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APPROXIMATE SOLUTION FOR MULTI-HIGHER NONLINEAR FRACTIONAL VOLTERRA- FREDHLOM INTEGRO-DIFFERENTIAL EQUATIONS

M. M. HAMOOD[1,3*], A. A.SHARIF[2,3], K. P. GHADLE[3], §

Abstract. In this paper finds semi-approximate solutions for nonlinear Integro-Differential Equations of fractional order (FIDEs) of the Volterra-Fredholm-Hammerstein (VF-H) type, where the higher-order fractional derivative is defined in the Caputo sense, by effectively using the Adomian Decomposition Method and the Modified Adomian Decomposition Method as computational techniques. This method involves converting VF-H type FIDEs into iterative algebraic equations. In this approach, the answer to these equations is the product of an endless series of terms, which usually converges to the solution given by the noise terms. For numerical reasons, a reduced number of terms is utilised when a closed-form solution cannot be found. Lastly, these notions are illustrated using examples.

Keywords: Volterra-Fredholm-Hammerstein, Caputo-fractional derivative, Adomian decomposition method, Modified Adomian decomposition method.

AMS Subject Classification: 34A08, 26A33, 37C25.

1. Introduction

Fractional derivative and fractional integration are introduced, along with some basic definitions and characteristics of these operators. The Caputo derivative, one of the most useful fractional derivatives, is a variant of the Riemann-Liouville operator. It efficiently handles initial value problems where the initial conditions are given in integer order, which is typical of most practical processes, [1, 2, 3, 9, 10, 11, 12], and physicists which provides an efficiency for the description of many practical dynamical arising in engineering and scientific disciplines such as, physics, biology, chemistry, economy, electromagnetic, control theory and viscoelasticity [13, 15, 17, 18, 19, 20].

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[1] Department of Mathematics, Taiz University, Taiz P.O. Box 6803, Yemen.
e-mail:mahamgh1@gmail.com, ORCID: https://orcid.org/0003-0657-2093.

[*] Corresponding author.

[2] Department of Mathematics, Hodeidah University, AL-Hudaydah, Yemen.
e-mail:abdul.sharef1985@hoduniv.net.ye; ORCID: https://orcid.org/0000-0001-9076-3615.

[3] Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, India.
e-mail: ghadle.maths@bamu.ac.in ,ORCID: https://orcid.org/0000-0003-3205-5498.

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Numerous academics have shown a keen interest in the Adomian Decomposition Method (ADM) and the Modified Adomian Decomposition Method (MADM) as potential solutions to issues in the applied sciences. Different types of differential equations, fractional-order derivatives, integral equations, and integro-differential equations can be solved using these approaches [4, 5, 6, 7, 8, 9, 10, 11].

The solution for the general form of variable coefficients in multi-higher nonlinear Integro-Fractional Differential Equations (FIDE) of the Volterra-Hammerstein (V-H) type is given in [12].

$$ {}_a^C\mathfrak{D}_{\mathfrak{t}}^{\mathfrak{w}_i} h(\mathfrak{t}) + \sum_{j=1}^{i-1} k_j(\mathfrak{t}) {}_a^C\mathfrak{D}_{\mathfrak{t}}^{\mathfrak{w}_j} h(\mathfrak{t}) + k_i(\mathfrak{t}) h(\mathfrak{t}) = u(\mathfrak{t}) + \sum_{\nu=0}^{n} \delta_\nu \int_a^{\mathfrak{t}} \kappa_\nu(\mathfrak{t},\mu) \mathcal{G}_\nu(\mu, {}_a^C\mathfrak{D}_{\mu}^{\mathfrak{q}_\nu} h(\mu)) d\mu $$

A general form of nonlinear fractional integro-differential equations (FIDE) of the Volterra Fredholm-Hammerstein (VF-H) type is being studied in this study with the purpose of providing a semi-analytic solution. Typically, we widened the scope of the equations. The following nonlinear fractional integralo-differential equations of the Volterra Fredhlom-Hammerstein type have been solved :

$$ \begin{aligned} {}_a^C\mathfrak{D}_{\mathfrak{t}}^{\mathfrak{w}_i} h(\mathfrak{t}) \quad &+ \quad \sum_{j=1}^{i-1} k_j(\mathfrak{t}) {}_a^C\mathfrak{D}_{\mathfrak{t}}^{\mathfrak{w}_j} h(\mathfrak{t}) + k_i(\mathfrak{t}) h(\mathfrak{t}) = u(\mathfrak{t}) \qquad (1) \\ &+ \quad \sum_{\nu=0}^{n} \left[\delta_\nu \int_a^{\mathfrak{t}} \kappa_\nu(\mathfrak{t},\mu) \mathcal{G}_\nu(\mu, {}_a^C\mathfrak{D}_{\mu}^{\mathfrak{q}_\nu} h(\mu)) d\mu + \delta_\nu^* \int_a^{T} \kappa_\nu^*(\mathfrak{t},\mu) \mathcal{G}_\nu^*(\mu, {}_a^C\mathfrak{D}_{\mu}^{\mathfrak{q}_\nu^*} h(\mu)) d\mu \right] \end{aligned} $$

with the initial conditions

$$ h^{(m)}(a) = h_m \in \mathbb{R}, m = 0, 1, ..., \iota - 1; \iota = \max\{\lceil \mathfrak{w}_i \rceil, \lceil \mathfrak{q}_n \rceil, \lceil \mathfrak{q}_n^* \rceil\}, \ \mathfrak{t} \in \psi = [0, T] \qquad (2) $$

where $h(\mathfrak{t})$ is the unknown function which is the solution of equation (1) under initial condition (2), as well as, the functions $\kappa_\nu, \kappa_\nu^* : \Upsilon \longrightarrow \mathbb{R}$ with $\Upsilon = \{(\mathfrak{t}, \mu) : 0 \leq \mu \leq \mathfrak{t} \leq T\}$; $\mathcal{G}_\nu, \mathcal{G}_\nu^* : \Upsilon^* \times \mathbb{R} \longrightarrow \mathbb{R}$ with $\Upsilon^* = [0, T]$ and $u(\mathfrak{t}) : [0, T] \longrightarrow \mathbb{R}$ is a continuous function.

The following is the outline of the paper:In Sect.2 provides the essential background information and definitions of fractional derivatives and fractional integration. Sect.3 covers the fundamental idea of the Adomian decomposition approach, whereas Sect.4 is devoted to the formulation of ADM and MADM for solving nonlinear FIDE of VF-H type. We provided examples in Sect.5 to illustrate our results. Lastly, includes a discussion for these methods in Sect.6.

2. Auxiliary Results

This section covers the fundamentals of fractional calculus theory , which will be utilized throughout the study [13, 14, 15, 16, 17, 21, 23, 24, 26].

Definition 2.1. [3, 10] *.Let $h \in \mathfrak{C}_k(\mathfrak{t})[\eta, \eta_0]$, $k(\mathfrak{t}) \geq -1$ and $\mathfrak{w} \in \mathbb{R}^+$. Then the (R-L) fractional integral operator of order $\mathfrak{w}$ of a function h, for $\mathfrak{w} > 0$ is defined by:*

$$ \mathfrak{I}_a^{\mathfrak{w}} h(\mathfrak{t}) = \frac{1}{\Gamma(\mathfrak{w})} \int_a^{\mathfrak{t}} (\mathfrak{t} - \sigma)^{\mathfrak{w}-1} h(\sigma) d\sigma, \ \mathfrak{w} > 0. \qquad (3) $$

Definition 2.2. [8, 10, 13] *The Caputo fractional derivative of order $\mathfrak{w}$ $(\upsilon - 1 < \mathfrak{w} < \upsilon)$ is defined as*

$$ {}^C\mathfrak{D}_a^{\mathfrak{w}} h(\mathfrak{t}) = \frac{1}{\Gamma(\upsilon - \mathfrak{w})} \int_a^{\mathfrak{t}} (\mathfrak{t} - \sigma)^{\upsilon - \mathfrak{w} - 1} h^{(\upsilon)}(\sigma) d\sigma, \ \mathfrak{w} > 0. \qquad (4) $$

Lemma 2.1. [8, 10], *For real numbers* $\mathfrak{w}, \mathfrak{q} > 0$ *and* h *is continuous on closed bounded interval* $[a, b]$*; we have*

