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APPROXIMATE CONTROLLABILITY OF SEMILINEAR CONTROL SYSTEMS IN HILBERT SPACES

N. I. MAHMUDOV[1], N. ŞEMI[1], §

Abstract. This paper deals with the approximate controllability of semilinear evolution systems in Hilbert spaces. Sufficient condition for approximate controllability have been obtained under natural conditions.

Keywords: Controllability, stochastic systems, fixed point.

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1. Introduction

We are given a probability space $(\Omega, \Im, P)$ together with a normal filtration $(\Im_t)_{t\geq 0}$. We consider three real separable spaces K, X and U, and Q-Wiener process on $(\Omega, \Im, P)$ with covariance linear bounded operator Q such that tr$Q < \infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in K, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k,\ \ k = 1, 2, ...$, and a sequence $\{\beta_k\}_{k\geq 1}$ of independent Brownian motions such that

$$\langle w\left(t\right), e\rangle = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \left\langle e_k, e\right\rangle \beta_k\left(t\right),\ e \in K,\ t \in [0, b],$$

and $\Im_t = \Im_t^w$, where $\Im_t^w$ is the sigma algebra generated by $\{w\left(s\right) : 0 \leq s \leq t\}$. Let $L_2^0 = L_2\left(Q^{1/2}K; X\right)$ be the space of all Hilbert-Schmidt operators from $Q^{1/2}K$ to X with the inner product $\langle \psi, \phi\rangle_{L_2^0} = tr\left[\psi Q \phi\right]$. $L^p\left(\Im_b, X\right)$ is the Banach space of all $\Im_b-$measurable square integrable variables with values in X. $L^p_{\Im}(0, b; X)$ is the Banach space of all p-square integrable and $\Im_t-$adapted processes with values in X. Let $C\left(0, b; L^p\left(\Im, X\right)\right)$ be the Banach space of continuous maps from $[0, b]$ into $L^p\left(\Im, X\right)$ satisfying the condition $\sup\left\{\mathbf{E}\left\|\varphi\left(t\right)\right\|^p : t \in [0, b]\right\} < \infty$. $\mathfrak{C}_p\left(0, b; X\right)$ is the closed subspace of $C\left(0, b; L^p\left(\Im, X\right)\right)$ consisting of measurable and $\Im_t-$adapted X-valued processes $\varphi \in C\left(0, b; L^p\left(\Im, X\right)\right)$ endowed with the norm $\left\|\varphi\right\|_{\mathfrak{C}_p} = \left(\sup\limits_{0\leq t\leq b} \mathbf{E}\left\|\varphi(t)\right\|_X^p\right)^{\frac{1}{p}}$.

Abstract semilinear differential equation serves as a formulation for many control systems described by partial or functional differential equations.Controllability theory for

[1] Department of Mathematics, Eastern Mediterranean University, Gazimagusa,
TRNC via Mersin 10, Turkey,
e-mail: nazim.mahmudov@emu.edu.tr and nidai.semi@emu.edu.tr

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abstract linear control systems in infinite-dimensional spaces is well-developed, and extensively investigated in the literature, see [1], [6], [17] and [23] and the references therein. Several authors have extended controllability concepts to infinite-dimensional systems represented by nonlinear evolution equations. The approximate controllability for the systems of differential equations has been investigated by several authors, see for instance [2]- [24].

This paper is devoted to the approximate controllability problems of the following semilinear control system

$$\begin{cases} dy(t) = [Ay(t) + (Bu)(t) + f(t, y(t))]\,dt + \int_0^t \sigma(r, y(r))\,dw(r), \\ y(0) = \xi, \quad 0 \le t \le b, \end{cases} \tag{1}$$

in a real Hilbert space $(X, \|\cdot\|)$. The meaning of all notations are listed in the following: A is the infinitesimal generator of a C_0-semigroup $\{S(t) : t \ge 0\}$, $u \in L^2_{\Im}(0, b; U)$ is a control function, U is a Hilbert space, B is a linear bounded operator from $L^2_{\Im}(0, b; U)$ to $L^2_{\Im}(0, b; X)$, $f : [0, b] \times X \to X$, $\sigma : [0, b] \times X \to L^0_2$.

