

AN INVESTIGATION OF THE FRACTIONAL DIRAC OPERATOR USING LAPLACE TRANSFORM

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ABSTRACT. In this paper, the fractional Dirac operator with Caputo's fractional derivative is considered. By using Laplace transform, the fractional Dirac operator reduces to an algebraic equation. Then by applying the inverse Laplace transform, we obtain the closed form of the characteristic function according to the two-parameters Mittag-Leffler function. By truncating the series of Mittag-Leffler function, the eigenvalues and the corresponding eigenfunctions are approximated. A convergence analysis for the proposed procedure is given. Finally, the efficiency and simplicity of the method are shown with some examples.

Keywords: One dimensional Dirac operator, Eigenvalue, Laplace transform, Caputo fractional derivative, Eigenfunction.

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1. INTRODUCTION

Relativistic quantum mechanics merges the principles of quantum mechanics and special relativity. In fact, the Dirac operator, a fundamental concept in quantum mechanics, plays a crucial role in describing relativistic particles. One dimensional Dirac equation and boundary value problems have been attracted a lot of attention during the last decades [3, 4, 10, 12, 13, 14, 17, 18, 22, 23]. This equation is an important equation with a wide range of applications in physics. It is used to describe the behavior of relativistic particles, such as electrons and quarks. In solid-state physics, it helps to understand electronic structures, graphene, and topological insulators. In nuclear and elementary particle physics, it is of utmost importance in quantum electrodynamics, electroweak theory, and quantum chromodynamics. Additionally, the Dirac equation finds applications in quantum field

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theory, condensed matter physics, and cosmology. The one-dimensional stationary Dirac system is an eigenvalue problem of the form

$$\ell[\mathbf{y}(x)] := B\mathbf{y}'(x) + \Omega(x)\mathbf{y}(x) = \lambda\mathbf{y}(x), \quad x \in (0, 1), \tag{1}$$

with boundary conditions

$$ay_1(0) + by_2(0) = 0, \tag{2}$$

$$cy_1(1) + dy_2(1) = 0, \tag{3}$$

where $a, b, c, d \in \mathbb{R}$, $a^2 + b^2 \neq 0$, $c^2 + d^2 \neq 0$, $\mathbf{y}(x) = (y_1(x), y_2(x))^T$,

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \Omega = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}.$$

Throughout this paper, $p_{ij} \in L^1(0, 1)$ are real and a parameter λ called spectral parameter. Using an appropriate and orthogonal transformation, we obtain

$$\Omega = \begin{bmatrix} p & 0 \\ 0 & r \end{bmatrix} \quad \text{or} \quad \Omega = \begin{bmatrix} p & q \\ q & -p \end{bmatrix},$$

where correspond to the canonical form of Dirac operator.

The canonical Dirac operator with the separated boundary conditions (2) and (3) is self-adjoint operator, so, all eigenvalues of the operator are real and the corresponding eigenfunctions of distinct eigenvalues are orthogonal [25]. In related to the Dirac equations computing the eigenvalues and eigenfunctions is one of the most important problem. Accordingly, researchers have explored a wide range of techniques to study this equation, including asymptotic iteration method, functional analysis, supersymmetric quantum mechanics, factorization method, quadratic algebra approach, Nikiforov-Uvarov method, quasi-exact solvability, and others [7, 8, 11]. Supersymmetric quantum mechanics, in particular, has proven to be a valuable tool in this area. While exact solutions are limited, the quasi-exact solvability method offers a partial solution by calculating a subset of the energy levels. For time-varying interactions, such as those found in atom-laser or heavy ion-atom collisions, the spectral method, particularly the Fourier grid method, is a more suitable approach [1, 5, 6, 7, 8, 11, 20].

One of the common methods is to use the fractional derivative instead of the classical derivative in the fractional problems [2, 23]. In this paper, we use Caputo's fractional derivative instead of the first order derivative in (1), and obtain the following Fractional Dirac Operator (FDO):

$$\ell_\gamma[\mathbf{y}(x)] := B {}_0^C \mathcal{D}_x^\gamma \mathbf{y}(x) + \Omega(x)\mathbf{y}(x) = \lambda\mathbf{y}(x), \quad x \in (0, 1) \tag{4}$$

where $1/2 < \gamma \leq 1$. We suppose that the entries p and q of the matrix Ω are real numbers. Eq. (4) together with conditions (2) and (3) is called the Fractional Dirac Problem (FDP). Our aim in solving the FDP is computing the eigenvalue λ and corresponding eigenfunction $\mathbf{y}(x)$. Here, we present a convenient computational algorithm to obtain the eigenvalues and eigenfunctions of the FDP by applying the Laplace transform method. By applying the inverse Laplace transform and using the two parameters Mittag-Leffler function the eigenvalues and eigenfunctions, are obtained numerically. Recently, the FDP has attracted several researchers. It should be noted that finding the analytical solution of this problem is very difficult and some times is impossible. Therefore, various numerical algorithms have been presented to approximate eigenvalues and eigenfunctions of FDP. For an example, a simple and suitable algorithm using the power series of solutions has been used in [24], and Pseudospectral method by using Chebyshev cardinal functions has been used in [23].

The rest of the paper is organized as follows: In Section 2, we state preliminary materials of fractional calculus. In Section 3, we obtain the analytic solution of FDP, in terms of

Mittag–Leffler function. In Section 4, we compute the eigenvalues of FDP, and state the convergence theorem. Some numerical examples are given in Section 5.

