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ON THE RELIABILITY OF k-OUT-OF-n: F SYSTEMS WITH EXCHANGEABLE COMPONENTS IN THE STRESS-STRENGTH SETUP

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Abstract. In this paper, we consider the reliability of k-out-of-n: F systems and its conditional form with exchangeable components in the stress-strength setup. We assume a random stress common to all the components in the system level. Applications of these results to illustrate the reliability for the system consisting of three components with the multivariate FGM and multivariate Marshall-Olkin distributions are also given.

Keywords: Reliability, Dependence, k-out-of-n:F system, Stress-strength model.

AMS Subject Classification: 62N05, 60K10

1. Introduction

An important method for improving the reliability of a system is to build redundancy into it. A common structure of redundancy is the k-out-of-n system. A k-out-of-n:F system consisting of n components works if and only if $(n-k+1)$ of the n components work and fails if k or more components fail. Thus, the system breaks down at the time of the kth component failure. It finds wide applications in various fields such as industrial and military systems. Some examples of such systems can be found in Kuo and Zuo (2003).

In the reliability theory, the stress-strength model is generally interested in the reliability of a component with strength X, which is under the random stress Y. When it is assumed that X and Y are independent random variables, then the component fails if the stress exceeds the strength of the component, i.e. $X < Y$. There are numerous papers on the reliability of a component which has been generally concerned with the probability $P(X > Y)$. Examples of such results and references can be found in Bhattacharyya and Johnson (1974, 1975), Greco and Ventura (2009), Hanagal (1999a), Johnson (1988), Kotz

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et al. (2003) and additional results can be found in Devanji and Rao (2001), Ebrahimi (1982), Eryılmaz (2008, 2010), Hanagal (1999b) for the reliability of systems in this setup.

In this paper, we consider the reliability of k-out-of-n: F systems with exchangeable components in the stress-strength setup. We assume that the stress and strengths are independent random variables and a random stress common to all the components. In Section 2, we present the reliability for the k-out-of-n: F system with exchangeable components in the stress-strength setup. We also give two examples for the 2-out-of-3: F system using the multivariate FGM and multivariate Marshall-Olkin models under the exponential distributed stress Y. In Section 3, we evaluate the conditional reliability of the system for the case of dependent components.

2. Reliability of the k-out-of-n: F System

The random vector $X_1, X_2, \ldots, X_n$ is exchangeable for each n, if the joint distribution

$$P(X_1 \leq x_1, \ldots, X_n \leq x_n) = P(X_{\pi(1)} \leq x_1, \ldots, X_{\pi(n)} \leq x_n),$$

for any finite permutation $\pi(1), \ldots, \pi(n)$ of the indices $\{1, 2, \ldots, n\}$. Then, the joint survival function is represented as

$$\overline{F}_{\mathbf{X}}(x_1, x_2, \ldots x_n) = P(X_1 > x_1, X_2 > x_2, \ldots, X_n > x_n).$$

Let us consider a k-out-of-n:F system with n exchangeable strengths represented by a random vector $X_1, X_2, \ldots, X_n$. If we assume that $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ are the ordered strengths of the components, the failure of the k-out-of-n:F system depends on the failure of the kth component. Hence, the strength of the system is represented by the corresponding kth order statistic $X_{k:n}$. The survival function of the system is given by

$$\begin{aligned} \overline{F}_{k:n}(t) &= P(X_{k:n} > t) \\ &= \sum_{i=n-k+1}^{n} (-1)^{i-n+k-1} \binom{i-1}{n-k} \binom{n}{i} \overline{F}_{\mathbf{X}}(\underbrace{t, \ldots, t}_{i}), \end{aligned} \tag{2.1}$$

for which $\overline{F}_{\mathbf{X}}(\underbrace{t, \ldots, t}_{i}) = P(\min(X_1, \ldots, X_i) > t)$, $1 \leq i \leq n$, (see David and Nagaraja, 2003). Navarro et al. (2006) have shown that $\overline{F}_{k:n}(t)$ with exchangeable component lifetimes is the generalized mixture of series and parallel systems. For some results on k-out-of-n systems taken into account the dependence among the lifetimes of components, see e.g. Bairamov and Parsi (2010), Eryılmaz (2011), Navarro and Balakrishnan (2010), Navarro et al. (2005), Navarro and Rychlik (2007).

Let us consider a k-out-of-n:F system with n exchangeable strengths and the strength of each component is subjected to a common random stress Y. Let also the random stress be independent of the random vector of the strengths.

