

COEFFICIENT BOUNDS AND FEKETE-SZEGÖ FUNCTIONAL PROBLEM FOR A NEW SUBCLASS OF M-FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS

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ABSTRACT. This paper introduces a novel subclass of m -fold symmetric bi-univalent functions denoted as $\mathcal{S}_{\Sigma_m}^{h,p}$. Precise coefficient estimates for terms $|a_{m+1}|, |a_{2m+1}|$ and the Fekete-Szegö functional are derived for functions belonging to this subclass. The results presented in this paper would generalize and improve some recent works of several earlier authors.

Keywords: Coefficient estimates, m -fold symmetric bi-univalent functions, Fekete-Szegö functional problems.

AMS Subject Classification:30C45, 30C80

1. INTRODUCTION

Let \mathcal{A} represent the set of analytic functions f defined on the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ normalized by the conditions $f(0) = f'(0) - 1 = 0$. Such functions can be expressed in the following series form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

The class $\mathcal{S} \subset \mathcal{A}$ consists of functions that are univalent (one-to-one) in Δ . (see [2, 9]). Every function $f \in \mathcal{S}$ has an inverse f^{-1} , which satisfies:

$$f^{-1}(f(z)) = z \quad (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4} \right).$$

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The inverse function f^{-1} can be expressed as:

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots \tag{2}$$

A function $f(z)$ is called bi-univalent if both f and f^{-1} are univalent in Δ , this subclass denoted by $\sigma_{\mathcal{B}}$. For a comprehensive overview and notable examples of functions in $\sigma_{\mathcal{B}}$, see [13]. In fact that this widely-cited work by Srivastava et al. [13] actually revived the study of analytic and bi-univalent functions in recent years and that it has led to a flood of papers on the subject by (for example) Srivastava et al. [12, 14, 17] and others [6, 10, 16, 20].

Among the widely studied subclasses of $\sigma_{\mathcal{B}}$ the subclass $\mathcal{H}_{\Sigma}(\beta)$ ($0 \leq \beta < 1$) and the bi-starlike subclass $\mathcal{S}^*(\beta)$ for $\beta(0 \leq \beta < 1)$ are of significant importance.(see [3, 13]). By definition, we have

$$\mathcal{H}_{\Sigma}(\beta) = \{f \in \sigma_{\mathcal{B}} : Re(f'(z)) > \beta, Re(g'(w)) > \beta (z, w \in \Delta)\}$$

and

$$\mathcal{S}^*(\beta) = \left\{ f \in \sigma_{\mathcal{B}} : Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta, \Re \left\{ \frac{wg'(w)}{g(w)} \right\} > \beta (z, w \in \Delta) \right\},$$

where the function $g = f^{-1}$ given by (2).

For any function $f \in \mathcal{S}$ the function $h(z)$, defined as:

$$h(z) = \sqrt[m]{f(z^m)} \tag{3}$$

is univalent and maps the unit disk Δ into a region with m-fold symmetry. A function f is said to be m-fold symmetric (see [8, 9]) if it satisfies the normalization:

$$f(z) = z + \sum_{n=1}^{\infty} a_{mn+1}z^{mn+1} \quad (z \in \Delta, m \in \mathbb{N}). \tag{4}$$

The class \mathcal{S}_m consists of m-fold symmetric univalent functions defined by the series (4), the class \mathcal{S} corresponds to functions with single-fold symmetry ($m=1$).

In [15] Srivastava et al. defined m-fold symmetric bi-univalent functions analogues to the concept of m-fold symmetric univalent functions. They gave some important results, such as each function $f \in \sigma_{\mathcal{B}}$ generates an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$. Furthermore, for the normalized form of f given by (4), they obtained the series expansion for f^{-1} as follows:

$$f^{-1}(w) = w + \sum_{n=1}^{\infty} A_{mn+1}w^{mn+1} \tag{5}$$

$$= w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1} \right]w^{3m+1} + \dots \tag{6}$$

We denote by Σ_m the class of m-fold symmetric bi-univalent functions in Δ . For $m = 1$, formula (6) coincides with formula (2) of the class $\sigma_{\mathcal{B}}$. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m} \right)^{\frac{1}{m}}, \left[\frac{1}{2} \log \left(\frac{1+z^m}{1-z^m} \right) \right]^{\frac{1}{m}} \text{ and } [-\log(1-z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1+w^m} \right)^{\frac{1}{m}}, \left(\frac{e^{2w^m} - 1}{e^{2w^m} + 1} \right)^{\frac{1}{m}} \text{ and } \left(\frac{e^{w^m} - 1}{e^{w^m}} \right)^{\frac{1}{m}},$$

respectively.