2. PRELIMINARIES

In this section, we state some preliminary materials of fractional calculus [15, 16, 19].

Definition 2.1. Let $g(t)$ on $(0, \infty)$ be a real-valued function and $\nu \in \mathbb{R}$. We say $g \in C_\nu$, if there exists $q \geq \nu$ such that $g(t) = t^q g_1(t)$, where $g_1 \in C(0, \infty)$. We say $g \in C_\mu^n$ if $g^{(n)} \in C_\nu$, $n \in \mathbb{N}$.

Definition 2.2. Let the function $g \in C_\nu[0, 1]$, for $\nu \geq -1$. Then, the fractional Riemann–Liouville integral of order $\gamma \geq 0$ of the function g is defined by the following formula

$${}_0I_t^\gamma g(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{g(x)}{(t-x)^{1-\gamma}} dx, \quad t \geq 0.$$

Definition 2.3. Let $g \in C_{-1}^m$, $m \in \mathbb{N}$. Then, the Caputo fractional derivative of order γ of g , is defined as

$${}_0^C D_t^\gamma g(t) = \begin{cases} {}_0I_t^{m-\gamma} g^{(m)}(t), & m-1 < \gamma < m, \\ \frac{d^m}{dt^m} g(t), & \gamma = m. \end{cases}$$

Thus, for $0 < \gamma < 1$, we have

$${}_0^C D_t^\gamma g(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{g'(x)}{(t-x)^\gamma} dx.$$

Definition 2.4. The one parameter Mittag–Leffler function is identified as

$$E_\gamma(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + 1)}, \quad \gamma \in \mathbb{C}, \quad \Re(\gamma) > 0,$$

and the two parameters Mittag–Leffler function is identified as

$$E_{\gamma, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + \beta)}, \quad \gamma, \beta \in \mathbb{C}, \quad \Re(\gamma) > 0,$$

where $\Re(\gamma)$ denotes the real part of γ .

Definition 2.5. A function $g(t)$ defined on $0 \leq t < \infty$ is called to be exponentially bounded of order $k \in \mathbb{R}$, if it fulfills an inequality presenting as

$$\|g(t)\| \leq M e^{kt},$$

for some real constant $M > 0$ and for all enough large values of t .

Definition 2.6. Suppose that $g(t)$ is an exponentially bounded function of order k . Then, the Laplace transform of g is defined as follows

$$G(s) = L(g(t)) = \int_0^{\infty} e^{-st} g(t) dt, \quad s \in \mathbb{C}.$$

Lemma 2.1. [16] The Laplace transform of the Caputo fractional derivative of order γ can be computed as

$$L({}_0^C D_t^\gamma(g(t))) = s^\gamma L(g(t)) - \sum_{i=0}^{n-1} s^{\gamma-i-1} g^{(i)}(0), \quad n-1 < \gamma \leq n.$$

Lemma 2.2. [16] For $k \in \{\mathbb{N} \cup 0\}$, $\gamma > 0$, and $\delta > 0$, we have

$$L(t^{\gamma k + \delta - 1} E_{\gamma, \delta}^{(k)}(at^\gamma)) = \frac{k! s^{\gamma - \delta}}{(s^\gamma - a)^{k+1}}, \quad \Re(s) > |a|^{\frac{1}{\gamma}}. \tag{5}$$

3. ANALYTIC SOLUTION OF THE FDP

In this section, we use the Laplace transform method to obtain the analytic solution of the FDP. By taking Laplace transform of both sides of the equation (4), we get:

$$Bs^\gamma L[\mathbf{y}(x)] - s^{\gamma-1} \mathbf{y}(0) + (\Omega - \lambda I)L[\mathbf{y}(x)] = 0,$$

then by taking the inverse Laplace transform for the case $\Omega = \begin{bmatrix} p & 0 \\ 0 & r \end{bmatrix}$, we obtain

$$\begin{aligned} \mathbf{y}(x) &= \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} \tag{6} \\ &= \begin{bmatrix} y_1(0)E_{2\gamma,1}(-(p-\lambda)(r-\lambda)x^{2\gamma}) + x^\gamma y_2(0)(r-\lambda)E_{2\gamma,\gamma+1}(-(p-\lambda)(r-\lambda)x^{2\gamma}) \\ y_2(0)E_{2\gamma,1}(-(p-\lambda)(r-\lambda)x^{2\gamma}) - x^\gamma y_1(0)(p-\lambda)E_{2\gamma,\gamma+1}(-(p-\lambda)(r-\lambda)x^{2\gamma}) \end{bmatrix}. \end{aligned}$$

By using the boundary condition (3), we find the characteristic function $\Delta(\lambda)$ as follows

$$\begin{aligned} \Delta(\lambda) &:= cy_1(1) + dy_2(1) \tag{7} \\ &= (cy_1(0) + dy_2(0))E_{2\gamma,1}(-(p-\lambda)(r-\lambda)) \\ &\quad + (cy_2(0)(r-\lambda) - dy_1(0)(p-\lambda))E_{2\gamma,\gamma+1}(-(p-\lambda)(r-\lambda)). \end{aligned}$$

It is easy to check that for $\gamma = 1$, the function $\Delta(\lambda)$ is an entire function of order $1/2$. All roots of this function are real and simple [25]. The roots of $\Delta(\lambda)$ are called eigenvalues of the problem. Using the boundary condition (2), the following states are valid:

Case I: Let $y_1(0) = 0$ and $y_2(0) \neq 0$, then the characteristic function of (4) is as follows

$$\begin{aligned} \Delta(\lambda) &= dy_2(0)E_{2\gamma,1}(-(p-\lambda)(r-\lambda)) \tag{8} \\ &\quad + cy_2(0)(r-\lambda)E_{2\gamma,\gamma+1}(-(p-\lambda)(r-\lambda)). \end{aligned}$$

Case II: Let $y_1(0) \neq 0$ and $y_2(0) = 0$, then

$$\begin{aligned} \Delta(\lambda) &= cy_1(0)E_{2\gamma,1}(-(p-\lambda)(r-\lambda)) \tag{9} \\ &\quad - dy_1(0)(p-\lambda)E_{2\gamma,\gamma+1}(-(p-\lambda)(r-\lambda)). \end{aligned}$$

Case III: Let $y_1(0) \neq 0$ and $y_2(0) \neq 0$, then the characteristic function of (4) is obtained as follows

$$\begin{aligned} \Delta(\lambda) &:= cy_1(1) + dy_2(1) \tag{10} \\ &= (cy_1(0) + dy_2(0))E_{2\gamma,1}(-(p-\lambda)(r-\lambda)) \\ &\quad + (cy_2(0)(r-\lambda) - dy_1(0)(p-\lambda))E_{2\gamma,\gamma+1}(-(p-\lambda)(r-\lambda)). \end{aligned}$$

Now consider the case $\Omega = \begin{bmatrix} p & q \\ q & -p \end{bmatrix}$. Taking Laplace transform of (4) and using Lemma 2.2 we obtain

$$\begin{aligned} \mathbf{y}(x) &= \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} \tag{11} \\ &= \begin{bmatrix} y_1(0)E_{2\gamma,1}((p^2 + q^2 - \lambda^2)x^{2\gamma}) + (qy_1(0) - (p + \lambda)y_2(0))x^\gamma E_{2\gamma,\gamma+1}((p^2 + q^2 - \lambda^2)x^{2\gamma}) \\ y_2(0)E_{2\gamma,1}((p^2 + q^2 - \lambda^2)x^{2\gamma}) - (qy_2(0) + (p - \lambda)y_1(0))x^\gamma E_{2\gamma,\gamma+1}((p^2 + q^2 - \lambda^2)x^{2\gamma}) \end{bmatrix}. \end{aligned}$$

Applying the boundary condition (3), we obtain

$$\begin{aligned}\Delta(\lambda) &:= cy_1(1) + dy_2(1) \\ &= (cy_1(0) + dy_2(0))E_{2\gamma,1}(p^2 + q^2 - \lambda^2) \\ &\quad + [(cq + d(p - \lambda))y_1(0) - (c(p + \lambda) - dq)y_2(0)]E_{2\gamma,\gamma+1}(p^2 + q^2 - \lambda^2).\end{aligned}\tag{12}$$

Using boundary condition (2), we have the following cases:

Case I: If $y_1(0) = 0$ and $y_2(0) \neq 0$, then the characteristic function of (4) is as follows

$$\begin{aligned}\Delta(\lambda) &= dy_2(0)E_{2\gamma,1}(p^2 + q^2 - \lambda^2) \\ &\quad - (c(p + \lambda) - dq)y_2(0)E_{2\gamma,\gamma+1}(p^2 + q^2 - \lambda^2).\end{aligned}\tag{13}$$

Case II: If $y_1(0) \neq 0$ and $y_2(0) = 0$, then the characteristic function of (4) is obtained as follows:

$$\begin{aligned}\Delta(\lambda) &= cy_1(0)E_{2\gamma,1}(p^2 + q^2 - \lambda^2) \\ &\quad - (cq + d(p - \lambda))y_1(0)E_{2\gamma,\gamma+1}(p^2 + q^2 - \lambda^2).\end{aligned}\tag{14}$$

Case III: If $y_1(0) \neq 0$ and $y_2(0) \neq 0$, then we find the following characteristic equation:

$$\begin{aligned}\Delta(\lambda) &:= (cy_1(0) + dy_2(0))E_{2\gamma,1}(p^2 + q^2 - \lambda^2) \\ &\quad + [(cq + d(p - \lambda))y_1(0) - (c(p + \lambda) - dq)y_2(0)]E_{2\gamma,\gamma+1}(p^2 + q^2 - \lambda^2).\end{aligned}\tag{15}$$

Now, we have the ability to employ the initial N terms of the corresponding power series of (7)-(15) to approximate the eigenvalues.

4. COMPUTING THE EIGENVALUES AND CONVERGENCE ANALYSIS

In the previous section, we obtained the characteristic function of Dirac problem according to Mittag Leffler function. Therefore the eigenvalues of (4) with boundary conditions (2)-(3) are zeros of two parameter Mittag–Leffler function or linear combination of these functions of order $(2\gamma, 1)$ and $(2\gamma, \gamma + 1)$. For computing the eigenvalues we truncate the series of Mittag–Leffler functions by a polynomial of order N and consider the roots of resulted polynomial as eigenvalues of fractional Dirac problem.

4.1. Convergence analysis. Here we discuss the convergence analysis of the proposed method. The following theorem implies that the convergence rate is proportional to the remainder term of truncated series.