(1) $\mathfrak{I}^{\mathfrak{w}}_{0+}\mathfrak{I}^{\mathfrak{q}}_{0+}h(\mathfrak{r}) = \mathfrak{I}^{\mathfrak{q}}_{0+}\mathfrak{I}^{\mathfrak{w}}_{0+}h(\mathfrak{r}) = \mathfrak{I}^{\mathfrak{w}+\mathfrak{q}}_{0+}h(\mathfrak{r})$
(2) $\mathfrak{I}^{\mathfrak{w}}_{0+}\, {}^{C}\mathfrak{D}^{\mathfrak{w}}_{0+}h(\mathfrak{r}) = h(\mathfrak{r}) - h(0),\ 0 < \mathfrak{w} < 1$
(3) ${}^{C}\mathfrak{D}^{\mathfrak{w}}_{0+}\mathfrak{I}^{\mathfrak{w}}_{0+}h(\mathfrak{r}) = h(\mathfrak{r}).$
(4) $\mathfrak{I}^{\mathfrak{w}}_{a}\, {}^{C}\mathfrak{D}^{\mathfrak{w}}_{a}h(\mathfrak{r}) = h(\mathfrak{r}) - \sum_{j=0}^{n-1} \frac{h^{(j)}(a)}{j!}(\mathfrak{r} - a)^{j}.$
(5) $\mathfrak{I}^{\mathfrak{w}}_{a}\{(\mathfrak{r} - a)^{\mathfrak{q}-1}\} = \frac{\Gamma(\mathfrak{q})}{\Gamma(\mathfrak{q}+\mathfrak{w})}(\mathfrak{r} - a)^{\mathfrak{q}+\mathfrak{w}-1}$

Lemma 2.2. [3]. *Let* $\mathfrak{F}$ *be continuous function on* $[\eta, \eta_1] \times [\eta, \eta_1]$. *Then for* $\mathfrak{w} \geq 0$

$$_{\eta}\mathfrak{I}^{\mathfrak{w}}_{\mathfrak{r}} \int_{\eta}^{\mathfrak{r}} \mathfrak{F}(\mathfrak{r}, \mu)d\mu = \int_{\eta}^{\mathfrak{r}} {}_{\mu}\mathfrak{I}^{\mathfrak{w}}_{\mathfrak{r}}\mathfrak{F}(\mathfrak{r}, \mu)d\mu,\ \mathfrak{r} \in [\eta, \eta_1]$$

3. The Adomian Decomposition method (ADM)

[9, 19, 20].The Adomian method defines the solution $h(\mathfrak{r})$ by the series

$$h(\mathfrak{r}) = \sum_{r=0}^{\infty} h_r(\mathfrak{r})$$

This technique focuses on finding the components $h_0(\mathfrak{r}), h_1(\mathfrak{r}), h_2(\mathfrak{r}), ...$ individually, where the components $h_r(\mathfrak{r}),\ r \geq 0$ have to be determined in a recursive way. Yet, the nonlinear component $\mathfrak{F}(h(\mathfrak{r}))$ and $\mathfrak{F}^*(h(\mathfrak{r}))$, can be written in the decomposed form:

$$\mathfrak{F}(h(\mathfrak{r})) = \sum_{m=0}^{\infty} A_m[h_0(\mathfrak{r}), h_1(\mathfrak{r}), h_2(\mathfrak{r}), ..., h_m(\mathfrak{r})]$$

and

$$\mathfrak{F}^*(h(\mathfrak{r})) = \sum_{m=0}^{\infty} B_m[h_0(\mathfrak{r}), h_1(\mathfrak{r}), h_2(\mathfrak{r}), ..., h_m(\mathfrak{r})]$$

where A_m and B_m are called the Adomian polynomials which depending on $h_0, h_1, h_2, ..., h_m$ and that are obtained for the nonlinearity $\mathfrak{F}(h(\mathfrak{r}))$ and $\mathfrak{F}^*(h(\mathfrak{r}))$ by formulas:

$$A_m(\mathfrak{r}) = \frac{1}{m!}\frac{d^m}{d\Theta^m}\left[\mathfrak{F}(\sum_{i=0}^{\infty} \Theta^i h_i)\right]_{\Theta=0},\ m = 0, 1, 2, ... \tag{5}$$

$$B_m(\mathfrak{r}) = \frac{1}{m!}\frac{d^m}{d\Theta^m}\left[\mathfrak{F}^*(\sum_{i=0}^{\infty} \Theta^i h_i)\right]_{\Theta=0},\ m = 0, 1, 2, ... \tag{6}$$

where Θ is a parameter introduced for convenience. The uniqueness of the Adomian polynomial is not required at all and we can apply the Taylor expansion of $\mathfrak{F}(h(\mathfrak{r}))$and $\mathfrak{F}^*(h(\mathfrak{r}))$ about the first component $h_0(\mathfrak{r})$ to generate the forms as follows:

$$\mathfrak{F}(h(\mathfrak{r})) = \sum_{m=0}^{\infty} A_m[h_0(\mathfrak{r}), h_1(\mathfrak{r}), h_2(\mathfrak{r}), ..., h_m(\mathfrak{r})] = \sum_{m=0}^{\infty} \frac{[h(\mathfrak{r}) - h_0(\mathfrak{r})]^m}{m!}\mathfrak{F}^{(m)}(h_0(\mathfrak{r}))$$

and

$$\mathfrak{F}^*(h(\mathfrak{r})) = \sum_{m=0}^{\infty} B_m[h_0(\mathfrak{r}), h_1(\mathfrak{r}), h_2(\mathfrak{r}), ..., h_m(\mathfrak{r})] = \sum_{m=0}^{\infty} \frac{[h(\mathfrak{r}) - h_0(\mathfrak{r})]^m}{m!}\mathfrak{F}^{*(m)}(h_0(\mathfrak{r}))$$

Through the use of above expansions, from the simple analytic nonlinearity $\mathfrak{F}(h(\mathfrak{r}))$ and $\mathfrak{F}^*(h(\mathfrak{r}))$, the Adomian polynomials A_m and B_m are arranged to have the forms:

$$\begin{aligned}
A_0 &= \mathfrak{F}(h_0) \\
A_1 &= h_1\mathfrak{F}'(h_0) \\
A_2 &= h_2\mathfrak{F}'(h_0) + \frac{1}{2!}h_1^2\mathfrak{F}''(h_0) \\
A_3 &= h_3\mathfrak{F}'(h_0) + h_1h_2\mathfrak{F}''(h_0) + \frac{1}{3!}h_1^3\mathfrak{F}'''(h_0) \\
A_4 &= h_4\mathfrak{F}'(h_0) + (\frac{1}{2!}h_2^2 + h_1h_3)\mathfrak{F}''(h_0) + \frac{1}{2!}h_1^2h_2\mathfrak{F}'''(h_0) + \frac{1}{4!}h_1^4\mathfrak{F}^{(4)}(h_0) \\
&\dots\dots
\end{aligned} \tag{7}$$

and

$$\begin{aligned}
B_0 &= \mathfrak{F}^*(h_0) \\
B_1 &= h_1\mathfrak{F}^{*'}(h_0) \\
B_2 &= h_2\mathfrak{F}^{*'}(h_0) + \frac{1}{2!}h_1^2\mathfrak{F}^{*''}(h_0) \\
B_3 &= h_3\mathfrak{F}^{*'}(h_0) + h_1h_2\mathfrak{F}^{*''}(h_0) + \frac{1}{3!}h_1^3\mathfrak{F}^{*'''}(h_0) \\
B_4 &= h_4\mathfrak{F}^{*'}(h_0) + (\frac{1}{2!}h_2^2 + h_1h_3)\mathfrak{F}^{*''}(h_0) + \frac{1}{2!}h_1^2h_2\mathfrak{F}^{*'''}(h_0) + \frac{1}{4!}h_1^4\mathfrak{F}^{*(4)}(h_0) \\
&\dots\dots
\end{aligned} \tag{8}$$