Several researchers have explored coefficient bounds for various subclasses of m -fold symmetric bi-univalent functions (see [1, 4, 6, 7, 11, 15, 17, 18, 19]).

Recently, Jadhav et al. [5] introduced two new subclasses $\mathcal{S}_{\Sigma_m}(\alpha)$ and $\mathcal{S}_{\Sigma_m}^*(\beta)$ of m -fold symmetric bi-univalent function. They derived coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ for functions belonging to these subclasses.

Definition 1.1. [5] A function $f \in \Sigma_m$ given by (4) is said to be in the class $\mathcal{S}_{\Sigma_m}(\alpha)$ if it satisfies the following conditions:

$$\left| \arg \left(\frac{z^3 f''(z)}{(f(z))^2} + 1 \right) \right| < \frac{\alpha\pi}{2}$$

and

$$\left| \arg \left(\frac{w^3 g''(w)}{(g(w))^2} + 1 \right) \right| < \frac{\alpha\pi}{2},$$

where $z, w \in \Delta$, $0 < \alpha \leq 1$, $m \in \mathbb{N}$ and $g = f^{-1}$.

Theorem 1.1 ([5]). Let $f \in \mathcal{S}_{\Sigma_m}(\alpha)$ be given by (4). Then

$$\begin{aligned} |a_{m+1}| &\leq \frac{2\alpha}{m(m+1)}, \\ |a_{2m+1}| &\leq \frac{\alpha(2-\alpha)}{m(2m-1)} \end{aligned}$$

and for some $k \in \mathbb{R}$,

$$|a_{2m+1} - k a_{m+1}^2| \leq \frac{\alpha}{m(2m-1)} + \begin{cases} 2T + \frac{T}{m} - 2G(k), & G(k) \leq 2T \\ \frac{T}{m} - 2T, & 2T \leq G(k) \leq \frac{T}{m} \\ 2G(k) - 2T - \frac{T}{m}, & G(k) \geq \frac{T}{m}, \end{cases}$$

where $G(k) = \frac{2k\alpha^2}{m^2(m+1)^2}$ and $T = \frac{\alpha(\alpha-1)}{(2m+1)(2m-1)}$.

Definition 1.2. [5] A function $f \in \Sigma_m$ given by (4) is said to be in the class $\mathcal{S}_{\Sigma_m}^*(\beta)$, if it satisfies the following conditions:

$$\Re \left(\frac{z^3 f''(z)}{(f(z))^2} + 1 \right) > \beta$$

and

$$\Re \left(\frac{w^3 g''(w)}{(g(w))^2} + 1 \right) > \beta,$$

where $z, w \in \Delta$, $0 \leq \beta < 1$, $m \in \mathbb{N}$ and $g = f^{-1}$.

Theorem 1.2 ([5]). Let $f \in \mathcal{S}_{\Sigma_m}^*(\beta)$ be given by (4). Then

$$\begin{aligned} |a_{m+1}| &\leq \frac{2(1-\beta)}{m(m+1)}, \\ |a_{2m+1}| &\leq \frac{1-\beta}{m(2m-1)} \end{aligned}$$

and for some $k \in \mathbb{R}$,

$$|a_{2m+1} - ka_{m+1}^2| \leq \left| \frac{(1-\beta)}{m(2m-1)} - \frac{4k(1-\beta)^2}{m^2(m+1)^2} \right|.$$

The primary goal of this paper is to define a new subclass, $\mathcal{S}_{\Sigma_m}^*(\beta)$, and to obtain coefficient estimates for the initial terms $|a_{m+1}|$, $|a_{2m+1}|$, and the general terms $|a_{nm+1}|$ (for $n \geq 2$). Additionally, we aim to improve and generalize certain recent results in the literature.

