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NORMALITY AND REGULARITY OF PYTHAGOREAN FUZZY CELLULAR SPACES

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Abstract. Normality and regularity are key separation axioms that helps to classify and understand the structure of topological spaces. This research article investigates the properties of normality and regularity within the context of Pythagorean fuzzy cellular spaces. Pythagorean fuzzy cellular space integrates Pythagorean fuzzy sets with cellular spaces, provide a robust framework for modeling and analyzing complex systems characterized by uncertainty and imprecision. In the the concepts of normality and regularity is defined formally in the context of Pythagorean fuzzy cellular space and explore their implications. This study establishes the theoretical foundations for analyzing normality and regularity in Pythagorean fuzzy cellular space, extending classical topological concepts to the fuzzy environment. In addition to it $PF_{cel}\mathfrak{q}$-normal, PF_{cel} ultra normal, PF_{cel} completely ultra normal, PF_{cel} quasi normal is defined in Pythagorean fuzzy cellular space and interrelations are explored.

Keywords: Pythagorean fuzzy cellular space, Pythagorean fuzzy cellular $\mathfrak{q}$ normal, PF_{cel} ultra normal, PF_{cel} quasi normal and PF_{cel} regular space.

AMS Subject Classification: $54A40$ and $54E55$.

1. Introduction

Topology, the mathematical study of shapes and spaces, is a rich and fascinating field that explores properties preserved under continuous transformations. Among the numerous concepts in topology, normality and regularity are two significant properties of topological spaces. These concepts are pivotal in understanding the structure and behavior of different types of spaces and play crucial roles in various theorems and applications.

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Before delving into normality and regularity, it is essential to grasp the basics of topological spaces. A topological space is a set paired with a topology, which is a collection of open sets that meet specific axioms. These axioms ensure that the union of open sets and the finite intersection of open sets are also open, and that both the entire set and the empty set are included in the topology. Topological spaces is the generalizations of more familiar geometric objects. For instance, the plane, the real line and higher-dimensional Euclidean spaces can all be endowed with standard topologies, making them topological spaces.

One of the most significant changes in science and mathematics in the twenty-first century has been the concept of uncertainty. This movement in science has been marked by a gradual transition from the traditional position, which considers uncertainty undesirable in research and should be avoided at all costs, to an alternative viewpoint, which is tolerant of uncertainty and believes that science cannot avoid it. In 1965, Zadeh [17] introduced the concept of fuzzy sets to mathematically express ambiguity and attempted to tackle such difficulties by assigning a specified grade of membership to each member of a given set. A fuzzy set can be mathematically defined by assigning a value denoting the grade of membership in the fuzzy set to each potential individual in the universe of discourse. Subsequently, Atanassov introduced the non-membership function. As an extension of the intuitionistic fuzzy set, Yager [16] introduced Pythagorean fuzzy sets, and Olgun [9] later established the foundation for Pythagorean fuzzy topological spaces.

Cellular spaces can be defined in any n-dimensional spaces and have multiple perspectives leading to automata. In this study, cellular spaces are defined using Pythagorean fuzzy sets, referred to as Pythagorean fuzzy cellular spaces. Normality is a stronger separation axiom than regularity. It guarantees a higher level of separation between disjoint closed sets, facilitating more refined constructions and proofs, such as Urysohn's Lemma and the Tietze Extension Theorem, which are fundamental results in topology. Regularity ensures a certain degree of separation between points and closed sets, which can be instrumental in constructing continuous functions and in the analysis of convergence and compactness within the space. Normality, one of the few separation axioms that can be defined purely by the properties of open and closed sets without reference to points, is examined alongside regularity within the context of Pythagorean fuzzy cellular spaces.

This study introduces $PF_{cel}\mathfrak{q}$-normal, PF_{cel} ultra normal, PF_{cel} completely ultra normal, PF_{cel} quasi normal in Pythagorean fuzzy cellular space and interrelations are explored. Further, Regularity of PF_{cel} space in investigated. The interrelations among these newly defined normality properties are systematically explored, highlighting their distinct characteristics and mutual dependencies within Pythagorean fuzzy cellular spaces.

2. Literature Review

This section provides an in-depth review of the existing literature related to the study. Tables 1 and 2 provide a detailed overview of previous research related to normality and regularity in fuzzy topological spaces.

Table 1. Literature review of Normality and Regularity in Fuzzy topology

Author	Year	Source	Findings
Hutton [6]	1975	Normality in fuzzy topological spaces	Normality in fuzzy topological space is introduced and Normality is characterised in terms of Urysohn's lemma in fuzzy topological spaces.

TABLE 2. Literature review of Normality and Regularity in Fuzzy topology

Author	Year	Source	Findings
Ghareeb [4]	2011	Normality of double topological spaces	The concepts of double fuzzy almost normal, double fuzzy normal, and double fuzzy mildly normal spaces are introduced within the framework of double fuzzy topological spaces, and their characteristic properties are explored.
Thabit et al [14]	2012	π-Normality in topological spcaes and its generalizations	This study introduced p-normality called π-normality, which lies between p-normality and almost p-normality.
Gathigi Stephen et al [13]	2013	Normality and Its Variants on Fuzzy Isotone Spaces	In this study, variants of normality are investigated, showing that in fuzzy isotone spaces, perfect normality implies complete normality, which in turn implies normality.
Balasubra maniyan [3]	2013	Mildly fuzzy normal spaces and some functions	Mildly normal spaces and several topological functions—namely, almost rgf-continuous, almost gf-continuous, rgf-open, fuzzy regular open, almost rgf-open, and almost gf-open—are defined. Additionally, fuzzy rc-preserving functions in fuzzy topological spaces are introduced, and the relationships between mildly fuzzy normal spaces and these new fuzzy topological functions are investigated.
Al-Qubati [1]	2017	On b-regularity and Normality in intuitionistic fuzzy topological spaces	Some new types of intuitionistic fuzzy bseparation axioms, which is intuitionistic fuzzy b-$T_i$space (for i = 3,4) and (intuitionistic Fuzzy b-Regular and Fuzzy b-normal spaces is defined and studied
Karthika [7]	2020	Fuzzy Ggp normal and regular spaces	this study introduces Ggp sets and studied normal and regular spaces
Ray et al [10]	2021	Separation axioms in mixed fuzzy topological spaces	Definitions for fuzzy-T_0, fuzzy-T_1, fuzzy-T_2 (or Hausdorff), fuzzy regular, and fuzzy normal spaces within mixed fuzzy topological spaces are provided. Additionally, the relationships among these various types of fuzzy spaces—fuzzy-T_0, fuzzy-T_1, fuzzy-T_2, fuzzy regular, and fuzzy normal—are explored.
Sivasangari [12]	2021	On e-Regularity and e-Normality in Intuitionistic fuzzy topological spaces	This study explores the separation axioms of e-open sets in intuitionistic fuzzy topological spaces.
Liang [8]	2022	Regularity and Normality of (L, M)-fuzzy topological spaces using residual implication	Along with the separation axiom this study extended to residual implication.

Table 3. Literature review of Normality and Regularity in Fuzzy topology

Author	Year	Source	Findings
Vivek et al [15]	2022	Regularity of the extensions of a double fuzzy topological space	Regularity and extensions in regular double fuzzy topological spaces are examined, and certain families of closed sets within these spaces and their extensions are investigated.
Saleh et al [11]	2023	On g-regularity and g-normality in Fuzzy soft topological spaces	The notions of generalized regularity, normality, and symmetric in fuzzy soft topological spaces via fuzzy soft generalized closed is introduced and studied.
Al-Qubati et al [2]	2022	On intuitionistic fuzzy β generalised α normal spaces	A new class of spaces called an intuitionistic fuzzy β generalised α normal spaces and investigate some of their properties. Some interesting characterizations such as intuitionistic fuzzy β generalised α-normality is hereditary property with respect to an open and intuitionistic fuzzy β generalised α- closed subspace.

