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# FIXED POINT THEOREMS IN $p-$SUMMABLE SYMMETRIC $n-$CONE NORMED SEQUENCE SPACES

AHMET SAHINER[1], TUBA YIGIT[2]

Abstract. In this study fixed point theorems and related concepts in summable symmetric cone normed sequence spaces are investigated.

Keywords: symmetric $n-$normed space; symmetric $n-$cone Banach space; fixed point theorems.

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## 1. Introduction

Gähler introduced the concepts of $2-$metric spaces, linear $2-$normed spaces and their topological structures [2]. Gunawan and Mashadi introduced the concepts of $n-$normed spaces and their topological structures [5]. Then Lewandowska defined generalized $2-$normed spaces and generalized symmetric $2-$normed spaces [3, 4].

In 2007, Guang and Xian [7] introduced the concept of cone metric space, replacing the set of real numbers by an ordered Banach space. They proved some fixed point theorems of contractive type mappings over cone metric spaces. Some of the articles in the literature dealt with the extension of certain fixed point theorems of cone metric spaces [8, 9, 10] and some others dealt with the structure of the spaces themselves [11, 12, 13, 14].

**Definition 1.1.** *Let $E$ be a real Banach space and $P$ be a subset of $E$. Then $P$ is called cone if*

*$(i)$ $P$ is closed, nonempty and $P \neq \{0\}$;*
*$(ii)$ $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers $a, b$;*
*$(iii)$ $P \cap (-P) = \{0\}$.*

We know that for given a cone $P \subset E$, a partial ordering $\leq$ with respect to $P$ can be defined by $x \leq y$ if and only if $y - x \in P$; $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x << y$ will stand for $y - x \in int\ P$, where $int\ P$ denotes the interior of $P$.

The cone $P$ is called normal if there is a number $M > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } ||x|| \leq M||y||.$$

The least positive number $M$ satisfying the above is called the normal constant of $P$ [8].

[1] Department of Mathematics, Suleyman Demirel University, 32260, Isparta, TURKEY.
e-mail: ahmetsahiner@sdu.edu.tr
[2] Department of Mathematics, Suleyman Demirel University, 32260, Isparta, TURKEY.
e-mail: cigdemtuba@hotmail.com
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The cone $P$ is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_k\}$ is a sequence such that

$$x_1 \leq x_2 \leq ... \leq x_n \leq ... \leq y$$

for some $y \in E$, then there is $x \in E$ such that $\lim_{k\to\infty} ||x_n - x|| = 0$. Equivalently the cone $P$ is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is normal cone.

In the following we always suppose $E$ is Banach space, $P$ is a cone in $E$ with int $P \neq \varnothing$ and $\leq$ is partial ordering with respect to $P$.

**Definition 1.2.** *A cone normed space is an ordered pair $(X, ||.||_c)$ where $X$ is a vector space over $\mathbb{R}$ and $||.||_c : X \to (E, P, ||.||)$ is a function satisfying:*
*(c1) $0 < ||x||_c$, for all $x \in X$;*
*(c2) $||x||_c = 0$ if and only if $x = 0$;*
*(c3) $||\alpha x||_c = |\alpha| \, ||x||_c$, for each $x \in X$ and $\alpha \in \mathbb{R}$;*
*(c4) $||x + y||_c \leq ||x||_c + ||y||_c$, $x, y \in X$.*

It is easy to see that each cone normed space is cone metric space. Namely, the cone metric is defined by $d(x, y) = ||x - y||_c$.

According to what we mentioned above, we say that a sequence $\{x_n\}$ of a cone normed space $(X, ||.||_c)$ over $(E, P, ||.||)$ is said to be convergent, if there exists $x \in X$ such that for all $c >> 0$, $c \in E$, there exists $n_0$ such that

$$||x - x_n||_c << c$$

for all $n \geq n_0$. Also, we say that $\{x_n\}$ is Cauchy if for each $c >> 0$, there exists $n_0$ such that

$$||x_m - x_n||_c << 0$$

for all $m, n \geq n_0$ [8].