For these cases the Adomian polynomials:

$$A_m(\mathfrak{r}) = A_m[h_0(\mathfrak{r}), h_1(\mathfrak{r}), h_2(\mathfrak{r}), \dots h_m(\mathfrak{r})],\ m \geq 1$$

and

$$B_m(\mathfrak{r}) = B_m[h_0(\mathfrak{r}), h_1(\mathfrak{r}), h_2(\mathfrak{r}), \dots h_m(\mathfrak{r})],\ m \geq 1$$

can be listed in general formula, [16]:

$$A_m(\mathfrak{r}) = \sum_{j=1}^{m} \zeta_m^j \mathfrak{F}^{(j)}(h_0)$$

and

$$B_m(\mathfrak{r}) = \sum_{j=1}^{m} \zeta_m^j \mathfrak{F}^{*(j)}(h_0),$$

where

$$\zeta_m^j = \begin{cases} h_m,\ j = 1 \\ \\ \frac{1}{m}\sum_{i=0}^{m-j}(i+1)h_{i+1}\zeta_{m-1-i}^{j-1},\ 2 \leq j \leq m \end{cases}$$

The Adomian Decomposition Method (ADM) can now have its convergence speeded up using a newly-developed technique. In order to show how quickly the solution converges, this new method frequently makes use of the "Noise Term Phenomenon" [13]. You can use the phenomena of noise terms with any functional equation, whether it's an integral or differential equation. Both $h_0(\mathfrak{r})$ and $h_1(\mathfrak{r})$, which are the same terms with the same signs but opposite signs, can be found in any component. Noise terms will also show up in the zeroth component if that component has the exact solution of the problem. Make that the remaining non-canceled terms satisfy the integral equation.

4. Methodology for Methods Analysis

Here, we apply the Adomian Decomposition Method and the Modified Adomian Decomposition Method to learn the general solution form of VF-H type multi-higher nonlinear FIDEs with the initial condition (2).

4.1. **Applying the ADM for Solving nonlinear FIDE of VF-H Type:**

Our approach begins by taking $\mathfrak{w}_i$-order of (R-L) fractional integral $\mathfrak{I}^{\mathfrak{w}_i}$ to both sides of equation (1) where $n_{\mathfrak{w}_i} = \lceil \mathfrak{w}_i \rceil$ and using lemma (2.2) and lemma (2.1, part-4) we obtain:

$$\begin{aligned} h(\mathfrak{t}) &= \mathfrak{I}^{\mathfrak{w}_i}u(\mathfrak{t}) + \sum_{l=0}^{n_{\mathfrak{w}_i}-1} \frac{h^{(l)}(a)}{l!}(\mathfrak{t}-a)^l - \mathfrak{I}^{\mathfrak{w}_i}\left(\sum_{j=1}^{i-1} k_j(\mathfrak{t}) {}_a^C\mathfrak{D}_{\mathfrak{t}}^{\mathfrak{w}_j} h(\mathfrak{t}) + k_i(\mathfrak{t})h(\mathfrak{t})\right) \qquad (9) \\ &+ \mathfrak{I}^{\mathfrak{w}_i}\left[\sum_{\nu=0}^{n}\left(\delta_\nu \int_a^{\mathfrak{t}} \kappa_\nu(\mathfrak{t},\mu)\mathcal{G}_\nu(\mu, {}_a^C\mathfrak{D}_\mu^{\mathfrak{q}_\nu} h(\mu))d\mu \right.\right. \\ &+ \left.\left. \delta_\nu^* \int_a^{T} \kappa_\nu^*(\mathfrak{t},\mu)\mathcal{G}_\nu^*(\mu, {}_a^C\mathfrak{D}_\mu^{\mathfrak{q}_\nu^*} h(\mu))d\mu\right)\right] \end{aligned}$$

Let the nonlinear terms $\mathfrak{F}_\nu(h(\mu)) = \mathcal{G}_\nu(\mu, {}_a^C\mathfrak{D}_\mu^{\mathfrak{q}_\nu} h(\mu))$ and $\mathfrak{F}^*{}_\nu(h(\mu)) = \mathcal{G}_\nu^*(\mu, {}_a^C\mathfrak{D}_\mu^{\mathfrak{q}_\nu^*} h(\mu))$ for all $\nu = 0, 1, ..., n$. We first express the linear term $h(\mathfrak{t})$ on the left side as an infinite series of components in the form:

$$h(\mathfrak{t}) = \sum_{r=0}^{\infty} h_r(\mathfrak{t}) \tag{10}$$

However, the nonlinear terms $\mathfrak{F}_\nu(h(\mu))$ and $\mathfrak{F}^*{}_\nu(h(\mu))$ at the right side will be represented by an infinite series of the Adomian polynomials in the form, for all $\nu = 0, 1, 2.......n$

$$\begin{aligned} \mathfrak{F}_\nu(h(\mu)) &= \sum_{m=0}^{\infty} A_m^\nu[h_0(\mu), h_1(\mu), ..., h_m(\mu)] \qquad (11) \\ \mathfrak{F}^*{}_\nu(h(\mu)) &= \sum_{m=0}^{\infty} B_m^\nu[h_0(\mu), h_1(\mu), ..., h_m(\mu)] \end{aligned}$$

Then, substituting equations (10) and (11) into equation (9) yields we advised the subsequent recursive formula (12):

$$\begin{aligned} h_0(\mathfrak{t}) &= \mathfrak{I}^{\mathfrak{w}_i}u(\mathfrak{t}) + \sum_{l=0}^{n_{\mathfrak{w}_i}-1} \frac{h^{(l)}(a)}{l!}(\mathfrak{t}-a)^l \\ h_{r+1}(\mathfrak{t}) &= -\left(\sum_{j=1}^{i-1} \mathfrak{I}^{\mathfrak{w}_i}\big(k_j(\mathfrak{t}) {}_a^C\mathfrak{D}_{\mathfrak{t}}^{\mathfrak{w}_j} h(\mathfrak{t})\big) + \mathfrak{I}^{\mathfrak{w}_i}\big(k_i(\mathfrak{t})h(\mathfrak{t})\big)\right) \qquad (12) \\ &+ \sum_{\nu=0}^{n}\left(\delta_\nu \int_a^{\mathfrak{t}} \mathfrak{I}^{\mathfrak{w}_i}\big(\kappa_\nu(\mathfrak{t},\mu)\big)A_r^\nu[h_0(\mu), h_1(\mu), ..., h_r(\mu)]d\mu \right. \\ &+ \left. \delta_\nu^* \int_a^{T} \mathfrak{I}^{\mathfrak{w}_i}\big(\kappa_\nu^*(\mathfrak{t},\mu)\big)B_r^\nu[h_0(\mu), h_1(\mu), ..., h_r(\mu)]d\mu\right), \text{ for all } r \geq 0 \end{aligned}$$

where

$$A_0^{\nu}[h_0]=\mathfrak{F}_{\nu}(h_0),\ A_1^{\nu}[h_0,h_1]=h_1\mathfrak{F}_{\nu}'(h_0),\ A_2^{\nu}[h_0,h_1,h_2]=h_2\mathfrak{F}_{\nu}'(h_0)+\frac{h_1^2}{2!}(\mathfrak{F}_{\nu}''(h_0)),...(13)$$

$$B_0^{\nu}[h_0]=\mathfrak{F}^*{}_{\nu}(h_0),\ B_1^{\nu}[h_0,h_1]=h_1\mathfrak{F}_{\nu}^{*'}(h_0),\ B_2^{\nu}[h_0,h_1,h_2]=h_2\mathfrak{F}_{\nu}^{*'}(h_0)+\frac{h_1^2}{2!}(\mathfrak{F}_{\nu}^{*''}(h_0)),...(14)$$

In practical computing for Adomian polynomials A_m^{ν} and B_m^{ν} we truncate the series after for positive finite number $\mathfrak{K}$. Thus:

$$A_m^{\nu}(\mathfrak{r})=\frac{1}{m!}\frac{d^m}{d\Theta^m}\Big[\mathfrak{F}_{\nu}\big(\sum_{k=0}^{\mathfrak{K}}\Theta^k h_k\big)\Big]_{\Theta=0},\ 0\le m\le\mathfrak{K},\ \nu=0,1,2,...,n \tag{15}$$

and

$$B_m^{\nu}(\mathfrak{r})=\frac{1}{m!}\frac{d^m}{d\Theta^m}\big[\mathfrak{F}_{\nu}^{*}(\sum_{k=0}^{\mathfrak{K}}\Theta^k h_k)\big]_{\Theta=0},\ 0\le m\le\mathfrak{K},\ \nu=0,1,2,...n, \tag{16}$$

Consequently, since it is not required to ascertain every term in the series in equation (10), we can approximate the solution using the following abbreviated series:

$$h(\mathfrak{r})\cong\hat{h}_{\mathfrak{K}}(\mathfrak{r})=\sum_{r=0}^{\mathfrak{K}}h_r(\mathfrak{r}),\ \mathfrak{K}\in\mathbb{Z}^+ \tag{17}$$

Recursively, the components $h_0, h_1, \dots, h_{\mathfrak{K}}$ are found by applying the noise terms technique or the formula (12). It is important to remember that the decomposition approach proposes using the function $u(\mathfrak{r})$ and the beginning conditions to define the zeroth component $h_0(\mathfrak{r})$. Recurrent derivation is used to obtain the remaining parts, which are $h_1, h_2, \dots, h_{\mathfrak{K}}$.