3. Motivation and Contribution of the Study

This study introduces a novel approach to normality and regularity by defining it within Pythagorean fuzzy cellular space. Normality and regularity helps to classify topological spaces based on their structural properties. This classification aids in understanding the inherent differences and similarities between various types of spaces. The normality property ensures the ability to separate disjoint closed sets using Pythagorean fuzzy cellular open sets. Similarly, regularity facilitates the separation of points and closed sets, which is vital for creating well-defined boundaries and classifications in applications. These concepts allows to analyze the behavior of spaces, especially interactions of the subsets and behaviour of functions within these spaces. In this study, normality is studied with Pythagorean fuzzy cellular spaces to study the interactions of the Pythagorean fuzzy cellular open sets and the Pythagorean fuzzy cellular functions like Pythagorean fuzzy cellular open function, Pythagorean fuzzy cellular closed function, Pythagorean fuzzy cellular clopen function behave in the Pythagorean fuzzy cellular spaces. The main contributions are as follows:

(i) Pythagorean fuzzy cellular is defined. Then Pythagorean fuzzy cellular space PF_{cel} space is established.
(ii) Normality in PF_{cel} space is introduced and q-normal, ultra normal, completely normal, quasi normal is defined in PF_{cel} space and interrelations are investigated.
(iii) Regularity in PF_{cel} space is defined. Then various properties of regularity is explored.

4. Preliminaries

Throughout this paper collection of Pythagorean fuzzy sets are denoted as $PF(\mathscr{X})$. This section includes the fundamental definitions needed for the study.

Definition 4.1. [16] A Pythagorean fuzzy set (PFS) R of a non-empty set $\mathscr{X}$ is a pair (μ_R, ν_R) where μ_R and ν_R are fuzzy sets of $\mathscr{X}$ in which ${\mu_R}^2\,(x)+\,{\nu_R}^2\,(x) \leq 1$ for any $x \,\epsilon\, \mathscr{X}$ the fuzzy set μ_R, ν_R is the degree of belongingness and non-belongingness respectively.

Definition 4.2. [9]
Let $\mathscr{X} \neq \emptyset$ and τ be a family of PFS. If

(i) $0_{\mathscr{X}},\ 1_{\mathscr{X}}\ \epsilon\ \tau$
(ii) $R_i \subset\ \tau$, we have $\bigcup\ R_j\ \epsilon\ \tau$ where $j \in I$ and I is an index set .
(iii) $R_1, R_2\ \epsilon\ \tau$, we have $R_1 \bigcap\ R_2\ \epsilon\ \tau$, where $0_X =\ (0,1)$ and $1_X = (1,0)$, then τ is called a Pythagorean fuzzy topology(PFT) on $\mathscr{X}$. Then $(\mathscr{X}, \tau)$ is the Pythagorean fuzzy topological space (PFTS). Each member in the PFTS is the Pythagorean fuzzy open set (PFOS). The complement of PFOS is called Pythagorean fuzzy closed set (PFCS).

Definition 4.3. [9] Let $(\mathscr{X}, \tau)$ be the PFTS. Let $S = (\mu_S, \nu_S)$ and $R = (\mu_R, \nu_R)$ be any two PFS of a set $\mathscr{X}$. Then,

(i) $R \bigcup S = (max(\mu_R, \mu_S), min(\nu_R, \nu_S))$
(ii) $R \bigcap S = (min(\mu_R, \mu_S), max(\nu_R, \nu_S))$
(iii) $R^c = (\nu_R, \mu_R)$
(iv) $S \supset R$ or $R \subset S$ if $\mu_R \leq\ \mu_S$ and $\nu_R \geq \nu_S$.

Example 4.1. Let $(\mathscr{X}, \tau)$ be the PFTS. $R = \{(0.4, 0.5), (0.2, 0.5))\}$ and
$S = \{(0.6, 0.3), (0.3, 0.4)\}$ be the two Pythagorean fuzzy set on $\mathscr{X} = \{a, b\}$, then
$R \bigcup S = max\{(0.4, 0.5), (0.2, 0.5)\} \bigcup \{(0.6, 0.3), (0.3, 0.4)\} = \{(0.6, 0.4), (0.3, 0.4)\}$
$R \bigcap S = min\{(0.4, 0.5), (0.2, 0.5)\} \bigcap \{(0.6, 0.3), (0.3, 0.4)\} = \{(0.4, 0.5), (0.2, 0.5)\}$
$R^c = \{(0.5, 0.4), (0.5, 0.2)\}$
$R \subset S = \{(0.4, 0.5), (0.2, 0.5))\} \subset \{(0.6, 0.3), (0.3, 0.4)\}$
$\mu_R \leq \mu_S = 0.4 \leq 0.6, 0.2 \leq 0.3$
$\nu_R \geq \nu_S = 0.5 \geq 0.3, 0.5 \geq 0.4.$

Definition 4.4. [9] Let $(\mathscr{X}, \tau)$ be a PFTS and $R = (\mu_R, \nu_R)$ be a PFS in $\mathscr{X}$. Then the Pythagorean fuzzy closure and Pythagorean fuzzy interior are defined by,

(i) $cl(R)= \bigcap\{K | K$ is a PFCS in $\mathscr{X}$ and $R \subseteq K\}$
(ii) $int(R)= \bigcup\{G | G$ is a PFOS in $\mathscr{X}$ and $G \subseteq R\}$

Definition 4.5. [5] If for every family $\Upsilon = \{K_i | K_i \in PF(\mathscr{X}); i \in I\ \}$ there exists a countable family $\Psi = \{L_i | L_i \in PF(\mathscr{X}); i \in I\ \}$ such that $\Upsilon \subseteq \Psi$ and $cl(\bigcup K_i) = \bigcup L_i$ for every $i \in I$ where I is an index set, then Υ is said to be Pythagorean fuzzy cellular (in short PF_{cel}).

Definition 4.6. [5] A Pythagorean fuzzy topology τ_p on $\mathscr{X}$ is said to be Pythagorean fuzzy cellular space if for every family $\Upsilon = \{K_i | K_i \in \tau_p; i \in I\ \}$ there exists a countable family $\Psi = \{L_i | L_i \in \tau_p; i \in I\ \}$ such that $\Upsilon \subseteq \Psi$ and $cl(\bigcup K_i) = \bigcup L_i$ for every $i \in I$ where I is an index set. Then $(\mathscr{X}, \tau_{p_{cel}})$ is called PF_{cel} space. Every member of PF_{cel} space is called PF_{cel} open set and its complement is PF_{cel} closed set.