**Definition 1.3.** *Let $n \in \mathbb{N}$, $X$ be a real vector space of dimension $2 \leq d < \infty$, $E$ be a Banach space and $P \subset E$ be a cone. If the function*

$$\|\centerdot, ..., \centerdot\|_c : X \times X \times ... \times X \longrightarrow (E, P, \|.\|)$$

*satisfies the following four conditions*

$(i)$ $\|x_1, x_2, ..., x_n\|_c = 0 \Leftrightarrow x_1, x_2, ..., x_n$ linear dependent;
$(ii)$ $\|x_1, x_2, ..., x_n\|_c = \|x_2, x_1, ..., x_n\|_c$ invariant under permutation;
$(iii)$ $\|\alpha x_1, x_2, ..., x_n\|_c = |\alpha| \, \|x_1, x_2, ..., x_n\|_c$ for any $\alpha \in \mathbb{R}$;
$(iv)$ $\|x_1, x_2, ..., x_{n-1}, y + z\|_c \leq \|x_1, x_2, ..., x_{n-1}, y\|_c + \|x_1, x_2, ..., x_{n-1}, z\|_c$ then $(X, \|\centerdot, ..., \centerdot\|_c)$ is called an $n-$cone normed space.

**Definition 1.4.** [4] *Let $X$ be a real linear space. Denote by $\chi$ a non-empty subset $X \times X$ with the property $\chi = \chi^{-1}$ and such that the set $\chi^y = \{x \in X; (x, y) \in \chi\}$ is a linear subspace of $X$, for all $y \in X$.*

*A function $\|., .\| : \chi \to [0, \infty)$ satisfying the following conditions:*
*(S1) $\|x, y\| = \|y, x\|$ for all $(x, y) \in \chi$;*
*(S2) $\|x, \alpha y\| = |\alpha| \, \|x, y\|$ for any real number $\alpha$ and all $(x, y) \in \chi$;*
*(S3) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for $x, y, z \in X$ such that $(x, y), (x, z) \in \chi$;*
*will be called a generalized symmetric $2-$norm on $\chi$. The set $\chi$ is called a symmetric $2-$normed set. In particular, if $\chi = X \times X$, the function $\|\centerdot, ..., \centerdot\|$ will be called a generalized symmetric $2-$norm on $X$ and the pair $(X, \|., .\|)$ a generalized symmetric $2-$normed space.*

In this article, we introduce generalized symmetric $n-$cone normed spaces and generalized symmetric $n-$cone Banach spaces. The results expressing under what conditions a self-mapping $T$ of generalized symmetric $n-$cone Banach space $(l_p, \|\cdot,...,\cdot\|_c)$ has a unique fixed point is also given.

## 2. Main Results

**Definition 2.1.** *Let $X$ be a real linear space. Denote by $\chi$ a non-empty subset of $\underbrace{X \times X \times \ldots \times X}_{n \text{ times}}$ with the property $\chi = \chi^{-1}$ and such that the set $\chi^x = \{x \in X; (x, x_1, x_2, ..., x_{n-1}) \in \chi\}$ is a linear subspace of $X$, for all $x_1, x_2, ..., x_{n-1} \in X$.*

*A function $\|\centerdot,...,\centerdot\| : \chi \to [0;\infty)$ satisfying the following conditions:*

*$(S1)$ $\|x_1, x_2, ..., x_n\| = \|x_2, x_1, ..., x_n\|$ invariant under permutation;*

*$(S2)$ $\|\alpha x_1, x_2, ..., x_n\| = |\alpha| \|x_1, x_2, ..., x_n\|$ for any real number $\alpha$ and all $(x_1, x_2, ..., x_n) \in \chi$;*

*$(S3)$ $\|x_1, x_2, ..., x_{n-1}, y + z\| \leq \|x_1, x_2, ..., x_{n-1}, y\| + \|x_1, x_2, ..., x_{n-1}, z\|$ for $x_1, x_2, ..., x_{n-1}, y, z \in X$ such that $(x_1, x_2, ..., x_{n-1}, y), (x_1, x_2, ..., x_{n-1}, z) \in \chi$;*