Theorem 4.1. *Suppose that $\Delta(\lambda)$ is the characteristic function obtained in section 3 and $\tilde{\Delta}_N(\lambda) = \sum_{n=0}^N a_n \lambda^n$ is the truncated sum of $\Delta(\lambda)$. Let $\tilde{\lambda}$ and $\tilde{\Lambda}_N$ are the first positive zero of $\Delta(\lambda)$ and $\tilde{\Delta}_N(\lambda)$, respectively. If N tends to infinity then $\tilde{\Lambda}_N$ converges to $\tilde{\lambda}$ with convergent rate $|R_N(\lambda)|$, where $R_N(\lambda) = \sum_{n=N+1}^{\infty} a_n \lambda^n$ is the reminder term of the truncated sum.*

proof. In accordance with Hurwitzs theorem [9], for any sufficiently small $\varepsilon > 0$, there exists a number $M \in \mathbb{N}$, such that for every $N > M$, the function $\tilde{\Delta}_N$ has exactly m zeros (counting multiplicities) in the disk with radius ε centered at $\lambda = \tilde{\lambda}$, and

$$\lim_{N \rightarrow \infty} \tilde{\Lambda}_N = \tilde{\lambda}.$$

Hence, for $|\lambda - \tilde{\lambda}| < \varepsilon$ there exists $g(\lambda) \neq 0$, such that

$$\Delta_N(\lambda) = (\lambda - \Lambda_N)^m g(\lambda).$$

Since

$$\Delta(\tilde{\lambda}) = \sum_{n=0}^N a_n \tilde{\lambda}^n + R_N(\tilde{\lambda}) = 0,$$

we can write

$$(\tilde{\lambda} - \tilde{\Lambda}_N)^m g(\tilde{\lambda}) + R_N(\tilde{\lambda}) = 0.$$

As a result,

$$|(\tilde{\lambda} - \tilde{\Lambda}_N)^m| = \frac{|R_N(\tilde{\lambda})|}{|g(\tilde{\lambda})|},$$

and thus the rate of convergence depends on the reminder term $|R_N(\hat{z})|$.

5. NUMERICAL RESULTS

In this section, we display the efficiency and proficiency of the introduced technique by presenting some illustrative examples.

Example 5.1. Consider (4) with $\Omega(t) = 0$ and boundary conditions

$$y_1(0) = 0, \quad y_2(1) = 0.$$

For $\gamma = 1$ the exact eigenvalues of this problem are $\lambda_n = (n - \frac{1}{2})\pi$, $n \in \mathbb{Z}$. For $\frac{1}{2} < \gamma < 1$ we find the following results:

$$\begin{aligned} y_1(t) &= -\lambda t^\gamma E_{2\gamma, \gamma+1}(-\lambda^2 t^{2\gamma}) \\ &= -\lambda t^\gamma \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda t^\gamma)^{2k}}{\Gamma(2\gamma k + \gamma + 1)} \\ &= -\sin_\gamma(\lambda t), \end{aligned} \tag{16}$$

and

$$\begin{aligned} y_2(t) &= E_{2\gamma, 1}(-\lambda^2 t^{2\gamma}) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda t^\gamma)^{2k}}{\Gamma(2\gamma k + 1)} \\ &= \cos_\gamma(\lambda t), \end{aligned} \tag{17}$$

So

$$\Delta(\lambda) = E_{2\gamma, 1}(-\lambda^2) = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda)^{2k}}{\Gamma(2\gamma k + 1)}. \tag{18}$$

The roots of $\Delta(\lambda)$ are eigenvalues. The numerical approximation of the first 8 eigenvalues, for various values of N and γ , are presented in Table 1. Note that Λ_k^{50} are taken as exact values into account. Some of eigenfunctions are plotted in Figure 1.

Remark 5.1. The Dirac operator (4) with the boundary conditions (2)–(3) in $\gamma = 1$ is self-adjoint. Therefore, the eigenvalues of this problem for $\gamma = 1$ are real and the eigenfunctions corresponding to distinct eigenvalues are orthogonal. But for $\gamma \in (0.5, 1)$ the number of real eigenvalues decreases with decreasing γ . This is shown in Table 2.

TABLE 1. Absolute errors corresponding to the first 8 eigenvalues of Example 5.1 with $N = 14, 18$, and $\gamma = 1, 0.95, 0.9, 0.85$.

k	$ \Lambda_k - \Lambda_k^{14} $ $\gamma = 1$	$ \Lambda_k^{50} - \Lambda_k^{14} $ $\gamma = 0.95$	$ \Lambda_k^{50} - \Lambda_k^{14} $ $\gamma = 0.9$	$ \Lambda_k^{50} - \Lambda_k^{14} $ $\gamma = 0.85$
-4	$5.05e - 02$	$2.52e - 01$	$6.65e - 01$	1.20
-3	$2.52e - 06$	$3.13e - 05$	$4.72e - 04$	$9.20e - 03$
-2	$5.80e - 13$	$1.28e - 11$	$3.56e - 10$	$1.36e - 08$
-1	$2.87e - 27$	$2.01e - 25$	$1.55e - 23$	$1.35e - 21$
1	$2.87e - 27$	$2.01e - 25$	$1.55e - 23$	$1.35e - 21$
2	$5.80e - 13$	$1.28e - 11$	$3.56e - 10$	$1.36e - 08$
3	$2.52e - 06$	$3.13e - 05$	$4.72e - 04$	$9.20e - 03$
4	$5.05e - 02$	$2.52e - 01$	$6.65e - 01$	1.20