4.2. **Applying the MADM for Solving nonlinear FIDE of VF-H Type:**

The Adomian Decomposition Method presents solutions in an endless series of components, as mentioned earlier. If there are a few terms in the inhomogeneous term in equation (9), the components $h_r, r\ge 0$ can be simply calculated. Evaluating the components $h_r, r\ge 0$ becomes more laborious, nevertheless, when the inhomogeneous term includes two or more terms. The assumptions provided by Adomian were adjusted by Wazwaz [17, 18], which will further accelerate the convergence of the series solution and facilitate the computational process in the adjusted Adomian Decomposition Method. Using the MADM, or Modified Adomian Decomposition Strategy. Notably, the MADM is primarily dependent on dividing the inhomogeneous term into two or more components; as a result, it is not applicable when the inhomogeneous term comprises of the simplest possible term. Success in implementing this change is conditional on making the optimal assumptional decision between two or more functions, which can only be determined by trial and error. One drawback of this approach is that a rule that can assist with selecting appropriate subfunctions has not been identified as of yet.see [8, 17]. The following equation can be obtained by following the same steps as in ADM:

$$\begin{aligned}h(\mathfrak{r}) &= \Phi(\mathfrak{r})-\left(\sum_{j=1}^{i-1}\mathfrak{I}^{\mathfrak{w}_i}\big(k_j(\mathfrak{r})\,{}_a^C\mathfrak{D}_{\mathfrak{r}}^{\mathfrak{w}_j}h(\mathfrak{r})\big)+\mathfrak{I}^{\mathfrak{w}_i}\big(k_i(\mathfrak{r})h(\mathfrak{r})\big)\right) \\ &+ \sum_{\nu=0}^{n}\left(\delta_{\nu}\int_a^{\mathfrak{r}}\mathfrak{I}^{\mathfrak{w}_i}\kappa_{\nu}(\mathfrak{r},\mu)\mathcal{G}_{\nu}(\mu,{}_a^C\mathfrak{D}_{\mu}^{\mathfrak{q}_{\nu}}h(\mu))d\mu\right. \\ &+ \left.\delta_{\nu}^*\int_a^T\mathfrak{I}^{\mathfrak{w}_i}\kappa_{\nu}^*(\mathfrak{r},\mu)\mathcal{G}_{\nu}^*(\mu,{}_a^C\mathfrak{D}_{\mu}^{\mathfrak{q}_{\nu}^*}h(\mu))d\mu\right)\end{aligned} \tag{18}$$

where

$$\Phi(\mathfrak{t}) = \mathfrak{J}^{\mathfrak{w}_i} u(\mathfrak{t}) + \sum_{l=0}^{n_{\mathfrak{w}_i}-1} \frac{h^{(l)}(a)}{l!}(\mathfrak{t}-a)^l \tag{19}$$

We can set the function $\Phi(\mathfrak{t})$ as the finite sum of partial functions, namely

$$\Phi(\mathfrak{t}) = u_0(\mathfrak{t}) + u_1(\mathfrak{t}) + u_2(\mathfrak{t}) + ... + u_{\mathcal{N}}(\mathfrak{t}) \tag{20}$$

The components $h_0(\mathfrak{t})$ and $h_\rho(\mathfrak{t})$, where $\rho = 1, 2, ..., \mathcal{N}$, are the only ones for which a small modification is suggested in order to decrease the size of the calculations. We find that only the component $u_0(\mathfrak{t})$ can be assigned to the zeroth component $h_0(\mathfrak{t})$ using one part of $\Phi(\mathfrak{t})$, while the other part of $\Phi(\mathfrak{t})$ can be introduced to the component $h_1(\mathfrak{t})$ among different terms, say $u_1(\mathfrak{t})$. The MADM thus presents the revised recurrence relation:

$$\begin{aligned}
h_0(\mathfrak{t}) &= u_0(\mathfrak{t}) \\
h_\rho(\mathfrak{t}) &= u_\rho(\mathfrak{t}) - \left(\sum_{j=1}^{i-1} \mathfrak{J}^{\mathfrak{w}_i}\big(k_j(\mathfrak{t})\,{}_a^C\mathfrak{D}_{\mathfrak{t}}^{\mathfrak{w}_j} h(\mathfrak{t})\big) + \mathfrak{J}^{\mathfrak{w}_i}\big(k_i(\mathfrak{t})h(\mathfrak{t})\big)\right) \\
&+ \sum_{\nu=0}^{n}\left(\delta_\nu \int_a^{\mathfrak{t}} \mathfrak{J}^{\mathfrak{w}_i}\big(\kappa_\nu(\mathfrak{t},\mu)\big) A_{\rho-1}^{\nu}[h_0(\mu), h_1(\mu), ..., h_{\rho-1}(\mu)]d\mu\right. \\
&+ \left.\delta_\nu^* \int_a^{T} \mathfrak{J}^{\mathfrak{w}_i}\big(\kappa_\nu^*(\mathfrak{t},\mu)\big) B_{\rho-1}^{\nu}[h_0(\mu), h_1(\mu), ..., h_{\rho-1}(\mu)]d\mu\right), \text{ for all } 1 \le \rho \le \mathcal{N} \\
h_{r+1}(\mathfrak{t}) &= -\left(\sum_{j=1}^{i-1} \mathfrak{J}^{\mathfrak{w}_i}\big(k_j(\mathfrak{t})\,{}_a^C\mathfrak{D}_{\mathfrak{t}}^{\mathfrak{w}_j} h(\mathfrak{t})\big) + \mathfrak{J}^{\mathfrak{w}_i}\big(k_i(\mathfrak{t})h(\mathfrak{t})\big)\right) \\
&+ \sum_{\nu=0}^{n}\left(\delta_\nu \int_a^{\mathfrak{t}} \mathfrak{J}^{\mathfrak{w}_i}\big(\kappa_\nu(\mathfrak{t},\mu)\big) A_{r}^{\nu}[h_0(\mu), h_1(\mu), ..., h_{r}(\mu)]d\mu\right. \\
&+ \left.\delta_\nu^* \int_a^{T} \mathfrak{J}^{\mathfrak{w}_i}\big(\kappa_\nu^*(\mathfrak{t},\mu)\big) B_{r}^{\nu}[h_0(\mu), h_1(\mu), ..., h_{r}(\mu)]d\mu\right), \text{ for all } r \ge \mathcal{N}
\end{aligned} \tag{21}$$

Note that scheme (21) lowers to relation (12) respectively if inhomogeneous time period consists of a one-time period alone.

5. Illustrative Examples:

Here we will provide some examples to help you to understand our methods.