*will be called a generalized symmetric $n-$norm on $\chi$. The set $\chi$ is called a symmetric $n-$normed set. In particular, if*

$$\chi = \underbrace{X \times X \times \ldots \times X}_{n \text{ times}}$$

*the function $\|\centerdot,...,\centerdot\|$ will be called a generalized symmetric $n-$norm on $X$ and the pair $(X, \|\centerdot,...,\centerdot\|)$ a generalized symmetric $n-$normed space.*

**Definition 2.2.** *Let $X$ be a real linear space. Denote by $\chi$ a non-empty subset of $\underbrace{X \times X \times \ldots \times X}_{n \text{ times}}$ with the property $\chi = \chi^{-1}$ and such that the set $\chi^x = \{x \in X; (x, x_1, x_2, ..., x_{n-1}) \in \chi\}$ is a linear subspace of $X$, for all $x_1, x_2, ..., x_{n-1} \in X$.*

*A function $\|\centerdot,...,\centerdot\|_c : \chi \to (E, P, ||.||)$ satisfying the following conditions:*

*$(S1)$ $\|x_1, x_2, ..., x_n\|_c = \|x_2, x_1, ..., x_n\|_c$ invariant under permutation;*

*$(S2)$ $\|\alpha x_1, x_2, ..., x_n\|_c = |\alpha| \|x_1, x_2, ..., x_n\|_c$ for any real number $\alpha$ and all $(x_1, x_2, ..., x_n) \in \chi$;*

*$(S3)$ $\|x_1, x_2, ..., x_{n-1}, y + z\|_c \leq \|x_1, x_2, ..., x_{n-1}, y\|_c + \|x_1, x_2, ..., x_{n-1}, z\|_c$ for $x_1, x_2, ..., x_{n-1}, y, z \in X$ such that $(x_1, x_2, ..., x_{n-1}, y), (x_1, x_2, ..., x_{n-1}, z) \in \chi$;*

*will be called a generalized symmetric $n-$cone norm on $\chi$. The set $\chi$ is called a symmetric $n-$cone normed set. In particular, if*

$$\chi = \underbrace{X \times X \times \ldots \times X}_{n \text{ times}}$$

*the function $\|\centerdot,...,\centerdot\|_c$ will be called a generalized symmetric $n-$cone norm on $X$ and the pair $(X, \|\centerdot,...,\centerdot\|_c)$ a generalized symmetric $n-$cone normed space.*

**Example 2.1.** *Let $X = \mathbb{R}^n$, $E = \mathbb{R}^n$ and $P = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_i \geqslant 0,\ i = 1, ..., n\}$. Then the function $\|\centerdot,...,\centerdot\|_c : \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times ... \times \mathbb{R}^n}_{n \text{ times}} \longrightarrow (E, P, \|.\|)$ defined by*

$$\|x_1, x_2, ..., x_n\|_c = \left( \underbrace{A, ..., A}_{n \text{ times}} \right)$$

*where*

$$A = abs\left(\begin{vmatrix} x_{11} & x_{12} & \ldots & x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix}\right)$$

*is a generalized symmetric $n-$cone norm and $(X, \|\centerdot, ..., \centerdot\|_c)$ is a generalized symmetric $n-$cone normed space.*

If we fix $\{u_1, u_2, ..., u_n\}$ to be a basis for $X$, we can give the following lemma.

**Lemma 2.1.** *Let $(X, \|\centerdot, ..., \centerdot\|_c)$ be a generalized symmetric $n-$cone normed space. Then a sequence $\{x_m\}$ converges to $x$ in $X$ if and only if for each $c \in E$ with $c >> \theta$ ($\theta$ is zero element of $E$) there exists an $N = N(c) \in \mathbb{N}$ such that $n > N$ implies $||x_1, x_2, ..., x_{n-2}, x_k - x, u_i||_c << c$ for every $i = 1, 2, ..., n$.*