k	$ \lambda_k - \Lambda_k^{18} $ $\gamma = 1$	$ \Lambda_k^{50} - \Lambda_k^{18} $ $\gamma = 0.95$	$ \Lambda_k^{50} - \Lambda_k^{18} $ $\gamma = 0.9$	$ \Lambda_k^{50} - \Lambda_k^{18} $ $\gamma = 0.85$
-4	$6.53e - 06$	$1.50e - 04$	$4.70e - 03$	$1.30e - 01$
-3	$1.89e - 11$	$6.40e - 10$	$2.71e - 08$	$1.45e - 06$
-2	$7.21e - 20$	$5.39e - 18$	$8.88e - 16$	$7.59e - 14$
-1	$5.41e - 38$	$1.82e - 35$	$6.94e - 33$	$3.06e - 30$
1	$5.41e - 38$	$1.82e - 35$	$6.94e - 33$	$3.06e - 30$
2	$7.21e - 20$	$5.39e - 18$	$8.88e - 16$	$7.59e - 14$
3	$1.89e - 11$	$6.40e - 10$	$2.71e - 08$	$1.45e - 06$
4	$6.53e - 06$	$1.50e - 04$	$4.70e - 03$	$1.30e - 01$

TABLE 2. Positive real eigenvalues for Example 5.1 with $N = 50$ and different values of $\gamma = 0.5, 0.6, 0.7, 0.8, 0.9, 1$.

Λ_k^{50}	$\gamma = 0.5$	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$	$\gamma = 0.9$	$\gamma = 1$
1	-	1.60238767	1.47043442	1.45285081	1.49179760	1.57079633
2	-	6.34203286	9.60848311	3.74371391	4.10443300	4.71238898
3	-	-	-	5.44706586	6.47010847	7.85398163
4	-	-	-	7.48397145	8.78105193	10.9955743
5	-	-	-	8.45187239	10.9874639	14.1371669
6	-	-	-	14.7037535	13.1874417	17.2787596
7	-	-	-	-	⋮	⋮

Example 5.2. Consider Eq. (4) with $\Omega(t) = 0$ and boundary conditions

$$y_1(0) = 0, \quad y_1(1) = 0.$$

The exact eigenvalues of this problem for $\gamma = 1$ are $\lambda_n = n\pi$, $n \in \mathbb{Z}$. For $\frac{1}{2} < \gamma < 1$, the closed form of the characteristic function is as follows

$$\begin{aligned} \Delta(\lambda) &= y_2(0)E_{2,\gamma,1}(-(p - \lambda)(r - \lambda)) \\ &\quad - y_1(0)(p - \lambda)E_{2,\gamma,\gamma+1}(-(p - \lambda)(r - \lambda)). \end{aligned} \tag{19}$$

The roots of $\Delta(\lambda)$ are eigenvalues. The numerical approximation of the first 8 eigenvalues, for various values of N and γ , are presented in Table 3. Some of eigenfunctions are plotted in Figure 2.

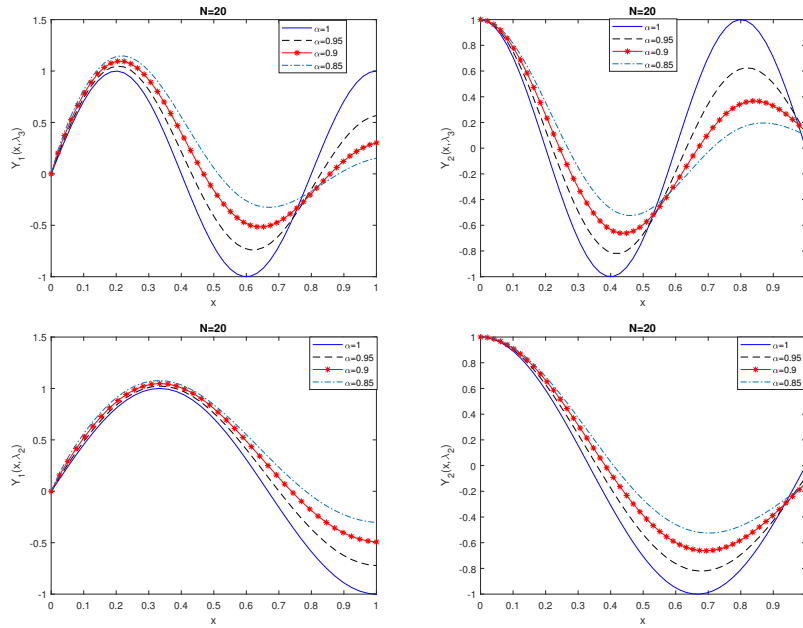


FIGURE 1. Eigenfunctions y_1 and y_2 corresponding to eigenvalues λ_2 and λ_3 of Example 5.1 for $N = 25$.

