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$\mathfrak{b}-\mathbf{m}_1$ DEVELOPABLE SURFACES OF BIHARMONIC NEW TYPE $\mathfrak{b}-$SLANT HELICES ACCORDING TO BISHOP FRAME IN THE SOL SPACE Sol^3

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Abstract. In this paper, we study $\mathfrak{b}-\mathbf{m}_1$ developable surfaces of biharmonic new type $\mathfrak{b}-$slant helix in the Sol^3. We characterize the $\mathfrak{b}-\mathbf{m}_1$ developable surfaces in terms of their Bishop curvatures. Finally, we find out their explicit parametric equations in the Sol^3.

Keywords: New type $\mathfrak{b}-$slant helix, Sol space, curvatures, $\mathfrak{b}-\mathbf{m}_1$ developable surface

AMS Subject Classification: 53A04, 53A10

1. Introduction

Developable surfaces are especially important to the home boatbuilder because they are often working with sheet materials like plywood, steel or aluminum. Developable surfaces can be formed from flat sheets without stretching, so the forces required to form sheet materials into developable surfaces are much less than for other surfaces. In some cases, particularly with plywood, the forces required to form non-developable surfaces could be so large that the material is damaged internally when it is formed. Another advantage of developable surfaces is that the development, or flattened out shape, of such a surface is exact, [2,3,13].

A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ [4-11].

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_{\phi}\mathcal{T}(\phi) + \mathrm{tr}R(\mathcal{T}(\phi), d\phi)\, d\phi, \tag{1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

This study is organised as follows: Firstly, we study $\mathfrak{b}-\mathbf{m}_1$ developable surfaces of biharmonic new type $\mathfrak{b}-$slant helix in the Sol^3. Secondly, we characterize the $\mathfrak{b}-\mathbf{m}_1$ developable surfaces in terms of their Bishop curvatures. Finally, we find explicit equations of $\mathfrak{b}-\mathbf{m}_1$ developable surfaces in the Sol^3.

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2. Riemannian Structure of Sol Space Sol^3

Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{Sol^3} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,$$

where (x, y, z) are the standard coordinates in $\mathbb{R}^3$ [11,12].

Note that the Sol metric can also be written as:

$$g_{Sol^3} = \sum_{i=1}^{3} \omega^i \otimes \omega^i,$$

where

$$\omega^1 = e^z dx, \quad \omega^2 = e^{-z} dy, \quad \omega^3 = dz,$$

and the orthonormal basis dual to the 1-forms is

$$\mathbf{e}_1 = e^{-z}\frac{\partial}{\partial x}, \quad \mathbf{e}_2 = e^{z}\frac{\partial}{\partial y}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}. \tag{2}$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g^3_{Sol} defined above the following is true:*

$$\nabla = \begin{pmatrix} -e_3 & 0 & e_1 \\ 0 & e_3 & -e_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3}$$

where the (i, j)-element in the table above equals $\nabla_{e_i} e_j$ for our basis

$$\{\mathbf{e}_k, k = 1, 2, 3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

Lie brackets can be easily computed as:

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, \quad [\mathbf{e}_2, \mathbf{e}_3] = -\mathbf{e}_2, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1.$$

The isometry group of Sol^3 has dimension 3. The connected component of the identity is generated by the following three families of isometries:

$$\begin{aligned} (x, y, z) &\to (x + c, y, z), \\ (x, y, z) &\to (x, y + c, z), \\ (x, y, z) &\to \left(e^{-c}x, e^{c}y, z + c\right). \end{aligned}$$

3. Biharmonic New Type $\mathfrak{b}$−Slant Helices in Sol Space Sol^3

Assume that $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame field along γ. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= \kappa\mathbf{n}, \\ \nabla_{\mathbf{t}}\mathbf{n} &= -\kappa\mathbf{t} + \tau\mathbf{b}, \\ \nabla_{\mathbf{t}}\mathbf{b} &= -\tau\mathbf{n}, \end{aligned} \tag{4}$$