Example 5.1. *Consider the following nonlinear (FIDE) of (VF-H) type:*

$$\begin{aligned}
&{}_0^C\mathfrak{D}_{\mathfrak{t}}^{1.6} h(\mathfrak{t}) = u(\mathfrak{t}) + \int_a^{\mathfrak{t}} (\mathfrak{t}-\mu)[{}_0^C\mathfrak{D}_{\mu}^{1.2} h(\mu)]^2 d\mu + \frac{1}{4}\int_a^1 (\mathfrak{t}-\mu)[{}_0^C\mathfrak{D}_{\mu}^{1.7} h(\mu)]d\mu \\
&h(0) = 0
\end{aligned} \tag{22}$$

where

$$u(\mathfrak{t}) = \frac{2}{\Gamma(1.5)}\mathfrak{t}^{0.5} + \frac{12}{5\Gamma^2(4.8)}\mathfrak{t}^{3.8}$$

while the exact solution:

$$h(\mathfrak{r}) = \frac{2.2\mathfrak{r}^{2.1}}{\Gamma(3.1)} + \left(\frac{12}{5\Gamma^2(4.8)}\right)\mathfrak{r}^{5.8}$$

From the problem we have:

$$\mathfrak{w}_i = 1.6,\ \mathfrak{q}_0 = 1.2,\ \mathfrak{q}_0^* = 1.7,\ \delta_0 = 1,\ \delta_0^* = \frac{1}{4}$$

First, using ADM:. *Applying the first part of recursive relation (12), we obtain:*

$$h_0(\mathfrak{r}) \cong \hat{h}_0(\mathfrak{r}) = \frac{2}{\Gamma(3.1)}\mathfrak{r}^{2.1} + \frac{12}{5\Gamma(4.8)\Gamma(6.4)}\mathfrak{r}^{5.4}$$

*Finding each of $A_0^0(\mathfrak{r})$ and $B_0^0(\mathfrak{r})$ by using equations (13) and (**??**) with $\nu = 0$, then we put $r = 0$, in the second part of recursive relation (12) to obtain $h_1(\mathfrak{r})$. Thus the approximate solution by the truncated series (17) using two iterate h_0 and h_1:*

$$\begin{aligned} h(\mathfrak{r}) \cong \hat{h}_1(\mathfrak{r}) &= h_0(\mathfrak{r}) + h_1(\mathfrak{r}) \\ &= \frac{2\mathfrak{r}^{2.1}}{\Gamma(3.1)} + \left(\frac{12}{5\Gamma(4.8)\Gamma(6.4)} + \frac{4\Gamma(2.8)}{\Gamma^2(1.9)\Gamma(6.4)}\right)\mathfrak{r}^{5.4} + \frac{12\Gamma(4.7)\mathfrak{r}^{7.3}}{20\Gamma(4.8)\Gamma(8.3)} \\ &+ \frac{144\Gamma(9.4)\mathfrak{r}^{12}}{25\Gamma^2(4.8)\Gamma^2(4.2)\Gamma(13)} + \frac{\mathfrak{r}^4}{2\Gamma(5)} + \frac{48\Gamma(6.1)\mathfrak{r}^{8.7}}{5\Gamma(1.9)\Gamma(4.8)\Gamma(4.2)\Gamma(9.7)} \end{aligned}$$

With the same steps, we obtain $\hat{h}_2(\mathfrak{r})$ Thus the approximate solution by the truncated series (17) using three iterate h_0, h_1 and h_2:

$$\begin{aligned} h(\mathfrak{r}) \cong \hat{h}_2(\mathfrak{r}) &= h_0(\mathfrak{r}) + h_1(\mathfrak{r}) + h_2(\mathfrak{r}) \\ &= \frac{2\mathfrak{r}^{2.1}}{\Gamma(3.1)} + \left(\frac{12}{5\Gamma(4.8)\Gamma(6.4)} + \frac{4\Gamma(2.8)}{\Gamma^2(1.9)\Gamma(6.4)}\right)\mathfrak{r}^{5.4} \\ &+ \frac{48\Gamma(6.1)\mathfrak{r}^{8.7}}{5\Gamma(1.9)\Gamma(4.8)\Gamma(4.2)\Gamma(9.7)} + \frac{144\Gamma(9.4)\mathfrak{r}^{12}}{25\Gamma^2(4.8)\Gamma^2(4.2)\Gamma(13)} + \frac{\mathfrak{r}^4}{2\Gamma(5)} \\ &+ \left(\frac{12\Gamma(4.7)}{20\Gamma(4.8)\Gamma(8.3)} + \frac{\Gamma(2.8)}{\Gamma^2(1.9)\Gamma(7.3)}\right)\mathfrak{r}^{7.3} + \frac{16\Gamma(2.8)\Gamma(6.1)\mathfrak{r}^{8.7}}{\Gamma^3(1.9)\Gamma(5.2)\Gamma(9.7)} \\ &+ \frac{2\Gamma(4.7)\mathfrak{r}^{8.3}}{\Gamma(1.9)\Gamma(3.8)\Gamma(9.3)} + \frac{3456\Gamma(9.4)\Gamma(16)\mathfrak{r}^{18.6}}{125\Gamma^2(4.8)\Gamma(5.2)\Gamma^2(4.2)\Gamma(11.8)\Gamma(19.6)} \\ &+ \left(\frac{192\Gamma(6.1)}{5\Gamma^2(1.9)\Gamma(4.8)\Gamma(4.2)\Gamma(8.5)} + \frac{96\Gamma(2.8)}{5\Gamma(4.8)\Gamma^2(1.9)\Gamma(5.2)}\right)\frac{\Gamma(9.4)\mathfrak{r}^{12}}{\Gamma(13)} \\ &+ \frac{1}{8\Gamma(6.9)}\mathfrak{r}^{5.9} + \frac{\Gamma(2.8)}{\Gamma^2(1.9)\Gamma(8.3)}\mathfrak{r}^{7.3} + \frac{3\Gamma(4.7)}{20\Gamma(4.8)\Gamma(10.2)}\mathfrak{r}^{9.2} \\ &+ \frac{576\Gamma(9.4)\Gamma(12.7)\mathfrak{r}^{15.3}}{25\Gamma^3(1.9)\Gamma^2(4.8)\Gamma^2(4.2)\Gamma(11.8)\Gamma(16.3)} + \frac{36\Gamma(9.4)\mathfrak{r}^{13.9}}{25\Gamma^2(4.8)\Gamma^2(4.2)\Gamma(14.9)} \\ &+ \left(\frac{48\Gamma(4.7)}{20\Gamma(1.9)\Gamma(4.8)\Gamma(7.1)} + \frac{24}{10\Gamma(4.8)\Gamma(5.2)\Gamma(3.8)}\right)\frac{\Gamma(8)\mathfrak{r}^{10.6}}{\Gamma(11.6)} \\ &+ \frac{288\Gamma(4.7)\Gamma(11.3)}{100\Gamma^2(4.8)\Gamma(5.2)\Gamma(7.1)\Gamma(15)}\mathfrak{r}^{14} + \frac{12\Gamma(6.1)\mathfrak{r}^{11.6}}{5\Gamma(1.9)\Gamma(4.8)\Gamma(4.2)\Gamma(12.6)} \\ &+ \frac{1152\Gamma(6.1)\Gamma(12.7)\mathfrak{r}^{15.3}}{25\Gamma(4.8)\Gamma(5.2)\Gamma(1.9)\Gamma(4.2)\Gamma(8.5)\Gamma(16.3)} \end{aligned}$$

A comparison of the exact and approximate analytical solutions $\hat{h}_0(\mathfrak{r}), \hat{h}_1(\mathfrak{r})$, *and* $\hat{h}_2(\mathfrak{r})$ *is shown in the following table and figure. Depending on the Root Mean square error.*

	Exact solution	***Approximate solutions***		
$\mathfrak{r}$	$h(\mathfrak{r})$	$\hat{h}_0(\mathfrak{r})$	$\hat{h}_1(\mathfrak{r})$	$\hat{h}_2(\mathfrak{r})$
0.0	*0.0000000000*	*0.0000000000*	*0.0000000000*	*0.0000000000*
0.1	*0.0079518958*	*0.0072289875*	*0.0072311907*	*0.0072311923*
0.2	*0.0340911383*	*0.0309914323*	*0.0310298264*	*0.0310299367*
0.3	*0.0798845321*	*0.0726167809*	*0.0728307328*	*0.0728321734*
0.4	*0.1461861626*	*0.1328667417*	*0.1336138266*	*0.1336236450*
0.5	*0.2336462137*	*0.2122958078*	*0.2143113030*	*0.2143582662*
0.6	*0.3428322602*	*0.3113467726*	*0.3159569573*	*0.3161354136*
0.7	*0.4742959507*	*0.4303931812*	*0.4397885561*	*0.4403626873*
0.8	*0.6286222206*	*0.5697626231*	*0.5873366320*	*0.5889613613*
0.9	*0.8064723779*	*0.7297512949*	*0.7605098912*	*0.7646591712*
1.0	*1.0086255423*	*0.9106339944*	*0.9616813705*	*0.9714229528*
RMSE		*0.3703*	*0.4378*	*0.4284*