Example 4.2. Let $\mathscr{X} = \{m, n\}$ $K_j \in PF(\mathscr{X})$ where $j = 1, 2, 3, ...8$,
$K_1(m) = (0.2, 0.8)$,$K_1(n) = (0.3, 0.7)$
$K_2(m) = (0.3, 0.7)$, $K_2(n) = (0.4, 0.7)$
$K_3(m) = (0.4, 0.7)$, $K_3(n) = (0.4, 0.5)$
$K_4(m) = (0.4, 0.8)$, $K_4(n) = (0.3, 0.5)$
$K_5(m) = (0.8, 0.2)$, $K_5(n) = (0.7, 0.3)$
$K_6(m) = (0.7, 0.3)$, $K_6(n) = (0.7, 0.4)$
$K_7(m) = (0.7, 0.4)$, $K_7(n) = (0.5, 0.4)$
$K_8(m) = (0.8, 0.4)$, $K_8(n) = (0.5, 0.3)$
$\tau = \{0_{\mathscr{X}}, 1_{\mathscr{X}}, K_1, K_2, K_3, ..., K_8\}$ is a Pythagorean fuzzy topology on $\mathscr{X}$. Hence $(\mathscr{X}, \tau)$ is

a Pythagorean fuzzy topological space. $\Upsilon = \{K_1, K_2\} \subseteq \Psi = \{K_1, K_2, K_3, K_4\}$, Here for every family Υ there exists a countable family Ψ satisfies the Pythagorean fuzzy cellular condition. $\tau_{pcel} = \{0_{\mathscr{X}}, 1_{\mathscr{X}}, K_1, K_2, K_3, ..., K_8\}$ is a Pythagorean fuzzy cellular topology on $\mathscr{X}$. Hence $(\mathscr{X}, \tau_{pcel})$ is the Pythagorean fuzzy cellular space.

Definition 4.7. [5] Let $(\mathscr{X}, \tau_{p_{cel}})$ be a PF_{cel} space and $\mathscr{Y} \subset \mathscr{X}$ and $\mathcal{X}_{\mathscr{Y}}$ be the characteristic function of $\mathscr{Y}$. Then $\tau_{p_{cel}}(\mathscr{Y}) = \{K \bigcap \mathcal{X}_{\mathscr{Y}}, K \in \tau_{p_{cel}}\}$ is a PF_{cel} subspace. $(\mathscr{Y}, \tau_{p_{cel}}(\mathscr{Y}))$ is called the PF_{cel} subspace of $(\mathscr{X}, \tau_{p_{cel}})$. If $\mathcal{X}_{\mathscr{Y}}$ is PF_{cel} open set then $(\mathscr{Y}, \tau_{p_{cel}}(\mathscr{Y}))$ is called the PF_{cel} open subspace of $(\mathscr{X}, \tau_{p_{cel}})$.

Definition 4.8. [5] Let $(\mathscr{X}, \tau_{p_{cel}})$ be a PF_{cel} space. the Pythagorean fuzzy cellular closure (PF_{cel} cl) and Pythagorean fuzzy cellular interior (PF_{cel} int) of a PFS is defined by,
$PF_{cel}\ cl(K) = \bigcap\{N : K \leq N; N \text{ is } PF_{cel} \text{ closed in } (\mathscr{X}, \tau_{p_{cel}})\}$
$PF_{cel}\ int(K) = \bigcup\{M : M \leq K; M \text{ is } PF_{cel} \text{ open in } (\mathscr{X}, \tau_{p_{cel}})\}$

Definition 4.9. [5] Let $(\mathscr{X}, \tau_{pcel})$ and $(\mathscr{Y}, \sigma_{pcel})$ be any two PF_{cel} spaces. A function $\phi_{pcel} : (\mathscr{X}, \tau_{pcel}) \to (\mathscr{Y}, \sigma_{pcel}) PF_{cel}$ is said to be PF_{cel} continuous if for every PF_{cel} open set K in $(\mathscr{Y}, \sigma_{pcel})$, $\phi^{-1}_{pcel}(K) \in (\mathscr{X}, \tau_{pcel})$ is PF_{cel} open set in $(\mathscr{X}, \tau_{pcel})$.

5. Normality of Pythagorean fuzzy cellular spaces

This section provides the detailed study of normality of Pythagorean fuzzy cellular spaces. $\mathfrak{q}$ normal, PF_{cel} completely ultra normal, PF_{cel} Ultra normal and PF_{cel} Quasi normal are defined in Pythagorean fuzzy cellular spaces and the properties are discussed.

Definition 5.1. If $K = (\mu_K(x), \nu_K(x))$ and $L = (\mu_L(x), \nu_L(x))$ be two PF_{cel} in $(\mathscr{X}, \tau_{pcel})$. Let K and L is called quasi coincident (for short KqL) if only if there exists an element $x \in \mathscr{X}$ such that $\mu_K(x) > \nu_K(x)$ or $\nu_K(x) < \mu_L(x)$.

Proposition 5.1. Let K and L be two PF_{cel} in $(\mathscr{X}, \tau_{pcel})$ then

(i) $K\ \tilde{q}\ L$ if and only if $K \subseteq L^c$
(ii) $K\ q\ L$ if and only if $K \nsubseteq L^c$

Proof. The proof is straightforward. □

Definition 5.2. A PF_{cel} space $(\mathscr{X}, \tau_{pcel})$ is said to be PF_{cel} $\mathfrak{q}$-normal if for any PF_{cel} closed sets K_1 and K_2 with $K_1\ \tilde{q}\ K_2$, there exists PF_{cel} open sets L_1 and L_2 such that $K_1 \subseteq L_1$, $K_2 \subseteq L_2$ and $L_1\ \tilde{q}\ L_2$.

Definition 5.3. Let $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} space. Then (X, τ_{pcel}) is said to be PF_{cel} ultra normal if for any PF_{cel} closed set R and PF_{cel} open set N in (X, τ_{pcel}) with $R \subseteq N$, there exists a PF_{cel} clopen set S such that $R \subseteq S \subseteq N$.

Definition 5.4. Let $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} space. Then $(\mathscr{X}, \tau_{pcel})$ is said to be PF_{cel} completely ultra normal if for any two PF_{cel} R and N with $PF_{cel}\ cl(R) \subseteq R$, $R \subseteq PF_{cel}\ int(R)$, then there exists a PF_{cel} clopen set S such that $R \subseteq S \subseteq N$.

Proposition 5.2. Let $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} space. Then the following statements are equivalent:

(i) $(\mathscr{X}, \tau_{pcel})$ is a PF_{cel} $\mathfrak{q}$-normal space.
(ii) For each PF_{cel} closed set K and for each PF_{cel} open set R with $K \subseteq R$, there exists a PF_{cel} open set L where $K \subseteq L$ such that $PF_{cel}\ cl(L) \subseteq R$.
(iii) For each PF_{cel} closed sets K_1 and K_2 with $K_1\ \tilde{q}\ K_2$ there exists a PF_{cel} open set L with $K_1 \subseteq L$ such that $PF_{cel}\ cl(L)\ \tilde{q}\ K_2$.

(iv) For each PF_{cel} closed sets K_1 and K_2 with $K_1 \tilde{q} K_2$, there exists PF_{cel} open sets L and M where $K_1 \subseteq L$ and $K_2 \subseteq M$ such that $PF_{cel}\ cl(L) \tilde{q} PF_{cel}\ cl(M)$.