TABLE 3. Absolute errors corresponding to first 8 eigenvalues of Example 5.2 with $N = 14, 18$ and $\gamma = 1, 0.95, 0.9, 0.85$

k	$ \lambda_k - \Lambda_k^{14} $ $\gamma = 1$	$ \Lambda_k^{50} - \Lambda_k^{14} $ $\gamma = 0.95$	$ \Lambda_k^{50} - \Lambda_k^{14} $ $\gamma = 0.9$	$ \Lambda_k^{50} - \Lambda_k^{14} $ $\gamma = 0.85$
-4	$4.37e - 01$	$9.26e - 01$	1.51	—
-3	$1.78e - 04$	$2.20e - 03$	$4.90e - 02$	—
-2	$6.49e - 10$	$1.08e - 08$	$2.02e - 07$	$3.75e - 06$
-1	$3.10e - 19$	$1.50e - 17$	$8.88e - 16$	$9.45e - 14$
1	$3.10e - 19$	$1.50e - 17$	$8.88e - 16$	$9.45e - 14$
2	$6.49e - 10$	$1.08e - 08$	$2.02e - 07$	$3.75e - 06$
3	$1.78e - 04$	$2.20e - 03$	$4.90e - 02$	—
4	$4.37e - 01$	$9.26e - 01$	1.51	—

k	$ \lambda_k - \Lambda_k^{18} $	$ \Lambda_k^{50} - \Lambda_k^{18} $	$ \Lambda_k^{50} - \Lambda_k^{18} $	$ \Lambda_k^{50} - \Lambda_k^{18} $
-4	$3.30e - 04$	$6.90e - 03$	$1.10e - 01$	$4.05e - 01$
-3	$4.61e - 09$	$1.48e - 07$	$7.40e - 06$	$1.10e - 03$
-2	$6.44e - 16$	$3.28e - 14$	$1.80e - 12$	$9.87e - 11$
-1	$1.19e - 27$	$2.38e - 25$	$6.70e - 23$	$3.16e - 20$
1	$1.19e - 27$	$2.38e - 25$	$6.70e - 23$	$3.16e - 20$
2	$6.44e - 16$	$3.28e - 14$	$1.80e - 12$	$9.87e - 11$
3	$4.61e - 09$	$1.48e - 07$	$7.40e - 06$	$1.10e - 03$
4	$3.30e - 04$	$6.90e - 03$	$1.10e - 01$	$4.05e - 01$

Example 5.3. Consider the FDO (4) with $\Omega = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and boundary conditions $y_1(1) = 0, y_2(0) = 0$. By using (9), we get

$$\Delta(\lambda) = cy_1(0)E_{2\gamma,1}((3 - \lambda)(\lambda - 2)).$$

TABLE 4. Comparison of the absolute errors corresponding to the first 4 positive eigenvalues of Examples 5.2 and Example 4.1 in [23] for $\gamma = 1$.

k	$ \lambda_k - \Lambda_k^{25} $	$ \lambda_k - \Lambda_k^{30} $	$ \lambda_k - \Lambda_k^{35} $	$ \lambda_k - \Lambda_k^{40} $
1	$2.44e - 18$	$2.44e - 18$	$2.44e - 18$	$2.44e - 18$
2	$4.87e - 18$	$4.87e - 18$	$4.87e - 18$	$4.87e - 18$
3	$2.22e - 17$	$2.12e - 17$	$2.12e - 17$	$2.12e - 17$
4	$4.02e - 12$	$1.06e - 17$	$9.74e - 18$	$9.74e - 18$

k	$ \lambda_k - \Lambda_k^{10} [23]$	$ \lambda_k - \Lambda_k^{15} [23]$	$ \lambda_k - \Lambda_k^{17} [23]$	$ \lambda_k - \Lambda_k^{19} [23]$
1	$9.18e - 08$	$5.47e - 16$	$9.71e - 19$	$1.41e - 19$
2	$2.36e - 06$	$2.24e - 09$	$2.08e - 11$	$1.42e - 13$
3	$2.99e - 03$	$1.64e - 08$	$2.67e - 09$	$4.17e - 11$
4	$6.89e - 03$	$4.22e - 05$	$1.76e - 05$	$1.51e - 06$

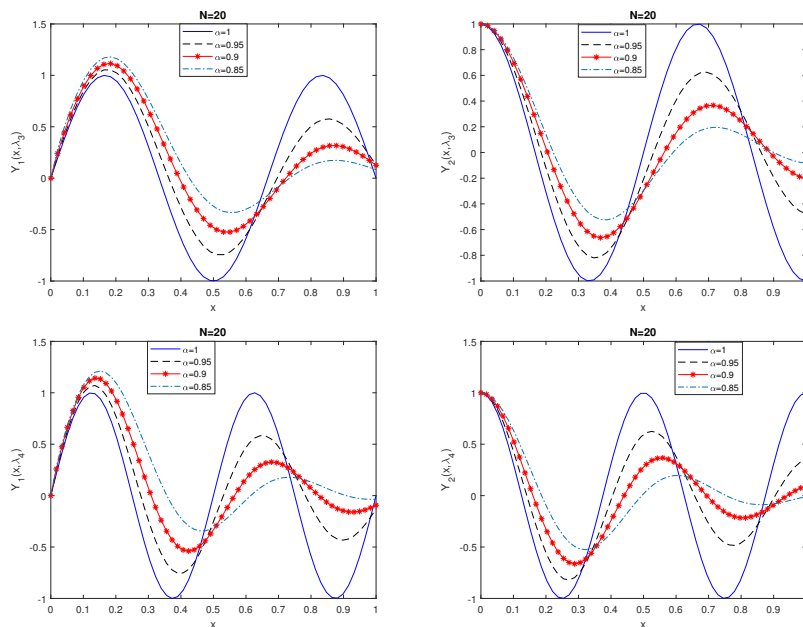


FIGURE 2. Eigenfunctions y_1 and y_2 corresponding to eigenvalues λ_3 and λ_4 by taking $N = 30$ for Example 5.2.