Denote the solution of (1) corresponding to a control u by $y(\cdot; u)$. Then $y(b; u)$ is the state value at the terminal time b. Introduce the set

$$R_b(f) = \left\{ y(b; u) : u \in L^2_{\Im}(0, b; U) \right\},$$

which is called the reachable set of system (1) at terminal time b, its closure in $L^2(\Im_b, X)$ is denoted by $\overline{R_b(f)}$.

Definition 1. *System (1) is said to be approximately controllable on* $[0, b]$ *if* $\overline{R_b(f)} = L^2(\Im_b, X)$.

2. Assumptions

Throughout the paper we impose the following assumptions:

- (A1) $(f, \sigma) : [0, b] \times X \to X \times L^0_2$ is locally Lipschitz continuous in y uniformly in $t \in [0, b]$: there exists a constant $L > 0$ such that

$$\|f(t, y_1) - f(t, y_2)\| + \|\sigma(t, y_1) - \sigma(t, y_2)\|_{L^0_2} \le L \|y_1 - y_2\|$$

  for any $t \in [0, b]$.
- (A2) There exists $L_1 > 0$ such that for all $(t, y) \in [0, b] \times X$

$$\|f(t, y)\| + \|\sigma(t, y)\|_{L^0_2} \le L_1 (1 + \|y\|)$$

- (A3) For any $p \in L^2_{\Im}(0, b; X)$, there exists a function $q \in \overline{\mathrm{Im}(B)}$ such that $\Xi p = \Xi q$, where $\Xi : L^2_{\Im}(0, b; X) \to L^0_2$ is defined as follows

$$\Xi p = \int_0^b S(b - s)\, p(s)\, ds, \quad p \in L^2_{\Im}(0, b; X).$$

The assumption (A3) was introduced by Naito in [15]. Let $N = \ker \Xi = \left\{ p \in L^2_{\Im}(0, b; X) : \Xi p = 0 \right\}$ and let G be an orthogonal projection operator from $L^2_{\Im}(0, b; X)$ into $N^{\perp}$ and $\mathrm{Im}\, B$ be the range of B. It follows from (A3) that $\{x + N\} \cap \overline{\mathrm{Im}\, B} \ne \emptyset$ for any $x \in N^{\perp}$. Therefore, the operator $P : N^{\perp} \to \overline{\mathrm{Im}\, B}$ defined by

$$Px = x^*,$$

where $x^* \in \{x + N\} \cap \overline{\mathrm{Im}\, B}$ and $\|x^*\| = \min \left\{ \|y\| : y \in \{x + N\} \cap \overline{\mathrm{Im}\, B} \right\}$ is well defined. The operator P is bounded [15].

3. Approximate controllability

This section provides the main results and several lemmas that will be used to prove the main results.

Under the assumptions (A1) and (A2), for any control $u \in L^2_{\Im}(0, b; U)$ the system (1) has a unique mild solution. This mild solution is defined as a solution of the following integral equation:

$$y(t;u) = S(t)\xi + \int_0^t S(t-s)[(Bu)(s) + f(s, y(s))]\,ds + \int_0^t S(t-s)\int_0^s \sigma(r, y(r))\,dw(r)\,ds, \quad 0 \le t \le b. \tag{2}$$

Similarly, for any $z \in L^2_{\Im}(0, b; X)$, the following integral equation

$$x(t;z) = x(t) = S(t)\xi + \int_0^t S(t-s)[z(s) + f(s, x(s))]\,ds + \int_0^t S(t-s)\int_0^s \sigma(r, x(r))\,dw(r)\,ds, \quad 0 \le t \le b \tag{3}$$

has a unique mild solution $x(\cdot;z)$. Therefore, the following operator $W : L^2_{\Im}(0, b; X) \to \mathfrak{C}_2(0, b; X)$ can be defined $(Wz)(\cdot) = x(\cdot;z)$.