Proof. $(i) \Rightarrow (ii)$ Let $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} $\mathfrak{q}$-normal space. Let K be a PF_{cel} closed set, R be a PF_{cel} open set with $K \subseteq R$, $K \tilde{q} R^c$. Also K and R^c are PF_{cel} closed sets. Since $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} $\mathfrak{q}$-normal space, there exists PF_{cel} open sets L and M with $K \subseteq L$ and $R^c \subseteq M$ such that $L \tilde{q} M$ This implies that $L \subseteq M^c$. Hence $PF_{cel}\ cl(L) \subseteq PF_{cel}\ cl(M^c) = M^c$, since $R^c \subseteq M, M^c \subseteq R$. Hence $PF_{cel}\ cl(L) \subseteq M^c \subseteq R$. Therefore $PF_{cel}\ cl(L) \subseteq R$.
$(ii) \Rightarrow (iii)$ Let K_1 and K_2 be a PF_{cel} closed sets with $K_1 \subseteq K_2$ where K_2^c is PF_{cel} open set . Therefore by ii), there exists a PF_{cel} open set L with $K_1 \subseteq L$ such that $PF_{cel}\ cl(L) \subseteq K^c$. This implies $PF_{cel}\ cl(L) \tilde{q} K_2$.
$(iii) \Rightarrow (iv)$ Let K_1 and K_2 be a PF_{cel} closed sets with $K_1 \tilde{q} K_2$. By (iii) there exists a PF_{cel} open set L with $K_1 \subseteq L$ such that $PF_{cel}\ cl(L) \tilde{q} K_2$. Again by (iii), there exists a PF_{cel} open set M with $K_2 \subseteq M$ such that $PF_{cel}\ cl(M) \tilde{q} PF_{cel}\ cl(L)$.
$(iv) \Rightarrow (i)$ Suppose that K_1 and K_2 are PF_{cel} closed sets with $K_1 \tilde{q} K_2$. By (iv) there exists PF_{cel} open sets L and M with $K_1 \subseteq L$ and $K_2 \subseteq M$ such that $PF_{cel}\ cl(L) \tilde{q} PF_{cel}\ cl(M)$ which implies $L \tilde{q} M$. Therefore, $(\mathscr{X}, \tau_{pcel})$ is a PF_{cel} $\mathfrak{q}$-normal space. □

Definition 5.5. Let $(\mathscr{X}, \tau_{pcel})$ and $(\mathscr{Y}, \sigma_{pcel})$ be any two PF_{cel} spaces. A function ϕ_{pcel} : $(\mathscr{X}, \tau_{pcel}) \to (\mathscr{Y}, \sigma_{pcel}) PF_{cel}$ is said to be PF_{cel} clopen function if $\phi_{pcel}(R) \in (\mathscr{Y}, \sigma_{pcel})$ is PF_{cel} clopen set in $(\mathscr{Y}, \sigma_{pcel})$ for each PF_{cel} clopen set $R \in (\mathscr{X}, \tau_{pcel})$.

Definition 5.6. Let $(\mathscr{X}, \tau_{pcel})$, $(\mathscr{Y}, \sigma_{pcel})$ be any two PF_{cel} spaces. A function ϕ_{pcel} : $(\mathscr{X}, \tau_{pcel}) \to (\mathscr{Y}, \sigma_{pcel})$ is said to be PF_{cel} injective(one-to-one) function, if for every Pythagorean fuzzy cellular K in $(\mathscr{X}, \tau_{pcel})$ there exists a Pythagorean fuzzy cellular L in $(\mathscr{Y}, \sigma_{pcel})$ such that $\phi_{pcel}(K) = L$ then $K = L$.

Definition 5.7. Let $(\mathscr{X}, \tau_{pcel})$, $(\mathscr{Y}, \sigma_{pcel})$ be any two PF_{cel} spaces. A function ϕ_{pcel} : $(\mathscr{X}, \tau_{pcel}) \to (\mathscr{Y}, \sigma_{pcel})$ is said to be PF_{cel} surjective(onto) function, if for every L in $(\mathscr{Y}, \sigma_{pcel})$ there exists atleast one K in $(\mathscr{X}, \tau_{pcel})$such that $\phi_{pcel}(L) = M$.

Definition 5.8. Let $(\mathscr{X}, \tau_{pcel})$, $(\mathscr{Y}, \sigma_{pcel})$ be any two PF_{cel} spaces. A function ϕ_{pcel} : $(\mathscr{X}, \tau_{pcel}) \to (\mathscr{Y}, \sigma_{pcel})$ is said to be PF_{cel} injective(one-to-one) function, if for every Pythagorean fuzzy cellular K in $(\mathscr{X}, \tau_{pcel})$ there exists a Pythagorean fuzzy cellular L in $(\mathscr{Y}, \sigma_{pcel})$ such that $\phi_{pcel}(K) = L$ then $K = L$.

Definition 5.9. Let $(\mathscr{X}, \tau_{pcel})$, $(\mathscr{Y}, \sigma_{pcel})$ be any two PF_{cel} spaces. A function ϕ_{pcel} : $(\mathscr{X}, \tau_{pcel}) \to (\mathscr{Y}, \sigma_{pcel})$ is said to be PF_{cel} surjective(onto) function, if for every L in $(\mathscr{Y}, \sigma_{pcel})$ there exists atleast one K in $(\mathscr{X}, \tau_{pcel})$such that $\phi_{pcel}(L) = M$.

Definition 5.10. Let $(\mathscr{X}, \tau_{pcel})$, $(\mathscr{Y}, \sigma_{pcel})$ be any two PF_{cel} spaces. A function ϕ_{pcel} : $(\mathscr{X}, \tau_{pcel}) \to (\mathscr{Y}, \sigma_{pcel})$ is said to be PF_{cel} bijective function, if it is both PF_{cel} injective function and PF_{cel} surjective function.

Proposition 5.3. Let $(\mathscr{X}, \tau_{pcel})$, $(\mathscr{Y}, \sigma_{pcel})$ be any two PF_{cel} spaces, $\phi_{pcel} : (\mathscr{X}, \tau_{pcel}) \to (\mathscr{Y}, \sigma_{pcel}) PF_{cel}$ continuous, PF_{cel} surjective and PF_{cel} clopen function. If $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} ultra normal $(\mathscr{Y}, \sigma_{pcel})$ is also PF_{cel} ultra normal.

Proof. Let $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} ultra normal. Let R be a PF_{cel} closed set and N be a PF_{cel} open set and $R \subseteq N$. Since ϕ_{pcel} is PF_{cel} continuous. $\phi_{pcel}^{-1}(R)$ and $\phi_{pcel}^{-1}(N)$ are PF_{cel} closed set and PF_{cel} open set respectively. Thus $\phi_{pcel}^{-1}(R) \subseteq \phi_{pcel}^{-1}(N)$. Since $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} ultra normal, there exists a $S \in PF_{cel}$ clopen set in $(\mathscr{X}, \tau_{pcel})$ such that $\phi_{pcel}^{-1}(R) \subseteq S \subseteq \phi_{pcel}^{-1}(N)$. Since ϕ_{pcel} is PF_{cel} surjective,

$$R = \phi_{pcel}[\phi_{pcel}^{-1}](R) \subseteq \phi_{pcel}(S) \subseteq \phi_{pcel}[\phi_{pcel}^{-1}(N)] = N.$$

Since ϕ_{pcel} is a PF_{cel} clopen function, $\phi_{pcel}(S)$ is a PF_{cel} clopen set in $(\mathscr{Y}, \sigma_{pcel})$. Hence $(\mathscr{Y}, \sigma_{pcel})$ is a PF_{cel} ultra normal space. □

Proposition 5.4. Let $(\mathscr{X}, \tau_{pcel})$, $(\mathscr{Y}, \sigma_{pcel})$ be any two PF_{cel} spaces, $\phi_{pcel} : (\mathscr{X}, \tau_{pcel}) \to (\mathscr{Y}, \sigma_{pcel})$ PF_{cel} continuous, PF_{cel} surjective and PF_{cel} clopen function. If $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} completely ultra normal $(\mathscr{Y}, \sigma_{pcel})$ is also PF_{cel} completely ultra normal.