*Proof.* We prove necessity since sufficiency is clear. In this case there exists $N = N(c) \in \mathbb{N}$ such that $n > N$ implies $||x_1, x_2, ..., x_{n-2}, x_k - x, u_i||_c << \frac{c}{n \max|\alpha_i|}$ for every $i = 1, 2, ..., n$. Since $\{u_1, u_2, ..., u_n\}$ is a basis for $X$, every $y$ can be written of the form $y = \alpha_1 u_1 + \alpha_2 u_2 + ... + \alpha_n u_n$ for some $\alpha_1, \alpha_2, ..., \alpha_n$ in $\mathbb{R}$. Hence

$$\begin{aligned} & ||x_1, x_2, ..., x_{n-2}, x_k - x, y||_c \\ \leq\ & |\alpha_1| \, ||x_1, x_2, ..., x_{n-2}, x_k - x, u_1||_c + ... + |\alpha_n| \, ||x_1, x_2, ..., x_{n-2}, x_k - x, u_n||_c \\ \ll\ & c. \end{aligned}$$

□

This gives us the following.

**Lemma 2.2.** *Let $(X, \|\centerdot, ..., \centerdot\|_c)$ be a generalized symmetric $n-$cone normed space. Then a sequence $\{x_m\}$ converges to $x$ in $X$ if and only if $\lim\limits_{n\to\infty} \max ||x_1, x_2, ..., x_{n-2}, x_k - x, u_i||_c = \theta$.*

Now we are ready to define a norm with respect to the basis $\{u_1, u_2, ..., u_n\}$ on $X$. The function $\|\centerdot, ..., \centerdot\|_{\infty}^{c} : X^n \to (E, P, \|.\|)$ defined by

$$\|\centerdot, ..., \centerdot\|_{\infty}^{c} := \max\{\|x_1, x_2, ..., x_n, u_i\|_c : i = 1, 2, ..., n\}$$

is a cone norm on $X$.

Note that if we choose another basis $\{v_1, v_2, ..., v_n\}$ then resulting $\|\centerdot, ..., \centerdot\|_{\infty}^{c}$ will be equivalent to the one defined with respect to the basis $\{u_1, u_2, ..., u_n\}$.

**Lemma 2.3.** *Let $(X, \|\centerdot, ..., \centerdot\|_c)$ be a generalized symmetric $n-$cone normed space. Then a sequence $\{x_m\}$ converges to $x$ in $X$ if and only if for each $c \in E$ with $c >> \theta$ ($\theta$ is zero element of $E$) there exists an $N = N(c) \in \mathbb{N}$ such that $n > N$ implies $||x_1, x_2, ..., x_{n-1}, x_k - x||_{\infty}^{c} << c$.*

**Definition 2.3.** *Let $\|\centerdot, ..., \centerdot\|_{\infty}^{c} : X^n \to (E, P, \|.\|)$ and $r \in E$ with $r >> \theta$. Then the set*

$$B_{\{u_1, u_2, ..., u_n\}}(x; r) = \left\{ y : ||x_1, x_2, ..., x_{n-1}, y - x||_{\frac{\infty}{c}} << r \right\}$$

*is called (open) ball centered at $x$, with radius $r$.*

Then we have following:

**Lemma 2.4.** *Let $(X, \|\centerdot, ..., \centerdot\|_c)$ be a generalized symmetric $n-$cone normed space. Then a sequence $\{x_m\}$ converges to $x$ in $X$ if and only if for each $r \in E$ with $r >> \theta$ ($\theta$ is zero element of $E$) there exists an $N = N(r) \in \mathbb{N}$ such that $n > N$ implies $||x_1, x_2, ..., x_{n-1}, x_k - x||_{\infty}^{c} \in B_{\{u_1,u_2,...,u_n\}}(x; r)$.*

**Theorem 2.1.** *Any symmetric $n-$cone normed space $X$ is a cone normed space and its topology agrees with the norm generated by $\|\centerdot, ..., \centerdot\|_{\infty}^{c}$.*

Now we introduce the notions of symmetric $n-$cone normed space of the sequence space $l_p$, $1 \leq p \leq \infty$, consisting of all sequences $x = (x_k)$ such that $\sum_k |x_k|^p < \infty$ and prove some fixed point theorems.