2. THE CLASS $\mathcal{S}_{\Sigma_m}^{h,p}$

In this section, we introduce the class $\mathcal{S}_{\Sigma_m}^{h,p}$.

Definition 2.1. Let $h, p : \Delta \rightarrow \mathbb{C}$ be analytic functions satisfying:

$$\min\{\Re(h(z)), \Re(p(z))\} > 0 \quad (z \in \Delta) \quad \text{and} \quad h(0) = p(0) = 1.$$

A function $f(z)$, represented as in equation (4), belongs to the class $\mathcal{S}_{\Sigma_m}^{h,p}$, if the following conditions hold:

$$\left(\frac{z^3 f''(z)}{(f(z))^2} + 1 \right) \in h(\Delta) \tag{7}$$

and

$$\left(\frac{w^3 g''(w)}{(g(w))^2} + 1 \right) \in p(\Delta), \tag{8}$$

where $z, w \in \Delta$ and $g = f^{-1}$.

Remark 2.1. Various choices for the functions h and p produce notable subclasses of $\mathcal{S}_{\Sigma_m}^{h,p}$. For example, If we let

$$h(z) = p(z) = \left(\frac{1+z^m}{1-z^m} \right)^\alpha = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \dots \quad (0 < \alpha \leq 1),$$

then the conditions of Definition 2.1 are satisfied. In this case, any function $f \in \mathcal{S}_{\Sigma_m}^{h,p}$, satisfies:

$$\left| \arg \left(\frac{z^3 f''(z)}{(f(z))^2} + 1 \right) \right| < \frac{\alpha\pi}{2}$$

and

$$\left| \arg \left(\frac{w^3 g''(w)}{(g(w))^2} + 1 \right) \right| < \frac{\alpha\pi}{2}.$$

In this case we say that f belongs to the class $\mathcal{S}_{\Sigma_m}(\alpha)$.

On the other hand, if we take

$$h(z) = p(z) = \frac{1 + (1-2\beta)z^m}{1-z^m} = 1 + 2(1-\beta)z^m + 2(1-\beta)z^{2m} + \dots \quad (0 \leq \beta < 1).$$

the conditions are also satisfied. In this case, for $f \in \mathcal{S}_{\Sigma_m}^{h,p}$, we have

$$\Re \left(\frac{z^3 f''(z)}{(f(z))^2} + 1 \right) > \beta$$

and

$$\Re \left(\frac{w^3 g''(w)}{(g(w))^2} + 1 \right) > \beta.$$

In this case we say that f belongs to the class $\mathcal{S}_{\Sigma_m}^*(\beta)$.

Remark 2.2. For one-fold symmetric bi-univalent functions, we denote the class $\mathcal{S}_{\Sigma_m}^{h,p} = \mathcal{S}_{\sigma}^{h,p}$. Special cases of this subclass illustrated below:

- (A) By putting $h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^\alpha$, then the class $\mathcal{S}_{\sigma}^{h,p}$ reduces to the class $\mathcal{M}_{\sigma}(\alpha)$.
- (B) By putting $h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z}$ and $\lambda = 1$, then the class $\mathcal{S}_{\Sigma}^{h,p}$ reduces to the class $\mathcal{M}_{\sigma}^*(\beta)$.

We now derive bounds for the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in the subclass $\mathcal{S}_{\Sigma_m}^{h,p}$.