Proof. Let $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} completely ultra normal. Let R and $N \in (\mathscr{Y}, \sigma_{pcel})$ be such that $PF_{cel}\ cl(R) \subseteq N$ and $R \subseteq PF_{cel}\ int(N)$ and $\phi_{pcel}^{-1} PF_{cel}\ cl(R) \subseteq \phi_{pcel}^{-1}(N)$, $\phi_{pcel}^{-1}(R) \subseteq \phi_{pcel}^{-1} PF_{cel}\ int(N)$. Since ϕ_{pcel}^{-1} is PF_{cel} continuous, $\phi_{pcel}^{-1} PF_{cel}\ cl(R)$ is PF_{cel} open set in $(\mathscr{X}, \tau_{pcel})$. Now $\phi_{pcel}^{-1} cl(R)$ is PF_{cel} closed set such that $\phi_{pcel}^{-1}(R) \subseteq \phi_{pcel}^{-1} PF_{cel}\ cl(R)$. Therefore by (iii) of Proposition 5.3 $PF_{cel}\ cl[\phi_{pcel}^{-1}(R)] \subseteq \phi_{pcel}^{-1} PF_{cel}\ cl(R)$ $\subseteq \phi_{pcel}^{-1}(N)$. Similarly, $\phi_{pcel}^{-1}(R) \subseteq \phi_{pcel}^{-1}(PF_{cel} int(N))$.
Therefore, $\phi_{pcel}^{-1}(R) \subseteq \phi_{pcel}^{-1}(PF_{cel}\ int(N)_{\subseteq} PF_{cel}\ int[\phi_{pcel}^{-1}(N)]$. Since$(\mathscr{X}, \tau_{pcel})$ is PF_{cel} completely ultra normal, there exists a PF_{cel} clopen set S such that $\phi_{pcel}^{-1}(\delta) \subseteq S \subseteq \phi_{pcel}^{-1}(N)$. Since ϕ_{pcel} is PF_{cel} surjective,

$$R = \phi_{pcel}(\phi_{pcel}^{-1}(R)) \subseteq \phi_{pcel}(S) \subseteq \phi_{pcel}(\phi_{pcel}^{-1})(N) = N$$
$$\Rightarrow R \subseteq \phi_{pcel}(S) \subseteq N.$$

Thus $PF_{cel}\ cl(R) = \phi_{pcel}(\phi_{pcel}^{-1}(PF_{cel}\ cl(R)))$ since ϕ_{pcel} is PF_{cel} surjective. $PF_{cel}\ cl(R)$ $\subseteq \phi_{pcel}(\phi_{pcel}^{-1}(N)) = N$ and $R = \phi_{pcel}[\phi_{pcel}^{-1}(R)]$, since ϕ_{pcel} is PF_{cel} onto $R \subseteq \phi_{pcel}(\phi_{pcel}^{-1}(PF_{cel}\ int(N))) = PF_{cel}\ int(N)$. Therefore $PF_{cel}\ cl(R) \subseteq N$ and $R \subseteq PF_{cel}$ $int(N)$. Also, since ϕ_{pcel} is a PF_{cel} clopen function, $\phi_{pcel}(S)$ is a ϕ_{pcel} clopen set in $(\mathscr{Y}, \sigma_{pcel})$. Thus $(\mathscr{Y}, \sigma_{pcel})$ is also PF_{cel} completely ultra normal. □

Definition 5.11. Let $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} space. Then $(\mathscr{X}, \tau_{pcel})$ is said to be PF_{cel} quasi normal if for any PF_{cel} closed set R and PF_{cel} open set N with $R \subseteq N$, there exists a PF_{cel} open set S such that $R \subseteq PF_{cel}\ int(S) \subseteq PF_{cel}\ cl(S) \subseteq N$.

Definition 5.12. Let $(\mathscr{X}, \tau_{pcel})$ and $(\mathscr{Y}, \sigma_{pcel})$ be any two PF_{cel} spaces. Any function $\phi_{pcel} : (\mathscr{X}, \tau_{pcel}) \to (\mathscr{Y}, \sigma_{pcel})$ is said to be a PF_{cel} slight open function if $PF_{cel}\ \phi_{pcel}(T)$ is PF_{cel} clopen set in $(\mathscr{Y}, \sigma_{pcel})$ for every PF_{cel} open set T in $(\mathscr{X}, \tau_{pcel})$.

Proposition 5.5. Let $(\mathscr{X}, \tau_{pcel})$ and $(\mathscr{Y}, \sigma_{pcel})$ be any two PF_{cel} spaces. Let ϕ_{pcel} : $(\mathscr{X}, \tau_{pcel}) \to (\mathscr{Y}, \sigma_{pcel})$ be a PF_{cel} continuous and PF_{cel} slight open function. If $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} quasi normal if and only if $(\mathscr{Y}, \sigma_{pcel})$ is PF_{cel} ultra normal.

Proof. Let R be a PF_{cel} closed set and N be a PF_{cel} open set in $(\mathscr{Y}, \sigma_{pcel})$ be such that $R \subseteq N$. Since ϕ_{pcel} is PF_{cel} continuous $\phi_{pcel}^{-1}(R)$ and $\phi_{pcel}^{-1}(N)$ are PF_{cel} closed set and PF_{cel} open set in $(\mathscr{X}, \tau_{pcel})$ respectively. Thus $\phi_{pcel}^{-1}(R) \subseteq \phi_{pcel}^{-1}(N)$. Since $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} quasi normal, there exists a PF_{cel} open set S in $(\mathscr{X}, \tau_{pcel})$ such that $\phi_{pcel}^{-1}(R) \subseteq S \subseteq PF_{cel} cl(S) \subseteq \phi_{pcel}^{-1}(N)$. Since ϕ_{pcel} is PF_{cel} surjective.

$$R = \phi_{pcel}(\phi_{pcel}^{-1}(R)) \subseteq \phi_{pcel}(S) \subseteq \phi_{pcel}(PF_{cel} cl(S)) \subseteq \phi_{pcel}(\phi_{pcel}^{-1}(N)) = N$$

Clearly, $R \subseteq \phi_{pcel}(S) \subseteq S$. Since S is a PF_{cel} open set in $(\mathscr{X}, \tau_{pcel})$ and ϕ_{pcel} is a PF_{cel} slight open function, $\phi_{pcel}(S)$ is a PF_{cel} clopen set in $(\mathscr{Y}, \sigma_{pcel})$. Hence $(\mathscr{Y}, \sigma_{pcel})$ is a PF_{cel} ultra normal space.

Conversely, Since $(\mathscr{Y}, \sigma_{pcel})$ is a PF_{cel} ultra normal there exists a PF_{cel} clopen set M in $(\mathscr{Y}, \sigma_{pcel})$ such that $\phi^{-1}(R) \subseteq \phi^{-1}(M) \subseteq \phi^{-1}(N).\phi^{-1}(M)$ is clopen set in $(\mathscr{X}, \tau_{pcel})$. Since ϕ is onto. $\phi(\phi^{-1}(R)) \subseteq \phi(\phi^{-1}(M)) \subseteq \phi(\phi^{-1}(N))$ implies $R \subseteq M \subseteq N \in (X, \tau_{pcel})$. Clearly, $M = PF_{cel}\ int(M), M = PF_{cel}\ cl(M)$. This implies that $R \subseteq PF_{cel}(int)(R) \subseteq PF_{cel}(cl)(M) \subseteq N$ in $(\mathscr{X}, \tau_{pcel})$. Hence $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} quasi normal. □

Proposition 5.6. Let $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} space. Then the following statements are equivalent:

(i) $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} $\mathfrak{q}$-normal.
(ii) For each PF_{cel} closed set K and for each PF_{cel} open set N with $K \subseteq N$, there exists a PF_{cel} L such that $K \subseteq PF_{cel}\ int(L) \subseteq PF_{cel}\ cl(L) \subseteq N$.