The roots of $\Delta(\lambda)$ are the eigenvalues. The numerical approximation of the first 8 eigenvalues for various values of N and γ , are shown in the Table 5. The eigenfunctions are plotted in Figure 3.

Example 5.4. Consider the canonical FDO with

$$\Omega = \begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix},$$

and the following boundary conditions

$$y_1(0) - y_2(0) = 0, \quad y_1(1) + y_2(1) = 0.$$

According to (12), we obtain

$$\Delta(\lambda) = 2y_1(0)(E_{2\gamma,1}(25 - \lambda^2) - 4E_{2\gamma,\gamma+1}(25 - \lambda^2)).$$

The numerical approximations of the first 8 eigenvalues, for various values of N and γ , are presented in the Table 5. The eigenfunctions are plotted in Figure 4.

TABLE 5. Absolute errors corresponding to the first 8 eigenvalues for Example 5.3 with $N = 14, 18$ and $\gamma = 1, 0.95, 0.9, 0.85$.

k	$ \Lambda_k^{50} - \Lambda_k^{14} $ $\gamma = 1$	$ \Lambda_k^{50} - \Lambda_k^{14} $ $\gamma = 0.95$	$ \Lambda_k^{50} - \Lambda_k^{14} $ $\gamma = 0.9$	$ \Lambda_k^{50} - \Lambda_k^{14} $ $\gamma = 0.85$
-4	$5.04e - 02$	$2.52e - 01$	$6.64e - 01$	1.20
-3	$2.52e - 06$	$3.12e - 05$	$4.71e - 04$	$9.20e - 03$
-2	$5.77e - 13$	$1.27e - 11$	$3.53e - 10$	$1.35e - 08$
-1	$2.74e - 27$	$1.91e - 25$	$1.47e - 23$	$1.28e - 21$
1	$2.74e - 27$	$1.91e - 25$	$1.47e - 23$	$1.28e - 21$
2	$5.77e - 13$	$1.27e - 11$	$3.53e - 10$	$1.35e - 08$
3	$2.52e - 06$	$3.12e - 05$	$4.71e - 04$	$9.20e - 03$
4	$5.04e - 02$	$2.52e - 01$	$6.64e - 01$	1.20

k	$ \Lambda_k^{50} - \Lambda_k^{18} $	$ \Lambda_k^{50} - \Lambda_k^{18} $	$ \Lambda_k^{50} - \Lambda_k^{18} $	$ \Lambda_k^{50} - \Lambda_k^{18} $
-4	$6.52e - 06$	$1.50e - 04$	$4.70e - 03$	$1.30e - 01$
-3	$1.89e - 11$	$6.39e - 10$	$2.70e - 08$	$1.45e - 06$
-2	$7.17e - 20$	$5.35e - 18$	$4.44e - 16$	$7.57e - 14$
-1	$5.16e - 38$	$1.73e - 35$	$6.58e - 33$	$2.89e - 30$
1	$5.20e - 38$	$1.73e - 35$	$6.58e - 33$	$2.89e - 30$
2	$7.17e - 20$	$5.35e - 18$	$8.88e - 16$	$7.54e - 14$
3	$1.89e - 11$	$6.39e - 10$	$2.70e - 08$	$1.45e - 06$
4	$6.52e - 06$	$1.50e - 04$	$4.70e - 03$	$1.30e - 01$

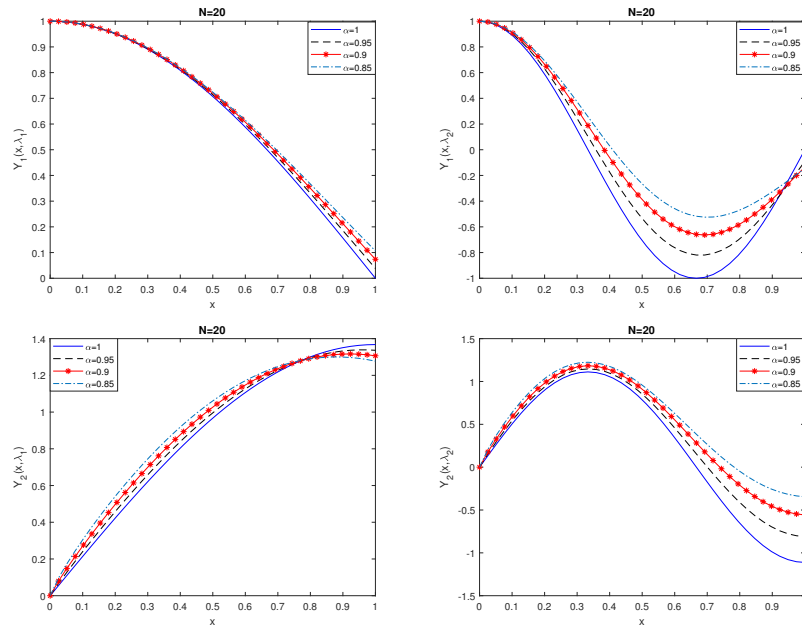


FIGURE 3. Eigenfunctions y_1 and y_2 corresponding to eigenvalues λ_2, λ_3 and λ_4 by taking $N = 20$ for Example 5.3.