Theorem 2.1. Let the function f given by (4) be in the class $\mathcal{S}_{\Sigma_m}^{h,p}$. Then

$$|a_{m+1}| \leq \min \left\{ \frac{|h_m|}{m(m+1)}, \sqrt{\frac{|h_{2m}| + |p_{2m}|}{2m(m+1)(2m-1)}} \right\},$$

$$|a_{2m+1}| \leq \min \left\{ \frac{|h_{2m}| + |p_{2m}|}{4m(2m+1)} + \frac{|h_m|^2}{2m^2(m+1)}, \frac{2m|h_{2m}| + |p_{2m}|}{2m(2m-1)(2m+1)} \right\}$$

and for some $k \in \mathbb{R}$,

$$|a_{2m+1} - ka_{m+1}^2| \leq \frac{1}{4m(2m+1)} \begin{cases} (1 + H(k)) |h_{2m}| + (1 - H(k)) |p_{2m}|, & |H(k)| \leq 1 \\ (1 + H(k)) |h_{2m}| + (H(k) - 1) |p_{2m}|, & |H(k)| \geq 1, \end{cases}$$

where $H(k) = \frac{(2m+1)(m+1-2k)}{(m+1)(2m-1)}$.

Proof. The proof involves expanding the terms in the Taylor-Maclaurin series for $|a_{m+1}|$ and $|a_{2m+1}|$. Indeed, by considering the relations (7) and (8), we have

$$1 + \frac{z^3 f''(z)}{(f(z))^2} = h(\Delta) \tag{9}$$

and

$$1 + \frac{w^3 g''(w)}{(g(w))^2} = p(\Delta), \tag{10}$$

In light of the following Taylor-Maclaurin series expansions for the functions h and p , we get

$$h(z) = 1 + h_m z^m + h_{2m} z^{2m} + h_{3m} z^{3m} + \dots \tag{11}$$

and

$$p(w) = 1 + p_m w^m + p_{2m} w^{2m} + p_{3m} w^{3m} + \dots \tag{12}$$

By substituting the relations (11) and (12) into (9) and (10), respectively, we get

$$m(m+1)a_{m+1} = h_m, \tag{13}$$

$$2m(2m+1)a_{2m+1} - 2m(m+1)a_{m+1}^2 = h_{2m}, \tag{14}$$

$$-m(m + 1)a_{m+1} = p_m \tag{15}$$

and

$$-2m(2m + 1)a_{2m+1} + 4m^2(m + 1)a_{m+1}^2 = p_{2m}. \tag{16}$$

Comparing the coefficients (13) and (15), we obtain

$$h_m = -p_m \tag{17}$$

and

$$2m^2(m + 1)^2 a_{m+1}^2 = h_m^2 + p_m^2. \tag{18}$$

Now, if we add (14) and (16), we get the following relation

$$2m(m + 1)(2m - 1)a_{m+1}^2 = h_{2m} + p_{2m}. \tag{19}$$

Therefore, from (17), (18) and (19), we have

$$a_{m+1}^2 = \frac{h_m^2 + p_m^2}{2m^2(m + 1)^2} \implies |a_{m+1}|^2 \leq \frac{|h_m|^2}{m^2(m + 1)^2} \tag{20}$$

and

$$a_{m+1}^2 = \frac{h_{2m} + p_{2m}}{2m(m + 1)(2m - 1)} \implies |a_{m+1}|^2 \leq \frac{|h_{2m}| + |p_{2m}|}{2m(m + 1)(2m - 1)}, \tag{21}$$

respectively.

Therefore, we find from the equations (20) and (21) that

$$|a_{m+1}| \leq \frac{|h_m|}{m(m + 1)}$$

and

$$|a_{m+1}| \leq \sqrt{\frac{|h_{2m}| + |p_{2m}|}{2m(m + 1)(2m - 1)}},$$

respectively. We have thus derived the desired bound on the coefficient $|a_{m+1}|$.

Next, we want to find the bound on the coefficient $|a_{2m+1}|$. Upon subtracting (16) from (14), we get

$$a_{2m+1} = \frac{h_{2m} - p_{2m}}{4m(2m + 1)} + \frac{(m + 1)}{2} a_{m+1}^2. \tag{22}$$

Putting the value of a_{m+1}^2 from (18) into (22), it follows that

$$a_{2m+1} = \frac{h_{2m} - p_{2m}}{4m(2m + 1)} + \frac{h_m^2 + p_m^2}{4m^2(m + 1)}.$$

Therefore, we conclude the following bound:

$$|a_{2m+1}| \leq \frac{|h_{2m}| + |p_{2m}|}{4m(2m + 1)} + \frac{|h_m|^2}{2m^2(m + 1)}. \tag{23}$$