TABLE 1. Values of $\mathfrak{r}$, $h(\mathfrak{r})$ $\hat{h}_0(\mathfrak{r})$, $\hat{h}_1(\mathfrak{r})$, and $\hat{h}_2(\mathfrak{r})$

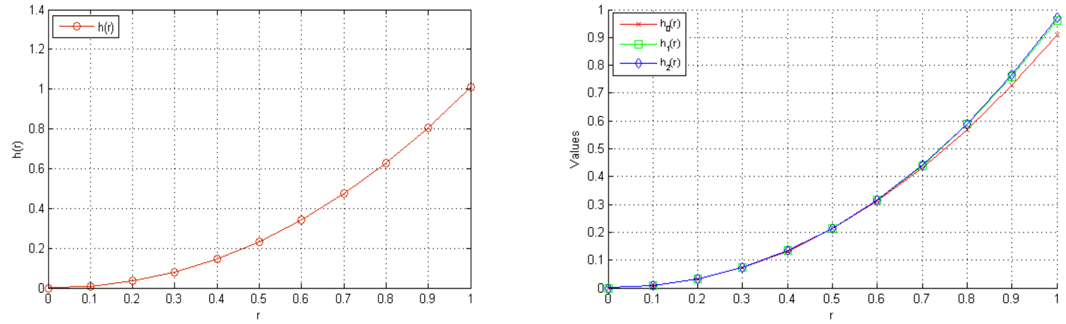

FIGURE 1. Comparison of Exact and Approximate Solutions

Second using MADM:.

From equations (19) ,we obtain

$$\Phi(\mathfrak{r}) = \mathfrak{I}_0^{1.6}\left[u(\mathfrak{r})\right] = \frac{2}{\Gamma(3.1)}\mathfrak{r}^{2.1} + \frac{12}{5\Gamma(4.8)\Gamma(6.4)}\mathfrak{r}^{5.4}$$

Let

$$u_0(\mathfrak{r}) = \frac{2}{\Gamma(3.1)}\mathfrak{r}^{2.1},\ u_1(\mathfrak{r}) = \frac{12}{5\Gamma(4.8)\Gamma(6.4)}\mathfrak{r}^{5.4}$$

Applying the first part of recursive relation (21), we obtain:

$$h_0(\mathfrak{r}) = u_0(\mathfrak{r}) = \frac{2}{\Gamma(3.1)}\mathfrak{r}^{2.1}$$

Finding each of $A_0^0(\mathfrak{r})$ *and* $B_0^0(\mathfrak{r})$ *by using equations (13) and (**??**) with* $\nu = 0$, *then we put* $\rho = 1$, *in the second part of recursive relation (21) to obtain* $h_1(\mathfrak{r})$. *Thus the* *approximate solution by the truncated series (17) using two iterate* h_0 *and* h_1*:*

$$\begin{aligned} h(\mathfrak{r}) \cong \hat{h}_1(\mathfrak{r}) &= h_0(\mathfrak{r}) + h_1(\mathfrak{r}) \\ &= \frac{2\mathfrak{r}^{2.1}}{\Gamma(3.1)} + \left[\frac{12}{5\Gamma(4.8)} + \frac{4\Gamma(2.8)}{\Gamma^2(1.9)}\right]\frac{\mathfrak{r}^{5.4}}{\Gamma(6.4)} + \frac{\mathfrak{r}^4}{2\Gamma(5)} \end{aligned}$$

Finding each of $A_1^0(\mathfrak{r})$ *and* $B_1^0(\mathfrak{r})$ *by using equations (13) and (??) with* $\nu = 0$, *then we put* $r = 1$, *in the third part of recursive relation (21) to obtain* $h_2(\mathfrak{r})$. *Thus the approximate solution by the truncated series (17) using three iterate* h_0, h_1 *and* h_2*:*

$$\begin{aligned} h(\mathfrak{r}) \cong \hat{h}_2(\mathfrak{r}) &= h_0(\mathfrak{r}) + h_1(\mathfrak{r}) + h_2(\mathfrak{r}) \\ &= \frac{2\mathfrak{r}^{2.1}}{\Gamma(3.1)} + \left[\frac{12}{5\Gamma(4.8)} + \frac{4\Gamma(2.8)}{\Gamma^2(1.9)}\right]\frac{\mathfrak{r}^{5.4}}{\Gamma(6.4)} + \frac{\mathfrak{r}^4}{2\Gamma(5)} \\ &+ \left(\frac{3}{5\Gamma(4.8)} + \frac{4\Gamma(2.8)}{\Gamma^2(1.9)} + \frac{2\Gamma(4.7)}{\Gamma(1.9)\Gamma(3.8)}\right)\frac{\mathfrak{r}^{7.3}}{\Gamma(8.3)} + \frac{1}{8\Gamma(6.9)}\mathfrak{r}^{5.9} \\ &+ \left(\frac{48}{5\Gamma(4.8)\Gamma(1.9)\Gamma(5.2)} + \frac{16\Gamma(2.8)}{\Gamma^3(1.9)\Gamma(5.2)}\right)\frac{\Gamma(6.1)\mathfrak{r}^{8.7}}{\Gamma(9.7)} \end{aligned}$$

With the same steps, we obtain $\hat{h}_3(\mathfrak{r})$ *:*

$$h(\mathfrak{r}) \cong \hat{h}_3(\mathfrak{r}) = h_0(\mathfrak{r}) + h_1(\mathfrak{r}) + h_2(\mathfrak{r}) + h_3(\mathfrak{r})$$

$$\begin{aligned} h(\mathfrak{r}) \cong \hat{h}_3(\mathfrak{r}) &= h_0(\mathfrak{r}) + h_1(\mathfrak{r}) + h_2(\mathfrak{r}) + h_3(\mathfrak{r}) \\ &= \frac{2\mathfrak{r}^{2.1}}{\Gamma(3.1)} + \left[\frac{12}{5\Gamma(4.8)} + \frac{4\Gamma(2.8)}{\Gamma^2(1.9)}\right]\frac{\mathfrak{r}^{5.4}}{\Gamma(6.4)} + \frac{\mathfrak{r}^4}{2\Gamma(5)} + \frac{1}{8\Gamma(6.9)}\mathfrak{r}^{5.9} \\ &+ \left(\frac{3}{5\Gamma(4.8)} + \frac{4\Gamma(2.8)}{\Gamma^2(1.9)} + \frac{2\Gamma(4.7)}{\Gamma(1.9)\Gamma(3.8)}\right)\frac{\mathfrak{r}^{7.3}}{\Gamma(8.3)} + \frac{\mathfrak{r}^{7.8}}{32}\frac{1}{\Gamma(8.8)} \\ &+ \left(\frac{48}{5\Gamma(4.8)\Gamma(1.9)\Gamma(5.2)} + \frac{16\Gamma(2.8)}{\Gamma^3(1.9)\Gamma(5.2)}\right)\frac{\Gamma(6.1)\mathfrak{r}^{8.7}}{\Gamma(9.7)} \\ &+ \left(\frac{12}{5\Gamma(4.8)} + \frac{4\Gamma(2.8)}{\Gamma^2(1.9)}\right)^2 \frac{\Gamma(6.1)}{\Gamma(8.5)\Gamma^2(5.2)}\frac{\Gamma(9.4)\mathfrak{r}^{12}}{\Gamma(13)} \\ &+ \left(\frac{192}{5\Gamma(4.8)\Gamma^2(1.9)\Gamma(5.2)} + \frac{64\Gamma(2.8)}{\Gamma^4(1.9)\Gamma(5.2)}\right)\frac{\Gamma(6.1)}{\Gamma(8.5)}\frac{\Gamma(9.4)\mathfrak{r}^{12}}{\Gamma(13)} \\ &+ \left(\frac{12}{5\Gamma(4.8)} + \frac{4\Gamma(2.8)}{\Gamma^2(1.9)}\right)\frac{1}{\Gamma(5.2)\Gamma(3.8)}\frac{\Gamma(8.3)}{\Gamma(7.1)}\frac{\Gamma(8)\mathfrak{r}^{10.6}}{\Gamma(11.6)} \\ &+ \left(\frac{12}{5\Gamma(1.9)\Gamma(4.8)} + \frac{4\Gamma(2.8)}{\Gamma^3(1.9)} + \frac{8\Gamma(4.7)}{\Gamma^2(1.9)\Gamma(3.8)}\right)\frac{\Gamma(8.3)}{\Gamma(7.1)}\frac{\Gamma(8)\mathfrak{r}^{10.6}}{\Gamma(11.6)} \\ &+ \left(\frac{1}{4\Gamma^2(3.8)} + \frac{1}{2\Gamma(1.9)} \cdot \frac{\Gamma(6.9)}{\Gamma(5.7)}\right)\frac{\Gamma(6.6)}{\Gamma(10.2)}\mathfrak{r}^{9.2} \\ &+ \left(\frac{3}{20\Gamma(4.8)} + \frac{4\Gamma(2.8)}{\Gamma^2(1.9)} + \frac{2\Gamma(4.7)}{\Gamma(1.9)\Gamma(3.8)}\right)\frac{\mathfrak{r}^{9.2}}{\Gamma(10.2)} \\ &+ \left(\frac{12\Gamma(6.1)}{5\Gamma(4.8)\Gamma(1.9)\Gamma(5.2)} + \frac{4\Gamma(2.8)\Gamma(6.1)}{\Gamma^3(1.9)\Gamma(5.2)}\right)\frac{\mathfrak{r}^{10.6}}{\Gamma(11.6)} \end{aligned}$$