Recall from [16] that the functions

$$||x_1, x_2, ..., x_n|| := \left[\frac{1}{n!}\sum_{j_1}...\sum_{j_n}|\det(x_{ij_k})|^p\right]^{1/p}$$

and

$$||x_1, x_2, ..., x_n||_\infty := \sup_{j_1}...\sup_{j_n}|\det(x_{ij_k})|$$

define a $n-$norm on $l_p$ for $1 \leq p \leq \infty$ and for $p = \infty$, respectively. Then we have the following:

If $X = l_p$, $E = \mathbb{R}^n$ and $P = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_i \geqslant 0,\ i = 1, ..., n\}$ then the functions $\|\centerdot, ..., \centerdot\|_p^c : \underbrace{l_p \times l_p \times ... \times l_p}_{n \text{ times}} \longrightarrow (E, P, \|.\|)$ and $\|\centerdot, ..., \centerdot\|_\infty^c : \underbrace{l_p \times l_p \times ... \times l_p}_{n \text{ times}} \longrightarrow (E, P, \|.\|)$ defined by

$$\|\centerdot, ..., \centerdot\|_p^c := \left(\underbrace{\alpha_1 B, ..., \alpha_n B}_{n \text{ times}}\right) \tag{1}$$

and

$$\|\centerdot, ..., \centerdot\|_\infty^c := \left(\underbrace{\alpha_1 C, ..., \alpha_n C}_{n \text{ times}}\right) \tag{2}$$

define a symmetric $n-$cone norm on $l_p$ for $1 \leq p < \infty$ and for $p = \infty$, respectively where

$$\begin{aligned} B\ &:= \left[\frac{1}{n!}\sum_{j_1}...\sum_{j_n}|\det(x_{ij_k})|^p\right]^{1/p}, \\ C\ &:= \sup_{j_1}...\sup_{j_n}|\det(x_{ij_k})| \end{aligned}$$

and $\alpha_i \geq 0$, $i = 1, 2, ..., n$.

Remember from [16] that for any $n-$normed space $X$ by using a derived norm, defined with respect to the set $\{a_1, a_2, ..., a_n\}$, where $a_i = (\delta_{ij})$, $i = 1, ..., n$ by

$$\|x\|_p^* := \left[\sum_{\{i_2,...,i_n\}\subseteq\{1,...,n\}}\|x, a_{i_2}, ..., a_{i_n}\|_p^p\right]^{1/p}$$

if $1 \leq p < \infty$, or

$$\|x\|_\infty^* := \sup_{\{i_2,...,i_n\}\subseteq\{1,...,n\}}\|x, a_{i_2}, ..., a_{i_n}\|_\infty$$

if $p = \infty$. The above facts allow us to define symmetric $n-$cone norms on $l_p$ by

$$\|x\|_{\frac{c}{p}}^{*} := \left( \alpha_1 \left[ \sum_{\{i_2,...,i_n\}\subseteq\{1,...,n\}} \|x, a_{i_2}, ..., a_{i_n}\|_p^p \right]^{1/p}, ..., \alpha_n \left[ \sum_{\{i_2,...,i_n\}\subseteq\{1,...,n\}} \|x, a_{i_2}, ..., a_{i_n}\|_p^p \right]^{1/p} \right)$$

and by

$$\|x\|_{\frac{c}{\infty}}^{*} := \left( \alpha_1 \sup_{\{i_2,...,i_n\}\subseteq\{1,...,n\}} \|x, a_{i_2}, ..., a_{i_n}\|_{\infty}, ..., \alpha_n \sup_{\{i_2,...,i_n\}\subseteq\{1,...,n\}} \|x, a_{i_2}, ..., a_{i_n}\|_{\infty} \right)$$

where $\alpha_i \geq 0$, $i = 1, 2, ..., n$ for $1 \leq p < \infty$ and for $p = \infty$, respectively. Remember also that