Proof. *i)* $\Rightarrow$ *ii)* Let K be a PF_{cel} closed set, N be a PF_{cel} open set such that $K \subseteq N$. Thus K and N^c are PF_{cel} closed sets in $(\mathscr{X}, \tau_{pcel})$. Since $(\mathscr{X}, \tau_{pcel})$ is a PF_{cel} $\mathfrak{q}$ normal, there exists L and R are PF_{cel} open set such that $K \subseteq L, K^c \subseteq R$ and $L\ \tilde{q}\ R$. Since $L\ \tilde{q}\ R, L \subseteq R^c$. Thus $PF_{cel}\ cl(L) \subseteq PF_{cel}\ cl(R^c) = R^c$. Since $N^c \subseteq R, R^c \subseteq N$. Therefore $PF_{cel}\ cl(L) \subseteq N$. Since $L \subseteq K$ and L is a PF_{cel} open set, $K \subseteq PF_{cel}\ int(L)$. Hence $K \subseteq PF_{cel}\ int(L) \subseteq PF_{cel}\ cl(L) \subseteq N$.
ii) $\Rightarrow$ *i)*. Suppose K_1 and K_2 are PF_{cel} closed sets with $K_1\ \tilde{q}\ K_2$. Thus $K_1 \subseteq K_2$. By hypothesis, there exists PF_{cel} L such that $K_1 \subseteq PF_{cel}\ int(L) \subseteq PF_{cel}\ cl(L) \subseteq K_2$. Since, $PF_{cel}\ cl(L) \subseteq K_2^c$, $K_2 \subseteq (cl(L))^c$. Also, since $PF_{cel}\ int(L), (PF_{cel}\ cl(L))^c$ are PF_{cel} open set, $K_1 \subseteq PF_{cel}\ int(L)$ and $K_2 \subseteq (PF_{cel}\ cl(L))^c$. Further $K_1\ \tilde{q}\ K_2$ implies $PF_{cel}\ int(L)\ \tilde{q}\ (PF_{cel}\ cl(L))^c$. Hence $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} $\mathfrak{q}$-normal space. □

Proposition 5.7. Let $(\mathscr{X}, \tau_{pcel})$ and $(\mathscr{Y}, \sigma_{pcel})$ be PF_{cel}. Let $\phi_{pcel} : (\mathscr{X}, \tau_{pcel}) \to (\mathscr{Y}, \sigma_{pcel})$ PF_{cel} continuous, PF_{cel} injective, PF_{cel} closed, PF_{cel} open function. If $(\mathscr{Y}, \sigma_{pcel})$ is PF_{cel} ultra normal, then $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} $\mathfrak{q}$-normal.

Proof. Let R be a PF_{cel} closed set and N be a PF_{cel} open set be such that $R \subseteq N$. Since ϕ_{pcel} is PF_{cel} closed and PF_{cel} open function, then $\phi_{pcel}(R)$ and $\phi_{pcel}(N)$ are PF_{cel} closed set and PF_{cel} open set respectively. Thus $\phi_{pcel}(R) \subseteq \phi_{pcel}(N)$. Since $(\mathscr{Y}, \sigma_{pcel})$ is PF_{cel} ultra normal, there exists a PF_{cel} clopen set S such that $\phi_{pcel}(R) \subseteq S \subseteq \phi_{pcel}(N)$.
From the PF_{cel} injectivity, $\phi_{pcel}, \phi_{pcel}^{-1}(\phi_{pcel}(R)) \subseteq \phi_{pcel}^{-1}(S) \subseteq \phi_{pcel}^{-1}(\phi_{pcel}(N))$ implies that $R \subseteq \phi_{pcel}^{-1}(S) \subseteq N$. Since ϕ_{pcel} is PF_{cel} continuous, $\phi_{pcel}^{-1}(S)$ is a PF_{cel} clopen set in $(\mathscr{X}, \tau_{pcel})$. Thus $R \subseteq PF_{cel}\ int(\phi_{pcel}^{-1}(S)) \subseteq PF_{cel}\ cl(\phi_{pcel}^{-1}(S)) \subseteq N$. Therefore by proposition 5.6, $(\mathscr{X}, \tau_{pcel})$ is a PF_{cel} $\mathfrak{q}$-normal. □

Remark 5.1. The converse of the Proposition 5.7 is proved through the Example 5.1

Example 5.1. Let $\mathscr{X} = [0, 1]$ be a non-empty set and $R_n = [\frac{n}{20n+2}, 1 - \frac{n}{20n+2}]$ be a Pythagorean fuzzy cellular defined on $\mathscr{X}$, where $n = 1, 2, 3, ...$ Let $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} ultra-normal space and $(\mathscr{Y}, \sigma_{pcel})$ be PF_{cel} $\mathfrak{q}$-normal space. Let $\phi_{pcel} : (\mathscr{X}, \tau_{pcel}) \to (\mathscr{Y}, \sigma_{pcel})$ be a Pythagorean fuzzy cellular continuous mapping defined as $\phi_{pcel}(a) = b$, $\phi_{pcel}\ (b) = a$. Here ϕ_{pcel} is a PF_{cel} continuous function. Clearly, every PF_{cel} $\mathfrak{q}$-normal space is PF_{cel} ultra normal.

Proposition 5.8. Let $(\mathscr{X}, \tau_{pcel})$, $(\mathscr{Y}, \sigma_{pcel})$ be two PF_{cel} spaces. Let $\phi_{pcel} : (\mathscr{X}, \tau_{pcel}) \to (\mathscr{Y}, \sigma_{pcel})$ PF_{cel} continuous, PF_{cel} surjective , PF_{cel} closed and PF_{cel} open function. If $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} quasi normal, then $(\mathscr{Y}, \sigma_{pcel})$ is PF_{cel} $\mathfrak{q}$-normal.

Proof. Let R be a PF_{cel} closed set and N be a PF_{cel} open set be such that $R \subseteq N$. Since ϕ_{pcel} is PF_{cel} continuous, $\phi^{-1}_{cel}(R)$ and $\phi^{-1}_{cel}(N)$ are PF_{cel} closed set and PF_{cel} open set in $(\mathscr{X}, \tau_{pcel})$ respectively. Thus $\phi^{-1}_{cel}(R) \subseteq \phi^{-1}_{cel}(N)$. Since $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} quasi normal. By definition of PF_{cel} quasi normal, there exists a PF_{cel} open set S such that $\phi^{-1}_{pcel}(R) \subseteq S \subseteq PF_{cel}cl(S) \subseteq \phi^{-1}_{pcel}(N)$. From the PF_{cel} surjectivity of ϕ_{pcel},

$$R = \phi_{pcel}[\phi^{-1}_{pcel}(R)] \subseteq \phi_{pcel}(S) \subseteq \phi^{-1}_{pcel}(PF_{cel}\ cl(N)) \subseteq \phi_{pcel}[\phi^{-1}_{pcel}(N)] = N$$