Lemma 2.1. *If $X_1, X_2, \ldots, X_n$ are the exchangeable strengths, the reliability of the k-out-of-n:F system under the random stress is*

$$R^Y_{k:n} = \int\limits_y \left(\sum_{i=n-k+1}^{n} (-1)^{i-n+k-1} \binom{i-1}{n-k} \binom{n}{i} \overline{F}_{\mathbf{X}}(\underbrace{y,\ldots,y}_{i}) \right) dG_Y(y),$$

where $G_Y(y)$ is the distribution function of the stress for $y > 0$.

Proof. From (2.1), we have

$$P(X_{k:n} > Y) = \int\limits_0^\infty \sum_{i=n-k+1}^{n} (-1)^{i-n+k-1} \binom{i-1}{n-k} \sum_{1 \le j_1 < \cdots < j_i \le n} P(X_{j_1} > Y, \ldots, X_{j_i} > Y \,| Y = y) dG_Y(y).$$

Then the proof follows noting that $R^Y_{k:n} = P(X_{k:n} > Y)$. □

Remark 2.2. *A special cases for $k = n$ and $k = 1$ are equivalent to a parallel and series systems, respectively. In these cases, Lemma 2.1 reduces to the reliability of the parallel and series systems under the random stress.*

Example 2.3. *Let X_1, X_2, X_3 have an exchangeable FGM distribution with the joint survival function*

$$\overline{F}_{\mathbf{X}}(x_1, x_2, x_3) = \prod_{i=1}^{3} \overline{F}_X(x_i) \left[1 + \alpha \left(\sum_{1 \le i_1 < i_2 \le 3} \left(1 - \overline{F}_X(x_{i_1})\right) \left(1 - \overline{F}_X(x_{i_2})\right) - \prod_{i=1}^{3} \left(1 - \overline{F}_X(x_i)\right) \right) \right],$$

where $x_i > 0$ and α is the common dependence parameter (see, Kotz et al., 2000). If we assume that X_1, X_2, X_3 have exponential marginals with common parameter λ_1 and the stress Y is also exponential with parameter λ_2, then we have for the reliability of a 2-out-of-3:F system from Lemma (2.1)

$$\begin{aligned} R^Y_{2:3} &= \int\limits_0^\infty \left(3\overline{F}_{\mathbf{X}}(y, y) - 2\overline{F}_{\mathbf{X}}(y, y, y) \right) dG_Y(y) \\ &= \frac{3\lambda_2(1+\alpha)}{2\lambda_1 + \lambda_2} - \frac{2\lambda_2(1+5\alpha)}{3\lambda_1 + \lambda_2} + \frac{9\lambda_2\alpha}{4\lambda_1 + \lambda_2} - \frac{2\lambda_2\alpha}{6\lambda_1 + \lambda_2}, \end{aligned}$$

where $0 < \alpha < 1/2$, $\lambda_1, \lambda_2 > 0$.

Figure 1 shows that when the λ_1 (λ_2) increases then the reliability of the system decreases (increases) as expected.

Example 2.4. *Let X_1, X_2, X_3 have Marshall and Olkin multivariate exponential distribution with the joint survival function*

$$\overline{F}_{\mathbf{X}}(x_1, x_2, x_3) = \exp\left(-\sum_{i=1}^{3} \lambda_i x_i - \lambda_0 \max(x_1, x_2, x_3) \right),$$

where $x_i > 0, \lambda_i > 0$ and $\lambda_0 \ge 0$ (see, Kotz et al., 2000).

The reliability of the 2-out-of-3:F system consisting of three components each have exponential (λ_1) marginals with the exponential stress Y with the parameter λ_2 is given by:

$$R^Y_{2:3} = \left(\frac{3\lambda_2}{2\lambda_1 + \lambda_2 + \lambda_0} - \frac{2\lambda_2}{3\lambda_1 + \lambda_2 + \lambda_0} \right).$$

Figure 2 shows that when the λ_1 increases then the reliability of the system decreases. $R^Y_{k:n}$ is a decreasing function of λ_0 for Marshall and Olkin multivariate exponential distribution.