Lemma 2. *For any $z_1, z_2 \in L^2_{\Im}(0, b; X)$ the following inequality holds:*

$$\mathbf{E}\|(Wz_1)(t) - (Wz_1)(t)\|^2 \le 3M\exp\left(3MLb^2(b+1)\right)\int_0^t \mathbf{E}\|z_1(s) - z_2(s)\|^2\,ds.$$

Proof. Let $z_1, z_2 \in L^2_{\Im}(0, b; X)$. Then

$$\mathbf{E}\|(Wz_1)(t) - (Wz_2)(t)\|^2 \le 3M\int_0^t \mathbf{E}\|z_1(s) - z_2(s)\|^2\,ds + 3MLb(b+1)\int_0^t \mathbf{E}\|(Wz_1)(s) - (Wz_2)(s)\|^2\,ds,$$

where $M = \sup\{\|S(t)\| : 0 \le t \le b\}$. By the Gronwall inequality we have

$$\begin{aligned}
&\mathbf{E}\|(Wz_1)(t) - (Wz_2)(t)\|^2 \\
&\le 3M\int_0^t \mathbf{E}\|z_1(s) - z_2(s)\|^2\,ds \\
&+ 3MLb(b+1)\int_0^t\int_0^s 3M\mathbf{E}\|z_1(\tau) - z_2(\tau)\|^2\,d\tau\exp(3MLb(b+1)(t-s))\,ds \\
&= 3M\int_0^t \mathbf{E}\|z_1(s) - z_2(s)\|^2\,ds - \int_0^t\int_0^s 3M\mathbf{E}\|z_1(\tau) - z_2(\tau)\|^2\,d\tau d_s\exp(3MLb(b+1)(t-s)) \\
&= 3M\int_0^t \mathbf{E}\|z_1(s) - z_2(s)\|^2\,ds - \int_0^s 3M\mathbf{E}\|z_1(\tau) - z_2(\tau)\|^2\,d\tau\exp(3MLb(b+1)(t-s))\,|_{s=0}^{s=t} \\
&+ 3M\int_0^t \exp(3MLb(b+1)(t-s))\,\mathbf{E}\|z_1(s) - z_2(s)\|^2\,ds \\
&\le 3M\exp\left(3MLb^2(b+1)\right)\int_0^t \mathbf{E}\|z_1(s) - z_2(s)\|^2\,ds.
\end{aligned}$$

□

By the definition of reachable set $R_b(0)$, for any $h \in R_b(0)$ there exists $u \in L^2_{\Im}(0,b;U)$ such that

$$h = S(b)\xi + \int_0^b S(b-s)(Bu)(s)\,ds.$$

Define an operator $\mathcal{J} : N^{\perp} \to N^{\perp}$ as follows

$$\mathcal{J}v = GBu - G\Gamma Pv, \qquad v \in N^{\perp}, \tag{4}$$

where $\Gamma : L^2_{\Im}(0,b;X) \to L^2_{\Im}(0,b;X)$ is the operator defined by

$$(\Gamma z)(t) = f(t,(Wz)(t)) + \int_0^t \sigma(r,(Wz)(r))\,dw(r).$$

For any $v \in N^{\perp}$, we have $Pv \in L^2_{\Im}(0,b;X)$, $\Gamma Pv \in L^2_{\Im}(0,b;X)$, and $G\Gamma Pv \in N^{\perp}$. Therefore, $\mathcal{J}$ is well defined.

Lemma 3. *The operator $\mathcal{J}$ defined by (4) has a unique fixed point in $N^{\perp}$.*

Proof. The proof is based on the classical Banach fixed point theorem for contractions. It is clear that $\mathcal{J}$ maps $N^{\perp}$ into itself. Let $v_1, v_2 \in N^{\perp}$. We show that there exists a natural number n such that$\mathcal{J}^n$ is a contraction mapping. Indeed,

$$\begin{aligned}
&\mathbf{E}\left\|\mathcal{J}v_1(t) - \mathcal{J}v_2(t)\right\|^2 \\
&\le \mathbf{E}\left\|(\Gamma Pv_1)(t) - (\Gamma Pv_2)(t)\right\|^2 \\
&\le L^2\mathbf{E}\left\|(WPv_1)(t) - (WPv_2)(t)\right\|^2 + L\int_0^t \mathbf{E}\left\|(WPv_1)(s) - (WPv_2)(s)\right\|^2 ds \\
&\le 3\left(L^2+L\right)bM\exp\left(3MLb^2(b+1)\right)\int_0^t \mathbf{E}\left\|(Pv_1)(s) - (Pv_2)(s)\right\|^2 ds \\
&\le 3\left(L^2+L\right)bM\exp\left(3MLb^2(b+1)\right)\|P\|^2\int_0^t \mathbf{E}\left\|v_1(s) - v_2(s)\right\|^2 ds \\
&= l\int_0^t \mathbf{E}\left\|v_1(s) - v_2(s)\right\|^2 ds.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{E}\left\|\mathcal{J}^2v_1(t) - \mathcal{J}^2v_2(t)\right\|^2 &\le l\int_0^t \mathbf{E}\left\|\mathcal{J}v_1(s) - \mathcal{J}v_2(s)\right\|^2 ds \\
&\le l^2\int_0^t\int_0^s \mathbf{E}\left\|v_1(r) - v_2(r)\right\|^2 drds \le l^2t\int_0^t \mathbf{E}\left\|v_1(s) - v_2(s)\right\|^2 ds.
\end{aligned}$$

