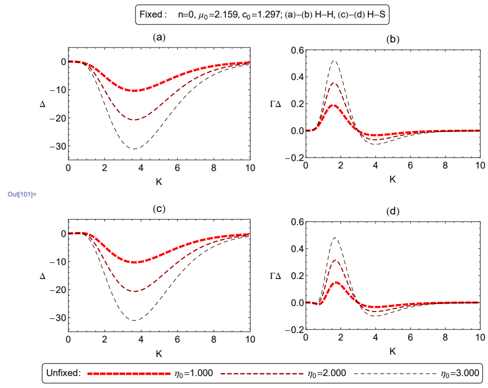

Figure 7. Effect of the parameter η_0 in the layer on the variation of Δ and $\Gamma\Delta$ as functions of K for H-H and H-S models, and for $n = 0$.

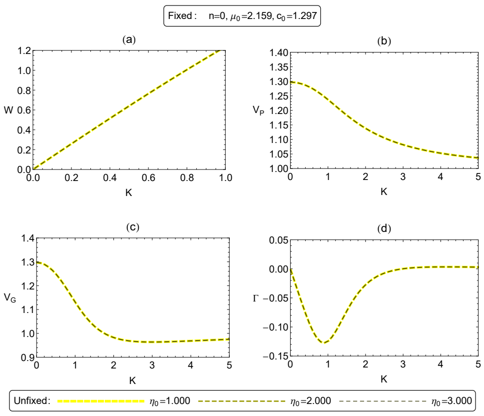

Figure 8. Effect of the parameter η_0 in the half-space on the variation of W, V_P, V_G, and Γ as functions of K for $n = 0$.

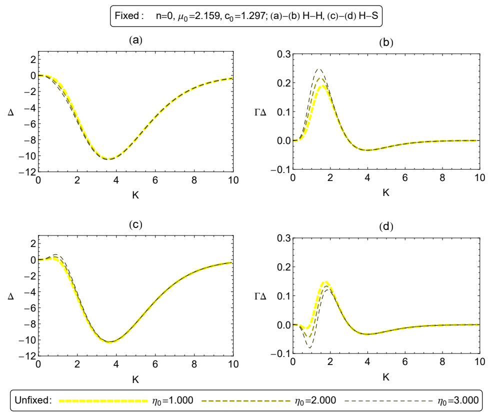

Figure 9. Effect of the parameter η_0 in the half-space on the variation of Δ and $\Gamma\Delta$ as functions of K for H-H and H-S models, and for $n = 0$.

5. Conclusion

In the present paper, a problem on the propagation of Love waves in a solid half-space covered by a solid layer is investigated by using the method of multiple scales. That is, one of the classical problems of linear elasticity is reconsidered. It is assumed that both layer and half-space consist of nonlinear, isotropic, homogeneous, hyper-elastic, and generalized neo-Hookean materials. Moreover, as done by Love, it is assumed that displacements and stresses are to be continuous at the interface between the layer and half-space, and the upper surface is to be free from traction, in addition to holding the radiation condition in the half-space. The present problem, due to adding nonlinearity, corresponds to the improved version of Love's study. Therefore, in the limiting case as the nonlinear parameters approach zero, the classical Love wave dispersion relation is obtained. Balancing nonlinearity and dispersion in the analysis, and taking Love's condition into account, the nonlinear self-modulation of Love waves is given by the nonlinear Schrödinger equation which has complicated coefficients. In the analysis, considering the signs of nonlinear parameters of nonlinear models, four different nonlinear models are taken into account. For each model, solitary Love wave solutions depend on the sign of the coefficients of the nonlinear Schrödinger equation. Therefore, the existence of bright and dark solitary Love wave solutions is, numerically, given in this layered half-space. In addition, the effects of the parameters of linear and nonlinear mediums on the functions of wave propagation are well observed for each model. Moreover, the proposed nonlinear model will be considered with heterogeneous materials in the future.

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