$$\|x\|_p \leq \|x\|_p^* \leq n^{1/p} \|x\|_p$$

for all $x \in l_p$, where $\|.\|_p$ is the usual norm on $l_p$. In particular, $\|.\|_{\infty} = \|.\|_{\infty}^*$. Hence, if we take $\alpha_i = 1$ for all $i = 1, 2, ..., n$ in (1) and (2) we have symmetric $n-$cone norms $\|\cdot, ..., \cdot\|_p^c := \left( \underbrace{B, ..., B}_{n \text{ times}} \right)$ and $\|\cdot, ..., \cdot\|_{\infty}^c := \left( \underbrace{C, ..., C}_{n \text{ times}} \right)$ of $l_p$ for $1 \leq p < \infty$ and for $p = \infty$, respectively. Thus we have

$$\left( \underbrace{\|x\|_p, ..., \|x\|_p}_{n \text{ times}} \right) \leq \|x\|_p^* \leq \left( \underbrace{n^{1/p} \|x\|_p, ..., n^{1/p} \|x\|_p}_{n \text{ times}} \right)$$

where $\|x\|_P = \left( \|x\|_p, ..., \|x\|_p \right)$ is usual $p-$norm-like symmetric cone norm on $\left( l_p, \|\cdot, ..., \cdot\|_p^c \right)$.

In order to show that $\left( l_p, \|\cdot, ..., \cdot\|_p^c \right)$ is complete we need following.

**Lemma 2.5.** *If a sequence in $l_p$ is convergent in the usual norm $\|.\|_p$ then it is convergent in symmetric $n-$cone norm $\|\cdot, ..., \cdot\|_p^c$. Similarly, if a sequence in $l_p$ is Cauchy with respect to $\|.\|_p$ then it is Cauchy with respect to $\|\cdot, ..., \cdot\|_p^c$.*

**Theorem 2.2.** $\left( l_p, \|\cdot, ..., \cdot\|_p^c \right)$ *is a symmetric $n-$cone Banach space.*

*Proof.* Let $\{x_m\}$ be a Cauchy sequence in $\left( l_p, \|\cdot, ..., \cdot\|_p^c \right)$. Then for each $c \in E$ with $c >> \theta$ there exists $N = N(c) \in \mathbb{N}$ such that $n > N$ implies $\|x_m - x_n, y\|_p^c << c$ for all $y$ in $l_p$ if and only if for each $c \in E$ with $c >> \theta$ there exists $N = N(c) \in \mathbb{N}$ such that $n > N$ implies $\|x_m - x_n\|_{\frac{c}{p}}^{*} << c$. This proves that $\{x_m\}$ is a Cauchy sequence in symmetric $n-$cone normed space $\left( l_p, \|\cdot, ..., \cdot\|_p^c \right)$ if and only if $\{x_m\}$ is a Cauchy sequence in $\left( l_p, \|.\|_{\frac{c}{p}}^{*} \right)$. □

**Theorem 2.3.** *Let $T$ a self-mapping of $l_p$ such that*

$$\|T_x - T_y, x_2, ..., x_n\|_p^c \leq K \|x - y, x_2, ..., x_n\|_p^c$$

*for all $x, y, x_2, ..., x_n$ in $X$, where $K \in (0, 1)$ is a constant. Then $T$ has a unique fixed point in* $\left( l_p, \|\cdot, ..., \cdot\|_p^c \right)$.

*Proof.* Clearly $T$ satisfies

$$\|T_x - T_y, a_{i_2}, ..., a_{i_n}\|_p^c \leq K \|x - y, a_{i_2}, ..., a_{i_n}\|_p^c$$

for all $x, y \in l_p$ and $\{i_2, ..., i_n\} \subseteq \{1, ..., n\}$, whence

$$\|T_x - T_y\|_{\substack{c\\p}}^* \leq K \|x - y\|_{\substack{c\\p}}^*$$

for all $x, y \in l_p$. Since $\left(l_p, \|.\|_{\substack{c\\p}}^*\right)$ is a cone Banach space $T$ must have a unique fixed point. □

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**Ahmet Sahiner** is presently employed as Associate Professor at Suleyman Demirel University, Isparta. He obtained her PhD from Erciyes University (Turkey). He is studying on functional analysis and new global optimization techniques. He has been an invited speaker of some of conferences.

**Tuba Yigit** is presently employed as mathematic teacher at a high school, Isparta. She obtained her PhD from Suleyman Demirel University (Turkey). She is studying on functional analysis.