Since ϕ_{pcel} is PF_{cel} continuous and $\phi_{pcel}(S)$ is PF_{cel} open function in (Y, σ_{pcel}),

$$R \subseteq PF_{cel}\ int(\phi_{pcel}(S)) \subseteq PF_{cel}cl(\phi_{pcel}(S)) \subseteq N$$

Therefore by Proposition 5.6 $(\mathscr{Y}, \sigma_{pcel})$ is a PF_{cel} $\mathfrak{q}$-normal space. □

Remark 5.2. The converse of the Proposition 5.8 is proved through the Example 5.2

Example 5.2. Let $\mathscr{X} = [0,1]$ be a non-empty set and $R_n = [\frac{n}{2n+2}, 1 - \frac{n}{2n+2}]$ be a Pythagorean fuzzy cellular defined on $\mathscr{X}$, where $n = 1, 2, 3, \ldots$. Let $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} quasi normal space and $(\mathscr{Y}, \sigma_{pcel})$ be PF_{cel} $\mathfrak{q}$-normal space. Let $\phi_{pcel} : (\mathscr{X}, \tau_{pcel}) \longrightarrow (\mathscr{Y}, \sigma_{pcel})$ be a Pythagorean fuzzy cellular continuous mapping defined as $\phi_{pcel}\ (a) = b$, $\phi_{pcel}\ (b) = a$. Here ϕ_{pcel} is a PF_{cel} continuous function. Clearly, every PF_{cel} $\mathfrak{q}$-normal space is PF_{cel} quasi normal.

Proposition 5.9. Let $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} space. Then the following statements hold:

(i) If $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} completely ultra normal, then $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} ultra normal.
(ii) If $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} quasi normal, then $(\mathscr{X}, \tau_{pcel})$ be PF_{cel} $\mathfrak{q}$-normal.
(iii) If $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} ultra normal, then $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} $\mathfrak{q}$- normal.

Proof. (i) Let R be a PF_{cel} closed set and N be a PF_{cel} open set in $(\mathscr{X}, \tau_{pcel})$ be such that $R \subseteq N$. Thus $PF_{cel}\ cl(N) \subseteq N$ and $R \subseteq PF_{cel}\ int(N)$. Since $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} completely ultra normal, there exists a PF_{cel} clopen set S such that $R \subseteq S \subseteq N$. Hence $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} ultra normal.

(ii) Let R be a PF_{cel} closed set and N be a PF_{cel} open set be such that $R \subseteq N$. Since $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} quasi normal, there exists a PF_{cel} open set S such that $PF_{cel}\ int(S) \subseteq PF_{cel}\ cl(S) \subseteq N$. This implies that there exists PF_{cel} S such that $R \subseteq PF_{cel}\ int(S) \subseteq PF_{cel}\ cl(S) \subseteq N$. Hence by Proposition 5.6, $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} $\mathfrak{q}$-normal.

(iii) Let R be a PF_{cel} closed set and N be a PF_{cel} open set such that $R \subseteq N$. Since $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} ultra normal, there exists a PF_{cel} clopen set S such that $R \subseteq S \subseteq N$. Since S is a PF_{cel} clopen set $S = PF_{cel}\ int(S)$, $S = PF_{cel}\ cl(S)$. Thus there exists PF_{cel} S such that $R \subseteq PF_{cel}\ int(S) = PF_{cel}\ cl(S) \subseteq N$. Hence by Proposition 5.6, $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} $\mathfrak{q}$- normal. □

Proposition 5.10. Let $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} space. If $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} completely ultra normal then $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} ultra normal.

Proof. Let K be a PF_{cel} closed set and M be a PF_{cel} open set in $(\mathscr{X}, \tau_{pcel})$ such that $K \subseteq M$. Thus $PF_{cel}\ cl(K) \subseteq M$ and $N \subseteq PF_{cel}\ int(M)$. Since $(\mathscr{X}, \tau_{pcel})$ is Pf_{cel} completely ultranormal, there exists a PF_{cel} clopen set such that $K \subseteq P \subseteq N$. Hence $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} ultra normal. □

Remark 5.3. The converse of the Proposition 5.10 is proved through the Example 5.3

Example 5.3. Let $\mathscr{X} = [0,1]$ be a non-empty set and $R_n = [\frac{n}{3n+2}, 1 - \frac{n}{3n+2}]$ be a Pythagorean fuzzy cellular defined on $\mathscr{X}$, where $n = 1, 2, 3, ...$ Let $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} completely ultra normal space and $(\mathscr{Y}, \sigma_{pcel})$ be PF_{cel} ultra normal space. Let ϕ_{pcel} : $(\mathscr{X}, \tau_{pcel}) \longrightarrow (\mathscr{Y}, \sigma_{pcel})$ be a Pythagorean fuzzy cellular continuous mapping defined as ϕ_{pcel} $(a) = b$, ϕ_{pcel} $(b) = a$. Here ϕ_{pcel} is a PF_{cel} continuous function. Clearly, every PF_{cel} $\mathfrak{q}$ ultra normal is PF_{cel} completely ultra normal.

6. Regularity of Pythagorean fuzzy cellular spaces

Definition 6.1. Let $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} space. Then $(\mathscr{X}, \tau_{pcel})$ is said to be PF_{cel} regular space if for any PF_{cel} closed set R in $(\mathscr{X}, \tau_{pcel})$ and a PF_{cel} L with $R\ \tilde{q}\ L$, there exists a PF_{cel} open set N and V such that $R \subseteq N, L \subseteq V$ and $N\ \tilde{q}\ V$.

Proposition 6.1. Let $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} space. Then the following statements are equivalent:

(i) $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} regular.
(ii) For each PF_{cel} L and PF_{cel} open set PF_{cel} R with $L\ q\ R$, there exists a PF_{cel} open set V with $L \subseteq V$ such that PF_{cel} $cl(V) \subseteq R$.

Proof. $(i) \Rightarrow (ii)$. Let L be a PF_{cel} and R be any PF_{cel} open set with $L\ q\ R$. By hypothesis, there exists PF_{cel} open set N and PF_{cel} V such that $L \subseteq V, R^c \subseteq N$ and $N\ \tilde{q}\ V$. Since $N\ \tilde{q}\ V$, $V \subseteq N^c$. Thus $PF_{cel}(V) \subseteq PF_{cel}$ $(N^c) = N^c$. Also, $R^c \subseteq N$ implies that $N^c \subseteq R$. Therefore PF_{cel} $cl(V) \subseteq R$.

$(ii) \Rightarrow (i)$ Let M be a PF_{cel} closed set and $L\ \tilde{q}\ M$ for PF_{cel} L. Thus M^c is PF_{cel} open set. By hypothesis, for each PF_{cel} L and PF_{cel} (M^c) with $L\ q\ M^c$, there exists a PF_{cel} open set V with $L \subseteq V$ such that PF_{cel} $cl(V) \subseteq M^c$. Then $M \subseteq (PF_{cel}\ cl(V))^c$. Also $(PF_{cel}\ cl(V))^c$ is PF_{cel} open. Thus $L \subseteq V$ and $M \subseteq (PF_{cel}\ cl(V))^c$. Also, $M \cup (PF_{cel}\ cl(M))^c \subseteq PF_{cel}\ cl(M) \cup [PF_{cel}\ cl(M)]^c \subseteq 1_X$. Therefore $M\ \tilde{q}\ [PF_{cel}\ cl(M)]^c$. Hence (X, τ_{pcel}) is PF_{cel} regular. □

Proposition 6.2. A PF_{cel} space is PF_{cel} regular if and only if for every PF_{cel} closed set M and PF_{cel} ,V with $V\ \tilde{q}\ M$ there exists PF_{cel} open sets W and L such that if $V \subseteq W$, $M \subseteq L$, then $W\ \tilde{q}\ PF_{cel}$ $cl(L)$.