Thus, it is obvious that

$$\begin{aligned}
\mathbf{E}\left\|\mathcal{J}^{n+1}v_1(t) - \mathcal{J}^{n+1}v_2(t)\right\|^2 &\le l\int_0^t \mathbf{E}\left\|\mathcal{J}^nv_1(s) - \mathcal{J}^nv_2(s)\right\|^2 ds \\
&\le l^{n+1}\int_0^t \frac{s^{n-1}}{(n-1)!}\int_0^s \mathbf{E}\left\|v_1(r) - v_2(r)\right\|^2 drds \\
&\le l^{n+1}\frac{t^n}{n!}\int_0^t \mathbf{E}\left\|v_1(s) - v_2(s)\right\|^2 ds,
\end{aligned}$$

and, consequently

$$\mathbf{E}\left\|\mathcal{J}^{n+1}v_1-\mathcal{J}^{n+1}v_2\right\|^2=\int_0^b\mathbf{E}\left\|\mathcal{J}^{n+1}v_1(t)-\mathcal{J}^{n+1}v_2(t)\right\|^2dt$$
$$\leq l^{n+1}\frac{b^{n+1}}{n!}\int_0^b\mathbf{E}\left\|v_1(s)-v_2(s)\right\|^2ds=l^{n+1}\frac{b^{n+1}}{n!}\mathbf{E}\left\|v_1-v_2\right\|^2.$$

It is known that $l^{n+1}\frac{b^{n+1}}{n!}<1$ for sufficiently large n. This results that $\mathcal{J}^{n+1}$ is a contraction mapping for sufficiently large n. Then $\mathcal{J}$ has a unique fixed point in $N^{\perp}$.

Similarly

$$\mathbf{E}\left\|\mathcal{J}v(t)\right\|^2\leq 2\mathbf{E}\left\|(Bu)(t)\right\|^2+2\mathbf{E}\left\|(\Gamma Pv)(t)\right\|^2$$
$$\leq 2\mathbf{E}\left\|(Bu)(t)\right\|^2+L_1\left(1+\mathbf{E}\left\|(WPv)(t)\right\|^2\right).$$

Now we state and prove the main result. □

Theorem 4. *Assume the assumptions (A1), (A2), (A3). Then the system (1) is approximately controllable on* $[0,b]$.

Proof. Note that the assumption (A3) implies the approximate controllability of the linear system associated with (1). Then $\overline{R_b(0)}=L^2(\Im_b,X)$ and to prove the approximate controllability of (1) it suffices to show that

$$R_b(0)\subset\overline{R_b(f)}.$$

In other words, we need to show that for any $\varepsilon>0$ and for any $h\in R_b(0)$, there exists $y_\varepsilon\in R_b(f)$ such that $\mathbf{E}\left\|y_\varepsilon-h\right\|^2<\varepsilon$. By Lemma 3 the operator $\mathcal{J}$ has a fixed point in $N^{\perp}$. So there exists $v^*\in N^{\perp}$ such that

$$\mathcal{J}v^*=GBu-G\Gamma Pv^*.$$

Recalling that $Pv^*\in(v^*+N)\cap\overline{\operatorname{Im}B}$, and G is the projection from $L^2(0,b;X)$ into $N^{\perp}$, we have

$$\int_0^bS(b-s)(Pv^*)(s)\,ds=\int_0^bS(b-s)v^*(s)\,ds,$$
$$\int_0^bS(b-s)Gp(s)\,ds=\int_0^bS(b-s)p(s)\,ds,$$
$$\int_0^bS(b-s)(Bu)(s)\,ds$$
$$=\int_0^bS(b-s)\left[\int_0^s\sigma(r,x(r;Pv^*))\,dw(r)+f(s,x(s;Pv^*))+v^*(s)\right]ds$$
$$=\int_0^bS(b-s)\left[\int_0^s\sigma(r,x(r;Pv^*))\,dw(r)+f(s,x(s;Pv^*))+(Pv^*)(s)\right]ds.$$