where κ is the curvature of γ and τ its torsion [14,15] and

$$\begin{aligned} g_{Sol^3}(\mathbf{t}, \mathbf{t}) &= 1, \; g_{Sol^3}(\mathbf{n}, \mathbf{n}) = 1, \; g_{Sol^3}(\mathbf{b}, \mathbf{b}) = 1, \\ g_{Sol^3}(\mathbf{t}, \mathbf{n}) &= g_{Sol^3}(\mathbf{t}, \mathbf{b}) = g_{Sol^3}(\mathbf{n}, \mathbf{b}) = 0. \end{aligned} \tag{5}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative, [1]. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= k_1\mathbf{m}_1 + k_2\mathbf{m}_2, \\ \nabla_{\mathbf{t}}\mathbf{m}_1 &= -k_1\mathbf{t}, \\ \nabla_{\mathbf{t}}\mathbf{m}_2 &= -k_2\mathbf{t}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} g_{Sol^3}(\mathbf{t},\mathbf{t}) &= 1,\ g_{Sol^3}(\mathbf{m}_1,\mathbf{m}_1) = 1,\ g_{Sol^3}(\mathbf{m}_2,\mathbf{m}_2) = 1, \\ g_{Sol^3}(\mathbf{t},\mathbf{m}_1) &= g_{Sol^3}(\mathbf{t},\mathbf{m}_2) = g_{Sol^3}(\mathbf{m}_1,\mathbf{m}_2) = 0. \end{aligned} \tag{7}$$

Here, we shall call the set $\{\mathbf{t},\mathbf{m}_1,\mathbf{m}_1\}$ as Bishop trihedra, k_1 and k_2 as Bishop curvatures and $\delta(s) = \arctan\frac{k_2}{k_1}$, $\tau(s) = \delta'(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$.

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{t} &= t^1e_1 + t^2e_2 + t^3e_3, \\ \mathbf{m}_1 &= m_1^1\mathbf{e}_1 + m_1^2\mathbf{e}_2 + m_1^3\mathbf{e}_3, \\ \mathbf{m}_2 &= m_2^1\mathbf{e}_1 + m_2^2\mathbf{e}_2 + m_2^3\mathbf{e}_3. \end{aligned} \tag{8}$$

To separate a new type slant helix according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as new type $\mathfrak{b}-$slant helix.

Theorem 3.1. *Let $\gamma : I \longrightarrow Sol^3$ be a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix. Then, the position vector of $\gamma$is*

$$\begin{aligned} \gamma(s) &= [\frac{\cos\mathcal{M}}{\mathcal{S}_1^2 + \sin^2\mathcal{M}}[-\mathcal{S}_1\cos[\mathcal{S}_1 s + \mathcal{S}_2] + \sin\mathcal{M}\sin[\mathcal{S}_1 s + \mathcal{S}_2]] + \mathcal{S}_4 e^{-\sin\mathcal{M}s + \mathcal{S}_3}]\mathbf{e}_1 \\ &\quad +[\frac{\cos\mathcal{M}}{\mathcal{S}_1^2 + \sin^2\mathcal{M}}[-\sin\mathcal{M}\cos[\mathcal{S}_1 s + \mathcal{S}_2] + \mathcal{S}_1\sin[\mathcal{S}_1 s + \mathcal{S}_2]] + \mathcal{S}_5 e^{\sin\mathcal{M}s - \mathcal{S}_3}]\mathbf{e}_2 \\ &\quad +[-\sin\mathcal{M}s + \mathcal{S}_3]\mathbf{e}_3, \end{aligned} \tag{9}$$

where $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5$ are constants of integration, [8].

We can use Mathematica in Theorem 3.1, yields

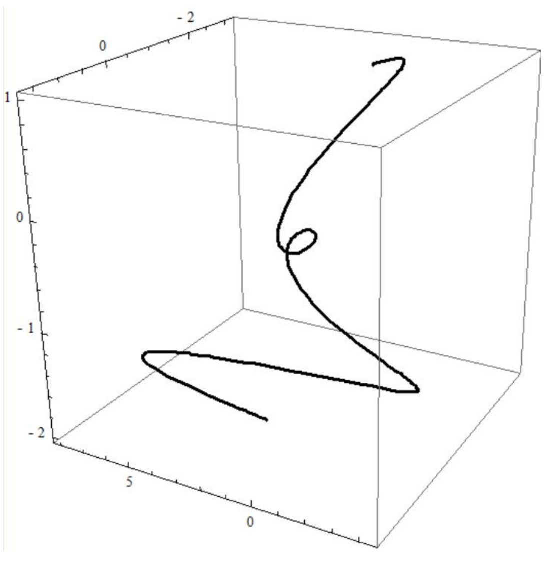

Figure 1.