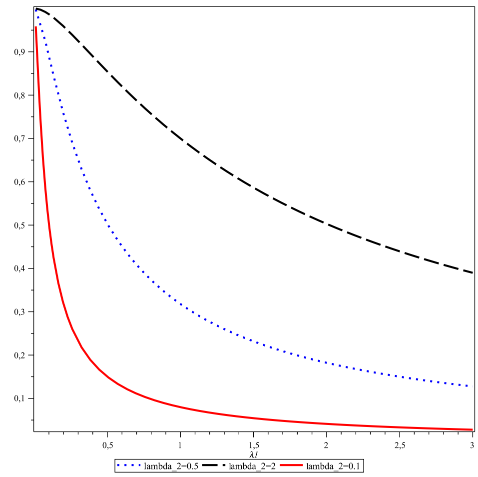

Figure 1. Reliability curves of the 2-out-of-3 system for FGM multivariate distribution with exponential marginals ($\alpha = 0.2$)

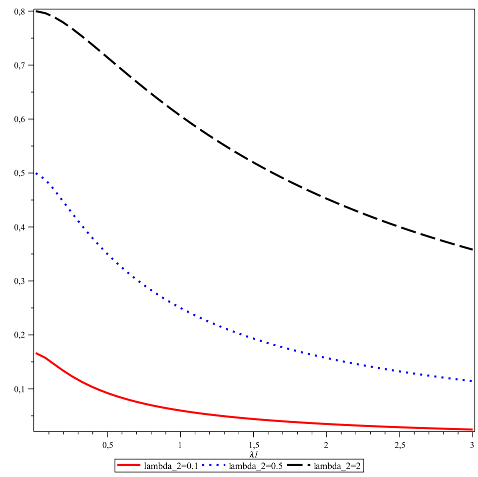

Figure 2. Reliability curves of the 2-out-of-3 system for Marshall-Olkin's multivariate exponential distribution ($\lambda_0 = 0.5$).

3. Conditional Reliability Under the Stress

For some systems consisting of n components which are subjected to a random stress, the operator may consider some maintenance or replacement procedure when all the components are working. The conditional random variable $(X_{k:n} - Y \mid X_{1:n} > Y)$ represents the remaining strength under the condition that all components are working under the stress.

Lemma 3.5. *If $X_1, X_2, \ldots, X_n$ are exchangeable strengths, the conditional reliability in a k-out-of-n:F system when all the components are working under the stress Y is*

$$S_{1,k:n}(x|y) = \frac{\int\limits_0^\infty \sum\limits_{i=n-k+1}^{n} \binom{n}{i} p_{i,n}(y, y+x) dG_Y(y)}{\int\limits_0^\infty \overline{F}_{\mathbf{X}}(\underbrace{y, \ldots, y}_{n}) dG_Y(y)},$$

where

$$\begin{aligned} p_{i,n}(y, y+x) =& \overline{F}_{\mathbf{X}}(\underbrace{y+x, \ldots, y+x}_{i}, \underbrace{y, \ldots, y}_{n-i}) \\ & - \sum_{j=1}^{n-i} (-1)^{j+1} \binom{n-j}{j} \overline{F}_{\mathbf{X}}(\underbrace{y+x, \ldots, y+x}_{i+j}, \underbrace{y, \ldots y}_{n-i-j}). \end{aligned}$$

Proof. Consider the random variable $X_{k:n} - Y | X_{1:n} > Y$ and its survival function with exchangeable strengths $X_1, X_2, \ldots, X_n$ given by

$$\begin{aligned} P(X_{k:n} - Y \quad > \quad x | X_{1:n} > Y) = & \frac{1}{\int\limits_0^\infty P(X_1 > y, \ldots X_n > y) dG_Y(y)} \int_0^\infty \sum_{i=n-k+1}^{n} \binom{n}{i} P\,(X_1 > y+x, \\ & \times \ldots, X_i > y+x, X_{i+1} \in (y, y+x], \ldots, X_n \in (y, y+x]) \, dG_Y(y) \end{aligned}$$

The probability in the nominator can be represented as

$$p_{i,n}(y, y+x) = P(A \cap B^c) = P(A) - P(A \cap B)$$

where $A \equiv \{X_1 > y+x, \ldots, X_i > y+x, X_{i+1} > y, \ldots, X_n > y\}$, $B \equiv \bigcup\limits_{j=1}^{n-i} \{X_j > y+x\}$. Thus the proof is completed. □

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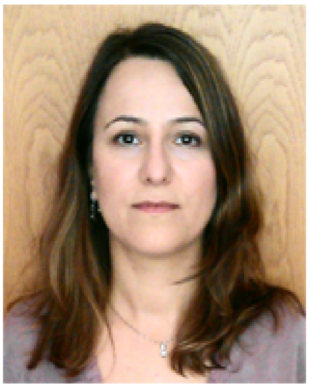

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