By substituting the value of a_{m+1}^2 from (19) into (22), we obtain

$$a_{2m+1} = \frac{(2m - 1)(h_{2m} - p_{2m}) + (2m + 1)(h_{2m} + p_{2m})}{4m(2m - 1)(2m + 1)}$$

which readily yields

$$|a_{2m+1}| \leq \frac{2m|h_{2m}| + |p_{2m}|}{2m(2m - 1)(2m + 1)}. \tag{24}$$

Hence, from (23) and (24), we get the desired estimate on the coefficient $|a_{2m+1}|$ as asserted in Theorem 2.1.

Finally, by using (22) for some $k \in \mathbb{R}$, we get

$$\begin{aligned} a_{2m+1} - ka_{m+1}^2 &= \frac{h_{2m} - p_{2m}}{4m(2m+1)} + \left(\frac{m+1-2k}{2} \right) a_{m+1}^2 \\ &= \frac{1}{4m(2m+1)} \left[h_{2m} - p_{2m} \right. \\ &\quad \left. + 2m(2m+1)(m+1-2k)a_{m+1}^2 \right]. \end{aligned} \quad (25)$$

From (25) and (19), we have

$$\begin{aligned} a_{2m+1} - ka_{m+1}^2 &= \frac{1}{4m(2m+1)} \left[h_{2m} - p_{2m} + \frac{(2m+1)(m+1-2k)}{(m+1)(2m-1)} (h_{2m} + p_{2m}) \right] \\ &= \frac{1}{4m(2m+1)} \left[\left(1 + \frac{(2m+1)(m+1-2k)}{(m+1)(2m-1)} \right) h_{2m} \right. \\ &\quad \left. + \left(\frac{(2m+1)(m+1-2k)}{(m+1)(2m-1)} - 1 \right) p_{2m} \right]. \end{aligned}$$

Let $H(k) := \frac{(2m+1)(m+1-2k)}{(m+1)(2m-1)}$, so

$$a_{2m+1} - ka_{m+1}^2 = \frac{1}{4m(2m+1)} \left[(1 + H(k)) h_{2m} + (H(k) - 1) p_{2m} \right].$$

Next, taking the absolute values we obtain

$$|a_{2m+1} - ka_{m+1}^2| \leq \frac{1}{4m(2m+1)} \left[|1 + H(k)| |h_{2m}| + |H(k) - 1| |p_{2m}| \right].$$

Then, we conclude that

$$|a_{2m+1} - ka_{m+1}^2| \leq \frac{1}{4m(2m+1)} \begin{cases} (1 + H(k)) |h_{2m}| + (1 - H(k)) |p_{2m}|, & |H(k)| \leq 1 \\ (1 + H(k)) |h_{2m}| + (H(k) - 1) |p_{2m}|, & |H(k)| \geq 1. \end{cases}$$

□

3. COROLLARIES AND CONSEQUENCES

By setting $h(z) = p(z) = \left(\frac{1+z^m}{1-z^m} \right)^\alpha$ in Theorem 2.1, we conclude the following result.

Corollary 3.1. *Let f given by (3) be in the class $\mathcal{S}_{\Sigma_m}(\alpha)$ ($0 < \alpha \leq 1$). Then*

$$\begin{aligned} |a_{m+1}| &\leq \min \left\{ \frac{2\alpha}{m(m+1)}, \alpha \sqrt{\frac{2}{m(m+1)(2m-1)}} \right\}, \\ |a_{2m+1}| &\leq \min \left\{ \frac{\alpha^2}{m(2m+1)} + \frac{2\alpha^2}{m^2(m+1)}, \frac{\alpha^2}{m(2m-1)} \right\} \end{aligned}$$

and for some $k \in \mathbb{R}$,

$$|a_{2m+1} - ka_{m+1}^2| \leq \frac{\alpha^2}{m(2m+1)} \begin{cases} 1, & \frac{(2m+1)|(m+1-2k|}{(m+1)(2m-1)} \leq 1 \\ \frac{(2m+1)|(m+1-2k|}{(m+1)(2m-1)}, & \frac{(2m+1)|(m+1-2k|}{(m+1)(2m-1)} \geq 1. \end{cases}$$

Remark 3.1. The bounds on $|a_{2m+1}|$ and $|a_{2m+1} - ka_{m+1}^2|$ given in Corollary 3.1 are better than those given in Theorem 1.1. Because

$$\frac{\alpha^2}{m(2m-1)} \leq \frac{\alpha(2-\alpha)}{m(2m-1)}.$$

By setting $h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m}$ in Theorem 2.1, we conclude the following result.