A comparison of the exact and approximate analytical solutions $\hat{h}_1(\mathfrak{r}), \hat{h}_2(\mathfrak{r})$, *and* $\hat{h}_3(\mathfrak{r})$ *is shown in the following table and figure. Depending on the Root Mean square error.*

	Exact solution	***Approximate solutions***		
$\mathfrak{r}$	$h(\mathfrak{r})$	$\hat{h}_1(\mathfrak{r})$	$\hat{h}_2(\mathfrak{r})$	$\hat{h}_3(\mathfrak{r})$
0.0	*0.0000000000*	*0.0000000000*	*0.0000000000*	*0.0000000000*
0.1	*0.0079518958*	*0.0072311907*	*0.0072311910*	*0.0072311910*
0.2	*0.0340911383*	*0.0310298259*	*0.0310298542*	*0.0310298552*
0.3	*0.0798845321*	*0.0728307227*	*0.0728311470*	*0.0728311877*
0.4	*0.1461861626*	*0.1336137379*	*0.1336168202*	*0.1336173945*
0.5	*0.2336462137*	*0.2143108149*	*0.2143257122*	*0.2143301889*
0.6	*0.3428322602*	*0.3159549635*	*0.3160102723*	*0.3160342506*
0.7	*0.4742959507*	*0.4397819297*	*0.4399525095*	*0.4400516355*
0.8	*0.6286222206*	*0.5873177109*	*0.5877760285*	*0.5881150504*
0.9	*0.8064723779*	*0.7604618119*	*0.7615685414*	*0.7625717419*
1.0	*1.0086255423*	*0.9615700015*	*0.9640242070*	*0.9666723986*
RMSE		*0.0297*	*0.0316*	*0.0332*

TABLE 2. Values of $\mathfrak{r}$, $h(\mathfrak{r})$, $\hat{h}_1(r)$, $\hat{h}_2(\mathfrak{r})$, and $\hat{h}_3(\mathfrak{r})$

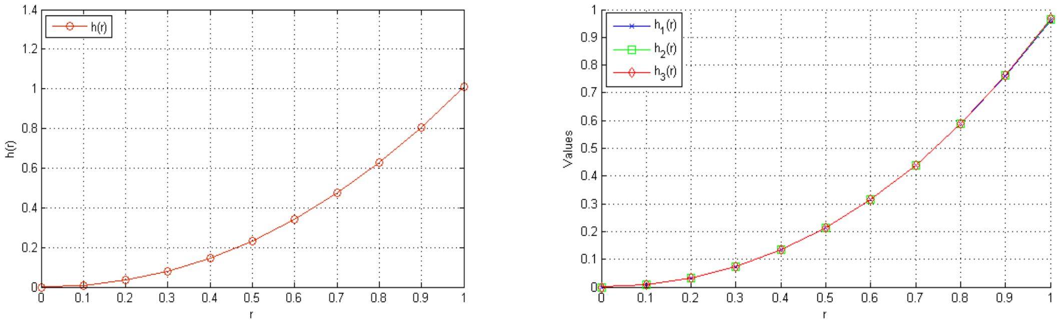

FIGURE 2. Comparison of Exact and Approximate Solutions

Summary. *The Adomian Decomposition Method (ADM) and the Modified Adomian Decomposition Method (MADM) reveals the following insights:*

- ***ADM vs. MADM****: Both methods are used to approximate the function* $h(\mathfrak{r})$. *The MADM generally converges faster, as seen by comparing the higher-order terms* $(\hat{h}_1, \hat{h}_2, \hat{h}_3)$ *in MADM to the equivalent terms in ADM.*
- ***Numerical Truncation****: In practical applications where a closed-form solution is not possible, both methods rely on truncation. MADM achieves more accurate results with fewer terms, making it computationally efficient.*

Example 5.2. *Consider the following nonlinear (FIDE) of (VF-H) type:*

$$
\begin{aligned}
&{}^{C}_{0}\mathfrak{D}^{0.3}_{\mathfrak{r}} h(\mathfrak{r}) = u(\mathfrak{r}) + \frac{1}{2}\int_a^{\mathfrak{r}} \mathfrak{r}^2 \mu^2 [{}^{C}_{0}\mathfrak{D}^{0.9}_{\mu} h(\mu)] d\mu - \int_a^{1} \mathfrak{r}^2 \mu [{}^{C}_{0}\mathfrak{D}^{1.1}_{\mu} h(\mu)] d\mu \\
&h(0) = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (23)
\end{aligned}
$$

where

$$u(\mathfrak{r}) = \frac{7}{\Gamma(2.3)}\mathfrak{r}^{1.3} + \frac{3}{4\Gamma(1.6)}\mathfrak{r}^{2.6} - \frac{1}{\Gamma(3.3)}\mathfrak{r}^{4.4}$$

while, the exact solution:

$$h(\mathfrak{r}) = \frac{8.3\mathfrak{r}^{1.6}}{\Gamma(2.3)\Gamma(2.6)} + \frac{3\mathfrak{r}^{2.9}}{4\Gamma(1.6)\Gamma(3.9)} - \frac{\mathfrak{r}^{4.7}}{\Gamma(3.3)\Gamma(5.7)}$$

From the problem we have:

$$\mathfrak{w}_i = 0.3,\ \mathfrak{q}_0 = 0.9,\ \mathfrak{q}_0^* = 1.1,\ \delta_0 = \frac{1}{2},\ \delta_0^* = -1$$

In the same way as in the first example, we will obtain the results for this example.

First using ADM:. *A comparison of the exact and approximate analytical solutions* $\hat{h}_0(\mathfrak{r}), \hat{h}_1(\mathfrak{r})$, *and* $\hat{h}_2(\mathfrak{r})$ *is shown in the following table and figure. Depending on the Root Mean square error.*

	Exact solution	***Approximate solutions***		
$\mathfrak{r}$	$h(\mathfrak{r})$	$\hat{h}_0(\mathfrak{r})$	$\hat{h}_1(\mathfrak{r})$	$\hat{h}_2(\mathfrak{r})$
0.0	*0.0000000000*	*0.0000000000*	*0.0000000000*	*0.0000000000*
0.1	*0.1251942523*	*0.1237283915*	*0.1237245050*	*0.1237245051*
0.2	*0.3803995837*	*0.3782549647*	*0.3781466908*	*0.3781466971*
0.3	*0.7297198617*	*0.7304275363*	*0.7296696038*	*0.7296697620*
0.4	*1.1596926298*	*1.1684505381*	*1.1654388224*	*1.1654403876*
0.5	*1.6625361510*	*1.6852664721*	*1.6764995681*	*1.6765087808*
0.6	*2.2330686927*	*2.2753760645*	*2.2544376949*	*2.2544766696*
0.7	*2.8675609722*	*2.9335629055*	*2.8899872064*	*2.8901182394*
0.8	*3.5631889397*	*3.6542134658*	*3.5723203744*	*3.5726919393*
0.9	*4.3177327121*	*4.4308841205*	*4.2887037712*	*4.2896267929*
1.0	*5.1293942544*	*5.2559894819*	*5.0244318545*	*5.0264920636*
RMSE		*0.0631461821*	*0.0345595298*	*0.0339407929*