4. $\mathfrak{b}-\mathbf{m}_1$ Developable Surfaces of New Type $\mathfrak{b}-$Slant Helices in Sol^3

To separate a $\mathbf{m}_1$ developable according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for this surface as $\mathfrak{b}-\mathbf{m}_1$ developable.

The purpose of this section is to study $\mathfrak{b}-\mathbf{m}_1$ developable of biharmonic new type $\mathfrak{b}-$slant helix in Sol^3.

The $\mathfrak{b}-\mathbf{m}_1$ developable of γ is a ruled surface

$$\mathcal{D}_{new}(s,u) = \gamma(s) + u\mathbf{m}_1. \tag{10}$$

Theorem 4.1. *Let $\gamma : I \longrightarrow Sol^3$ be a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix. Then, the equation of $\mathfrak{b}-\mathbf{m}_1$ developable of γ is*

$$\begin{aligned}\mathcal{D}_{new}(s,u) &= [\frac{\cos\mathcal{M}}{\mathcal{S}_1^2+\sin^2\mathcal{M}}[-\mathcal{S}_1\cos[\mathcal{S}_1 s+\mathcal{S}_2]+\sin\mathcal{M}\sin[\mathcal{S}_1 s+\mathcal{S}_2]] \\ &\quad +\mathcal{S}_4 e^{-\sin\mathcal{M}s+\mathcal{S}_3}+u\cos[\mathcal{S}_1 s+\mathcal{S}_2]]\mathbf{e}_1 \\ &\quad +[\frac{\cos\mathcal{M}}{\mathcal{S}_1^2+\sin^2\mathcal{M}}[-\sin\mathcal{M}\cos[\mathcal{S}_1 s+\mathcal{S}_2]+\mathcal{S}_1\sin[\mathcal{S}_1 s+\mathcal{S}_2]] \\ &\quad +\mathcal{S}_5 e^{\sin\mathcal{M}s-\mathcal{S}_3}-u\sin[\mathcal{S}_1 s+\mathcal{S}_2]]\mathbf{e}_2 \\ &\quad +[-\sin\mathcal{M}s+\mathcal{S}_3]\mathbf{e}_3,\end{aligned} \tag{11}$$

where $\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3,\mathcal{S}_4,\mathcal{S}_5$ are constants of integration.

Proof. We assume that γ is a unit speed new type $\mathfrak{b}-$slant helix.

Using Bishop formulas (6) and (2), we have

$$\mathbf{m}_1 = \cos[\mathcal{S}_1 s+\mathcal{S}_2]\,\mathbf{e}_1 - \sin[\mathcal{S}_1 s+\mathcal{S}_2]\,\mathbf{e}_2. \tag{12}$$

Substituting (12) to (10), we have (11). Thus, the proof is completed. □

We can prove the following interesting main result.

Theorem 4.2. *Let $\mathcal{D}_{new}$ be $\mathfrak{b}-\mathbf{m}_1$ developable of a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix. Then, the parametric equations of $\mathfrak{b}-\mathbf{m}_1$ developable are*

$$\begin{aligned}\boldsymbol{x}_{\mathcal{D}_{new}}(s,u) &= e^{\sin\mathcal{M}s-\mathcal{S}_3}[\frac{\cos\mathcal{M}}{\mathcal{S}_1^2+\sin^2\mathcal{M}}[-\mathcal{S}_1\cos[\mathcal{S}_1 s+\mathcal{S}_2]+\sin\mathcal{M}\sin[\mathcal{S}_1 s+\mathcal{S}_2]] \\ &\quad +\mathcal{S}_4 e^{-\sin\mathcal{M}s+\mathcal{S}_3}+u\cos[\mathcal{S}_1 s+\mathcal{S}_2]], \\ \boldsymbol{y}_{\mathcal{D}_{new}}(s,u) &= e^{-\sin\mathcal{M}s+\mathcal{S}_3}[\frac{\cos\mathcal{M}}{\mathcal{S}_1^2+\sin^2\mathcal{M}}[-\sin\mathcal{M}\cos[\mathcal{S}_1 s+\mathcal{S}_2]+\mathcal{S}_1\sin[\mathcal{S}_1 s+\mathcal{S}_2]] \\ &\quad +\mathcal{S}_5 e^{\sin\mathcal{M}s-\mathcal{S}_3}-u\sin[\mathcal{S}_1 s+\mathcal{S}_2]], \\ \boldsymbol{z}_{\mathcal{D}_{new}}(s,u) &= -\sin\mathcal{M}s+\mathcal{S}_3,\end{aligned} \tag{13}$$