Corollary 3.2. Let f given by (4) be in the class $\mathcal{S}_{\Sigma_m}^*(\beta)$ ($0 \leq \beta < 1$). Then

$$|a_{m+1}| \leq \min \left\{ \frac{2(1-\beta)}{m(m+1)}, \sqrt{\frac{2(1-\beta)}{m(m+1)(2m-1)}} \right\},$$

$$|a_{2m+1}| \leq \min \left\{ \frac{1-\beta}{m(2m+1)} + \frac{2(1-\beta)^2}{m^2(m+1)}, \frac{1-\beta}{m(2m-1)} \right\}$$

and for some $k \in \mathbb{R}$,

$$|a_{2m+1} - ka_{m+1}^2| \leq \frac{(1-\beta)}{m(2m+1)} \begin{cases} 1, & \frac{(2m+1)|(m+1-2k|}{(m+1)(2m-1)} \leq 1 \\ \frac{(2m+1)|(m+1-2k|}{(m+1)(2m-1)}, & \frac{(2m+1)|(m+1-2k|}{(m+1)(2m-1)} \geq 1. \end{cases}$$

Remark 3.2. The bounds on $|a_{2m+1}|$ and $|a_{2m+1} - ka_{m+1}^2|$ given in Corollary 3.2 are better than those given in Theorem 1.2. Because

$$\frac{1-\beta}{m(2m+1)} + \frac{2(1-\beta)^2}{m^2(m+1)} \leq \frac{1-\beta}{m(2m-1)}, \quad \beta \geq \frac{3m^2 - m - 1}{4m^2 - 1}.$$

By setting $m = 1$ in Corollary 3.1, we conclude the following result.

Corollary 3.3. Let the function f given by (1) be in the class $\mathcal{M}_\sigma(\alpha)$. Then

$$|a_2| \leq \alpha \quad , \quad |a_3| \leq \min \left\{ \frac{4\alpha^2}{3}, \alpha^2 \right\} = \alpha^2$$

and for some $k \in \mathbb{R}$,

$$|a_3 - ka_2^2| \leq \begin{cases} \frac{\alpha^2}{3}, & 3|1-k| \leq 1 \\ |1-k|\alpha^2, & 3|1-k| \geq 1. \end{cases}$$

By setting $m = 1$ in Corollary 3.2, we conclude the following result.

Corollary 3.4. Let the function f given by (1) be in the class $\mathcal{M}_\sigma^*(\beta)$. Then

$$|a_2| \leq \min \left\{ (1-\beta), \sqrt{1-\beta} \right\} = 1-\beta,$$

$$|a_3| \leq \begin{cases} (1 - \beta), & 0 \leq \beta \leq \frac{1}{3} \\ \frac{1-\beta}{3} + (1 - \beta)^2, & \frac{1}{3} \leq \beta < 1, \end{cases}$$

and for some $k \in \mathbb{R}$,

$$|a_3 - ka_2^2| \leq \begin{cases} \frac{(1-\beta)}{3}, & 3|1 - k| \leq 1 \\ |1 - k|(1 - \beta), & 3|1 - k| \geq 1. \end{cases}$$

4. CONCLUSIONS

In this paper, we introduce a new subclass $\mathcal{S}_{\Sigma_m}^{h,p}$ of analytic functions, characterized by m -fold symmetric as a foundational framework. It is worth noting that this subclass is a generalization of many well-known or new subclasses, mentioned in section 2. Moreover, by Theorem 2.1, we obtained sharp bounds of the coefficients for many well-known subclasses as consequences. That in certain cases our data has improved the results of others.

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