TABLE 3. Values of $\mathfrak{r}$, $h(\mathfrak{r})$ $\hat{h}_0(r)$, $\hat{h}_1(r)$, and $\hat{h}_2(r)$

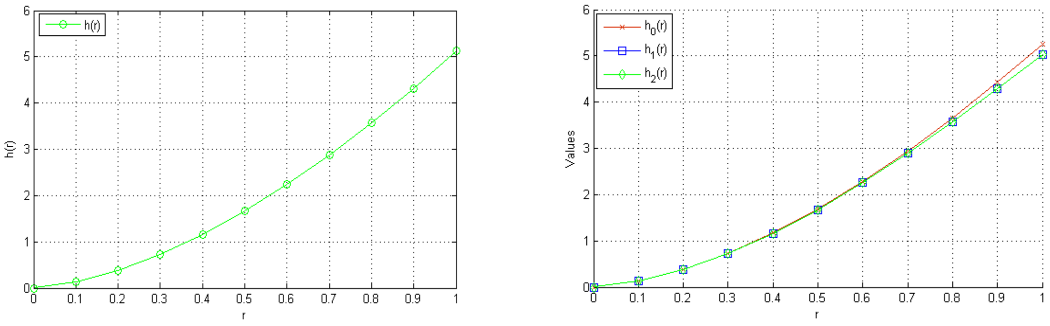

FIGURE 3. Comparison of Exact and Approximate Solutions

Second using MADM:. *A comparison of the exact and approximate analytical solutions* $\hat{h}_1(\mathfrak{r}), \hat{h}_2(\mathfrak{r})$, *and* $\hat{h}_3(\mathfrak{r})$ *is shown in the following table and figure . Depending on the Root Mean square error. The Root Mean Square Error (RMSE) values indicate the differences between the predicted values* $\hat{h}_1(\mathfrak{r})$, $\hat{h}_2(\mathfrak{r})$, $\hat{h}_3(\mathfrak{r})$ *and the actual values* $h(r)$. *Among the three approximations:*

	Exact solution	***Approximate solutions***		
$\mathfrak{r}$	$h(\mathfrak{r})$	$\hat{h}_1(\mathfrak{r})$	$\hat{h}_2(\mathfrak{r})$	$\hat{h}_3(\mathfrak{r})$
0.0	*0.0000000000*	*0.0000000000*	*0.0000000000*	*0.0000000000*
0.1	*0.1251942523*	*0.1232107527*	*0.1232107526*	*0.1232107526*
0.2	*0.3803995837*	*0.3744517687*	*0.3744517610*	*0.3744517609*
0.3	*0.7297198617*	*0.7183947923*	*0.7183946666*	*0.7183946656*
0.4	*1.1596926298*	*1.1416181918*	*1.1416170625*	*1.1416170453*
0.5	*1.6625361510*	*1.6360753310*	*1.6360682579*	*1.6360680905*
0.6	*2.2330686927*	*2.1960271842*	*2.1959932780*	*2.1959922030*
0.7	*2.8675609722*	*2.8168877315*	*2.8167557346*	*2.8167505552*
0.8	*3.5631889397*	*3.4946701208*	*3.4942341103*	*3.4942138896*
0.9	*4.3177327121*	*4.2256846810*	*4.2244218770*	*4.2243546492*
1.0	*5.1293942544*	*5.0063631341*	*5.0030761423*	*5.0028792310*
RMSE		*0.0345595298*	*0.0339407929*	*0.0339407929*

TABLE 4. Values of $\mathfrak{r}$,$h(\mathfrak{r})$, $\hat{h}_1(\mathfrak{r})$, $\hat{h}_2(\mathfrak{r})$, and $\hat{h}_3(\mathfrak{r})$

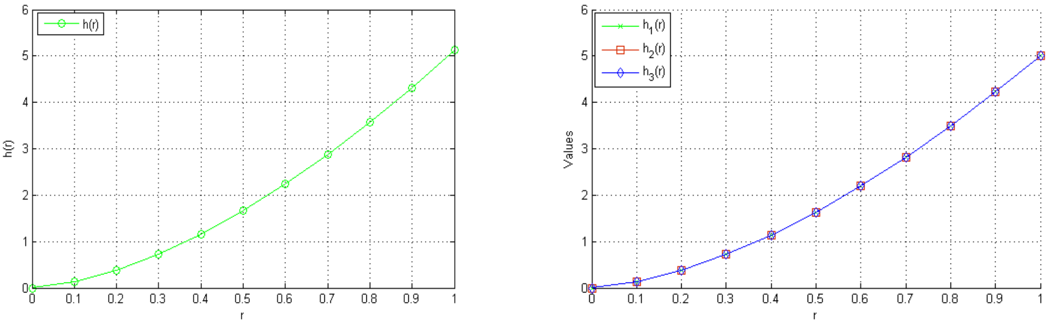

FIGURE 4. Comparison of Exact and Approximate Solutions

- $\hat{h}_2(\mathfrak{r})$ *has the lowest RMSE, suggesting it is the most accurate in approximating* $h(r)$.
- $\hat{h}_3(\mathfrak{r})$ *also shows a very low RMSE, similar to* $\hat{h}_2(\mathfrak{r})$, *indicating high accuracy.*
- $\hat{h}_1(\mathfrak{r})$, *while slightly less accurate, still provides a good approximation with a low RMSE.*

This analysis highlights the increasing accuracy from $\hat{h}_1(\mathfrak{r})$ *to* $\hat{h}_2(\mathfrak{r})$ *and* $\hat{h}_3(\mathfrak{r})$ *in approximating the values of* $h(\mathfrak{r})$.

Summary.

- *For both Modified Adomian and Adomian datasets,* $\hat{h}_2(r)$ *has the lowest RMSE, indicating it is the most accurate in approximating* $h(r)$.

- $\hat{h}_3(r)$ *in the Modified Adomian datasets also shows a very low RMSE, similar to* $\hat{h}_2(r)$*, indicating high accuracy.*
- *In the Adomian datasets,* $\hat{h}_0(r)$ *has the highest RMSE, indicating the least accurate approximation of* $h(r)$*.*

6. Conclusion

In this work, we use the Adomian Decomposition method (ADM) and the Modified Adomian Decomposition method (MADM) to solve nonlinear integro-fractional differential equations of the Volterra-Fredholm-Hammerstein type, both approximatively and exactly. We were able to efficiently and effectively develop an analytical and numerical solution to our problem using these techniques. It offers a faster convergent series solution that is more realistic. The Modified Adomian Decomposition Method (MADM) is generally better for achieving a solution due to its faster convergence and higher accuracy with fewer terms compared to the Adomian Decomposition Method (ADM). MADM provides a more efficient approach for solving nonlinear integro-fractional differential equations, making it a preferred method in applied sciences and engineering problems.

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Mrs. Maha. M. Hamood, working now as Assistant Professor, Taiz University, Taiz, Yemen, and she has completed post graduation in Mathematics with first class. Mrs. Maha has wide experience of 6 Yrs in teaching Mathematics. Her areas of interest are Differential Equations, Fractional Integro-Differential Equations, Integral Transform Technique etc. Mrs. Maha has more than 15 research papers are published in National & International peer-reviewed, Scopus, Web of Science, UGC index.

Mr. Abdulrahman A. Sharif, working now as Assistant Professor, Department of Mathematics, Hodeidah University, Al-Hudaydah, Yemen, and he has completed post graduation in Mathematics with first class. Mr. Sharif has wide experience of 6 Yrs in teaching Mathematics. His areas of interest are Differential Equations, Fractional Integro-Differential Equations, Integral Transform Technique etc. Mr. Sharif has more than 25 research papers are published in National & International peer-reviewed, Scopus, Web of Science, UGC index.

Kirtiwant P. Ghadle, working now as a Professor and Head Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, India. Total teaching experience 30 Yrs. Research experience 19 Yrs. His areas of interest are Thermoelasticity BVP, Operation Research, Integral Equations, Fractional Differential Equations. Prof. Ghadle has more than 170 research papers are published in National & International peerreviewed, Scopus index, web of science, UGC listed & ICI journals. Many national and international conferences are attended. Also, he is a member of the board of studies.