Proof. Let $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} regular space. Let M be a PF_{cel} closed set and V be a PF_{cel} with $V\ \tilde{q}\ M$. Since $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} regular space, there exists PF_{cel} open sets L and M such that $V \subseteq M$, $M \subseteq L$ and $W \subseteq L$. Since $(\mathscr{X}, \tau_{pcel})$ is a PF_{cel} regular space, there exists PF_{cel} open set W such that $V \subseteq M$ and $M \subseteq L$ and $W\ \tilde{q}\ L$. Since $W\ \tilde{q}\ L$ and $L \subseteq W^c$ implies PF_{cel} $cl(L) \subseteq PF_{cel}$ $cl(W^c)$ implies PF_{cel} $cl(L) \subseteq W^c$ implies PF_{cel} $cl(L) \cup W \subseteq 1_{\mathscr{X}}$. Therefore, $W\ \tilde{q}\ PF_{cel}$ $cl(L)$.

Conversely, for every PF_{cel} closed set M and PF_{cel}, V with $V\ \tilde{q}\ M$, there exists PF_{cel} open sets W and L such that $V \subseteq W$ and $M \subseteq L$, then $W\ \tilde{q}\ PF_{cel}$ $cl(L)$. Since $W\tilde{q}PF_{cel}$ $cl(L)$ if follows that $W \cup PF_{cel}$ $cl(L) \subseteq 1_{\mathscr{X}}$. Thus $W \cup L \subseteq W \cup PF_{cel}$ $cl(L) \subseteq 1_{\mathscr{X}}$. Therefore $W\ \tilde{q}\ L$. Hence $(\mathscr{X}, \tau_{pcel}) PF_{cel}$ regular space. □

Proposition 6.3. Let $(\mathscr{X}, \tau_{pcel})$ be a PF_{cel} space. Then the following statements are equivalent:

(i) $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} regular.
(ii) For each $PF_{cel}V$ and PF_{cel} open set R with $V \subseteq R$, there exists a PF_{cel} M open such that $V \subseteq W \subseteq PF_{cel}$ $cl(W) \subseteq R$.

(iii) For each $PF_{cel}V$ and PF_{cel} open set R with $V \subseteq R$, there exists a PF_{cel} open set W and $J = PF_{cel}\ int(K)$ where $K^c \in PF_{cel}$ open such that $V \subseteq J \subseteq PF_{cel}(J) \subseteq R$.

(iv) For each $PF_{cel}V$ and PF_{cel} closed set S with $V\ \tilde{q}\ S$, there exists PF_{cel} open sets W and R such that $V \subseteq W, S \subseteq R$ with $PF_{cel}\ cl(R)\ \tilde{q}\ PF_{cel}\ cl(W)$.

Proof. $(i) \Rightarrow (ii)$ Let R be a PF_{cel} open set and V be a PF_{cel} be such that $V \subseteq R$. Thus $R^c \in PF_{cel}$ closed. By (i) $V\ \tilde{q}\ R^c$. Since $(\mathscr{X}, \tau_{pcel})$ is PF_{cel} regular there exists a PF_{cel} open sets W and J such that $V \subseteq W, R^c \subseteq J$ and $W\ \tilde{q}\ J$. Since $W\ \tilde{q}\ J, W \subseteq J^c$. Since J is a PF_{cel} open set, $PF_{cel}\ cl(W) \subseteq PF_{cel}\ cl(J) \subseteq PF_{cel}\ cl(J^c) = J^c$. But $J^c \subseteq R$. Therefore $V \subseteq W \subseteq PF_{cel}\ cl(W) \subseteq R$.

$(ii) \Rightarrow (iii)$ Let R be a PF_{cel} open set and V is PF_{cel} such that $V \subseteq R$. By (ii) there exists a PF_{cel} open set W such that $V \subseteq W \subseteq PF_{cel}\ cl(W) = W \subseteq R$. Let $J = PF_{cel}\ int(K)$ where $PF_{cel}\ cl(W) = W$. Since $W \in PF_{cel}$ open $V \subseteq W = PF_{cel}\ int(W)$. Also, $W \subseteq PF_{cel}\ cl(M), V \subseteq PF_{cel}\ int(W) \subseteq PF_{cel}\ int(PF_{cel}\ cl(W)) = J$. Thus $V \subseteq J$. Therefore, $V \subseteq J \subseteq PF_{cel}\ cl(J) = PF_{cel}(PF_{cel}\ int(W)) \subseteq PF_{cel}\ cl(K)$. Since $PF_{cel}\ int(K) \subseteq K = PF_{cel}\ cl(PF_{cel}\ cl(W)) = PF_{cel}\ cl(W) \subseteq R$. Hence $V \subseteq J \subseteq PF_{cel}\ cl(V) \subseteq R$.

$(iii) \Rightarrow (iv)$ Let S be a PF_{cel} closed set and V be PF_{cel} with $V\ \tilde{q}\ V$. Thus $S^c \in PF_{cel}$ open $V \subseteq S^c$. By iii) there exists PF_{cel} open set J such that $V \subseteq J \subseteq PF_{cel}\ cl(J) \subseteq S$ where $J = PF_{cel}\ int(K)$ for some $K \in PF_{cel}$, $K^c \in PF_{cel}$ open. Again by hypothesis, there exists PF_{cel}, W such that $V \subseteq W \subseteq PF_{cel}\ cl(W) \subseteq J$. Let $R = (PF_{cel}\ cl(J))^c$. Then $PF_{cel}\ cl(J) \subseteq S^c$ implies that $S \subseteq (PF_{cel}\ cl(J))^c = R$. Thus $V \subseteq W, S \subseteq R$. But $R = [PF_{cel}\ cl(J)]^c$. Since $J \subseteq PF_{cel}\ cl(J), R \subseteq J^c$. Therefore, $PF_{cel}\ cl(R) \subseteq PF_{cel}\ cl(J^c) = J^c$, since J^c is PF_{cel} closed. $PF_{cel}\ cl(R) \subseteq (PF_{cel}(W))^c$ since $PF_{cel}(W) \subseteq J$. Thus $PF_{cel}\ cl(R) \cup PF_{cel}\ cl(W) \subseteq 1_X$. Hence $PF_{cel}\ cl(R)\ \tilde{q}\ PF_{cel}\ cl(W)$.
$(iv) \Rightarrow (i)$. The proof is apparent. □

7. Conclusion

Normality and regularity are fundamental properties in topology, forming the foundation for a deeper understanding of topological spaces and their behavior. This study extends the concepts of normality and regularity to the Pythagorean fuzzy domain. Specifically, it provides a comprehensive framework for understanding Pythagorean fuzzy cellular open sets within Pythagorean fuzzy cellular spaces by defining normality and regularity in this context. The study introduces several new types of normality in Pythagorean fuzzy cellular spaces, including $\mathfrak{q}$-normal, ultra-normal, completely ultra-normal, and quasi-normal. Additionally, it examines the properties of regularity within these spaces. These findings have potential applications in Pythagorean fuzzy cellular spaces, which are effective tools for modelling and analysing decision-making processes involving imprecise information. To support these advancements, the study emphasizes the importance of exploring open sets within these spaces, focusing on the separation axioms of normality and regularity for detailed investigation.

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