Finally,

$$h=S(b)\xi+\int_0^bS(b-s)\left[\int_0^s\sigma(r,x(r;Pv^*))\,dw(r)+f(s,x(s;Pv^*))+(Pv^*)(s)\right]ds$$
$$=x(b;Pv^*).$$

On the other hand there exists a sequence $u_n\in L^2_{\Im}(0,b;U)$ such that $Bu_n\to Pv^*$ as $n\to\infty$. This implies that

$$x(b;Bu_n)\to x(b;Pv^*)=h$$

as $n \to \infty$. Since $x(b; Bu_n) = y(b; u_n) \in R_b(f)$, we obtain that $h \in \overline{R_b(f)}$. This completes the proof of the theorem. □

4. Example

Let $X = L^2(0,\pi)$ and $e_n(x) = \sin(nx)$ for $n \geq 1$. Define $A : X \to X$ by $Ay = y''$ with domain

$$D(A) = \left\{ y \in X : y \text{ and } y' \text{ are absolutely continuous, } y'' \in X, y(0) = y(\pi) = 0 \right\}.$$

Then the operator

$$Ay = -\sum_{n=1}^{\infty} n^2 \langle y, e_n \rangle e_n, \quad y \in D(A),$$

and A generates strongly continuous semigroup $\{S(t) : t \geq 0\}$ defined by

$$S(t) = \sum_{n=1}^{\infty} e^{-n^2 t} \langle y, e_n \rangle e_n, \quad y \in X.$$

Define the space U by

$$U = \left\{ u : u = \sum_{n=2}^{\infty} u_n e_n, \ \|u\|^2 = \sum_{n=2}^{\infty} u_n^2 < \infty \right\}.$$

Define an operator $B : U \to X$ as follows:

$$Bu = 2u_2 e_1 + \sum_{n=2}^{\infty} u_n e_n.$$

Consider the following semilinear heat equation

$$\begin{cases} \dfrac{\partial y(t,x)}{\partial t} = \dfrac{\partial^2 y(t,x)}{\partial x^2} + Bu(t,x) + f(t, y(t,x)) + \int_0^t \sigma(s, y(s,x))\, dw(s), & 0 < t < b, 0 < x < \pi, \\ y(t,0) = y(t,\pi) = 0, \quad 0 \leq t \leq b, \\ y(t,x) = \xi(x), \quad 0 \leq x \leq \pi. \end{cases} \tag{5}$$

System (5) can be written in the abstract form (1). It follows from [16] that (A3) holds and the corresponding linear system of (5) is approximately controllable on $[0,b]$. Assuming that f and σ satisfy Lipschitz and growth conditions we may see that (A1) and (A2) are satisfied. It follows from Theorem 4 that system (5) is approximately controllable on $[0,b]$.

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Nazım İdrisoğlu Mahmudov was born in 1958 in Cebrayil Province, Azerbaijan. He received the B.Sc. and M.Sc. degrees in mathematics from the Baku State University, Baku, Azerbaijan, and the Ph.D. degree in mathematics in 1985 from Institute of Cybernetics of Azerbaijan Academy of Sciences, Baku. He is currently Professor at the Eastern Mediterranean University, T.R. North Cyprus. He has done research in stochastic optimal controls, controllability of linear and nonlinear systems, differential equations, approximation of positive linear operators. His research interests include the areas of optimal control theory, stochastic control, differential equations, approximation theory and number theory.

Nidai Şemi was born in 1958, in Nicosia, North Cyprus. He received his B.Sc. degree in Mathematics from Boğaziçi University, İstanbul, in 1984. and M.Sc and Ph.D. degrees in Mathematics, in 1992 and 1998, respectively, from Eastern Mediterranean University, Famagusta - North Cyprus. He is currently Assistant Professor at Eastern Mediterranean University, Famagusta - North Cyprus. He is also working as Academic Coordinator at Eastern Mediterranean University. His research areas are Business Mathematics, Probability Theory, Statistics, Stochastic Systems, Estimation Theory and Optimal Control.