where $\mathcal{S}_1,\mathcal{S}_2,\mathcal{S}_3,\mathcal{S}_4,\mathcal{S}_5$ are constants of integration.

Proof. The parametric equations of $\mathcal{D}_{new}$ can be found from (10), (11). This concludes the proof of Theorem. □

Thus, we proved the following:

Corollary 4.1. *Let $\gamma : I \longrightarrow Sol^3$ be a unit speed non-geodesic biharmonic new type $\mathfrak{b}-$slant helix and $\mathcal{D}_{new}$ its $\mathfrak{b}-\mathbf{m}_1$ developable surface in Sol space. Then, unit normal of $\mathfrak{b}-\mathbf{m}_1$ developable of γ is*

$$\mathbf{N}_{\mathcal{D}} = \sin\mathcal{M}\sin[\mathcal{S}_1 s+\mathcal{S}_2]\,\mathbf{e}_1 + \sin\mathcal{M}\cos[\mathcal{S}_1 s+\mathcal{S}_2]\,\mathbf{e}_2 + \cos\mathcal{M}\mathbf{e}_3, \tag{14}$$

where $\mathcal{S}_1, \mathcal{S}_2$ are constants of integration.

Proof. Assume that $\mathbf{N}_{\mathcal{D}}$ be the unit normal vector field on $\mathfrak{b}-\mathbf{m}_1$ defined by

$$\mathbf{N}_{\mathcal{D}} = \frac{\mathcal{D}_s \wedge \mathcal{D}_u}{\left| g_{Sol^3}\left(\mathcal{D}_s \wedge \mathcal{D}_u, \mathcal{D}_s \wedge \mathcal{D}_u\right)\right|^{\frac{1}{2}}}. \tag{15}$$

Using Bishop formulas (6) and (2), we have

$$\mathbf{m}_2 = \sin \mathcal{M} \sin\left[\mathcal{S}_1 s + \mathcal{S}_2\right] \mathbf{e}_1 + \sin \mathcal{M} \cos\left[\mathcal{S}_1 s + \mathcal{S}_2\right] \mathbf{e}_2 + \cos \mathcal{M} \mathbf{e}_3, \tag{16}$$

where $\mathcal{S}_1, \mathcal{S}_2$ are constants of integration.

Substituting (16) to (15), we have (14). This concludes the proof of corollary. □

We may use Mathematica in Theorem 4.2, yields

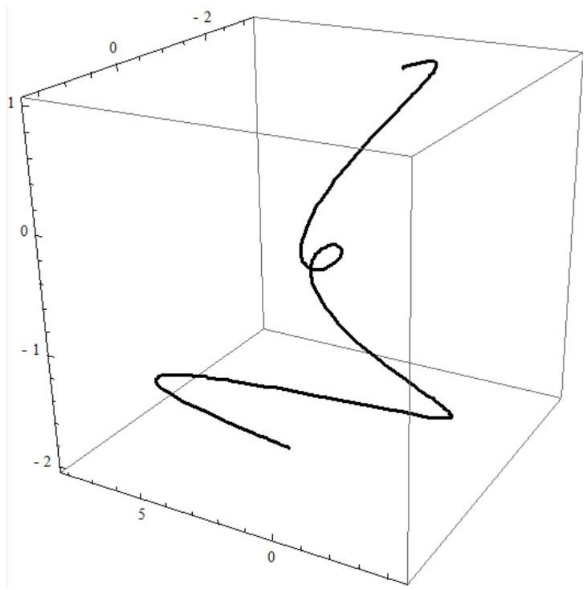

Figure 2.

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