TABLE 6. The first 8 eigenvalues error of Example 5.4 For $N = 12, 15$ and $\gamma = 1, 0.95, 0.9, 0.85$.

k	$ \Lambda_k^{50} - \Lambda_k^{12} $ $\gamma = 1$	$ \Lambda_k^{50} - \Lambda_k^{12} $ $\gamma = 0.95$	$ \Lambda_k^{50} - \Lambda_k^{12} $ $\gamma = 0.9$	$ \Lambda_k^{50} - \Lambda_k^{12} $ $\gamma = 0.85$
-4	$3.28e - 01$	$6.16e - 01$	$9.37e - 01$	1.32
-3	$5.17e - 05$	$3.21e - 04$	$2.30e - 2$	$1.77e - 01$
-2	$2.66e - 12$	$3.01e - 11$	$4.65e - 10$	$1.14e - 08$
-1	$1.87e - 12$	$8.71e - 11$	$3.55e - 09$	$1.24e - 07$
1	$1.87e - 12$	$8.71e - 11$	$3.55e - 09$	$1.24e - 07$
2	$2.66e - 12$	$3.01e - 11$	$4.65e - 10$	$1.14e - 08$
3	$5.17e - 05$	$3.21e - 04$	$2.30e - 2$	$1.77e - 01$
4	$3.28e - 01$	$6.16e - 01$	$9.37e - 01$	1.32

k	$ \Lambda_k^{50} - \Lambda_k^{15} $	$ \Lambda_k^{50} - \Lambda_k^{15} $	$ \Lambda_k^{50} - \Lambda_k^{15} $	$ \Lambda_k^{50} - \Lambda_k^{15} $
-4	$1.91e - 02$	$2.10e - 01$	—	—
-3	$1.32e - 08$	$1.64e - 07$	$2.33e - 06$	$3.31e - 05$
-2	$1.52e - 17$	$8.88e - 16$	$1.68e - 14$	$1.19e - 12$
-1	$1.20e - 17$	$2.22e - 15$	$3.05e - 13$	$3.81e - 11$
1	$1.20e - 17$	$2.22e - 15$	$3.05e - 13$	$3.81e - 11$
2	$1.52e - 17$	$8.88e - 16$	$1.68e - 14$	$1.19e - 12$
3	$1.32e - 08$	$1.64e - 07$	$2.33e - 06$	$3.31e - 05$
4	$1.91e - 02$	$2.10e - 01$	—	—

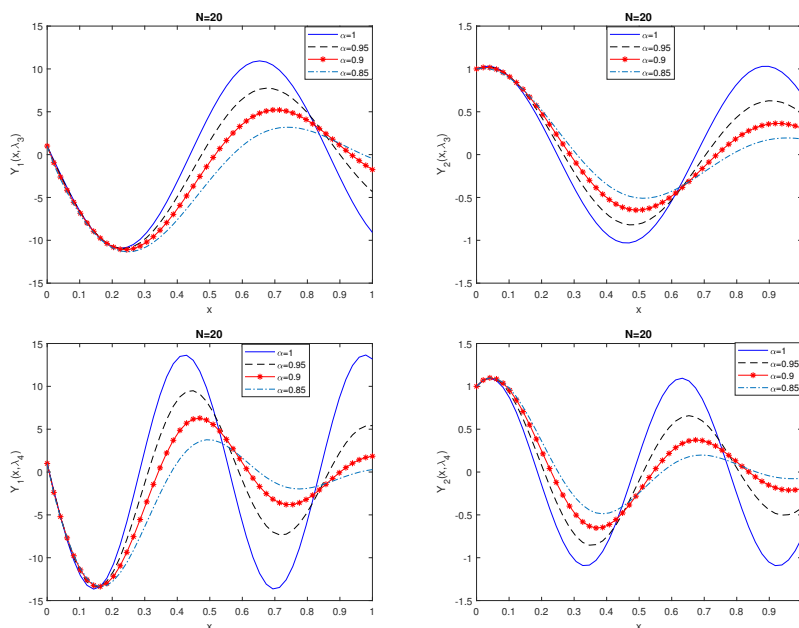


FIGURE 4. Eigenfunctions y_1 and y_2 corresponding to eigenvalues λ_3 and λ_4 by taking $N = 20$ for Example 5.4.

Remark 5.2. In Examples 5.1, 5.2, and 5.4 the characteristic functions are even functions. Thus the eigenvalues are as $\pm\lambda$. It is obvious that the results in Tables 1, 3 and 5 are also as $\pm\lambda$. Thus our numerical results confirm the analytic results.

6. CONCLUSIONS

In this work, we studied the fractional Dirac problem. We obtained the eigenvalues and eigenfunctions of the problem by using the inverse Laplace transform and the two parameters Mittag–Leffler function. Based on the convergence analysis of the method and the expansion of the Mittag–Leffler function, the error of the method is of order $O(\frac{1}{\Gamma(N\gamma+1)})$ which is a decreasing function of the parameter γ in the interval $(0.5, 1)$. Therefore, the method will have good results for values of γ close to 1. The results can be improved for other values of γ by increasing N . The graphs show that similar to classical Dirac problem, in the fractional case the eigenfunctions are oscillatory functions and the oscillations increase with respect to index k .

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