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A NEW LOOK AT q-HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. For complex parameters $a_i, b_j, q(i = 1, ..., r, j = 1, ..., s, b_j \in \mathbb{C}\backslash\{0, -1, -2, ...\}$, $|q| < 1)$, define the q-hypergeometric function ${}_r\Phi_s(a_1, ..., a_r; b_1, ..., b_s; q, z)$ by

$$ {}_r\Phi_s(a_i; b_j; q, z) = \sum_{n=0}^{\infty} \frac{(a_1, q)_n ... (a_r, q)_n}{(q, q)_n (b_1, q)_n ... (b_s, q)_n} z^n $$

$(r = s + 1; r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathcal{U})$ where $\mathbb{N}$ denote the set of positive integers and $(a, q)_n$ is the q-shifted factorial defined by

$$ (a, q)_n = \begin{cases} 1, & n = 0; \\ (1 - a)(1 - aq)(1 - aq^2)...(1 - aq^{n-1}), & n \in \mathbb{N}. \end{cases} $$

Recently, the authors [7] defined the linear operator $\mathcal{M}(a_i, b_j; q)f$. Using the operator $\mathcal{M}(a_i, b_j; q)f(z)f$, Aldweby and Darus [13] gave a new integral operator. In this work we highlight a result related to the new integral operator.

Keywords: q-hypergeometric functions, Integral operators.

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1. Introduction

The word 'hypergeometric series' is not an alien to any mathematicians or statisticians, be it in pure or applied studies. Numerous applications are seen in solving real problems as well as in fitting it right in solving problems defined in a complex plane. It is known at first as basic hypergeometric series which started essentially by Euler back in 1748 that emphasis on generating functions of partitions. Later, Gauss (1813) and Cauchy (1852) found several transformations and summations formulas related to basic hypergeometric series. A hundred years later after Euler discovery, Heine (1846) converted a simple notation

$$ lim_{q\to 1} \frac{1 - q^a}{1 - q} = a $$

into a systematic theory of basic hypergeometric series parallel to the theory of Gausss hypergeometric series.

Many great mathematicians have made important contributions to the basic hypergeometric series. For example, Andrews and Askey persistently convincing people how useful the summation and transformation formulas for basic hypergeometric series are in the theory of partitions and other disciplines (see also [1]).

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In basic hypergeometric series (or q-analogue of generalized hypergeometric series) there exist a fixed parameter $q \in \mathbb{C}$, which usually taken to satisfy $|q| < 1$.

For complex parameters $a_i, b_j, q(i = 1, ..., r, j = 1, ..., s, b_j \in \mathbb{C}\backslash\{0, -1, -2, ...\}$, $|q| < 1)$, we define the q-hypergeometric function ${}_r\Phi_s(a_1, ..., a_r; b_1, ..., b_s; q, z)$ by

$$ {}_r\Phi_s(a_i; b_j; q, z) = \sum_{n=0}^{\infty} \frac{(a_1, q)_n ... (a_r, q)_n}{(q, q)_n (b_1, q)_n ... (b_s, q)_n} z^n \tag{1} $$

$(r = s + 1; r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; z \in \mathcal{U})$ where $\mathbb{N}$ denote the set of positive integers and $(a, q)_n$ is the q-shifted factorial defined by

$$ (a, q)_n = \begin{cases} 1, & n = 0; \\ (1 - a)(1 - aq)(1 - aq^2)...(1 - aq^{n-1}), & n \in \mathbb{N}. \end{cases} $$

By using the ratio test, we should note that, if $|q| < 1$, the series (1) converges absolutely for $|z| < 1$ and $r = s + 1$. For more mathematical background of these functions, one may refer to [8].

Corresponding to a function ${}_r\mathcal{G}_s(a_i; b_j; q, z)$ defined by

$$ {}_r\mathcal{G}_s(a_i; b_j; q, z) = z \quad {}_r\Phi_s(a_i; b_j; q, z). \tag{2} $$

Recently, the authors [7] defined the linear operator $\mathcal{M}(a_i, b_j; q)f : \mathcal{A} \longrightarrow \mathcal{A}$ by

$$ \begin{aligned} \mathcal{M}(a_i, b_j; q)f(z) &= {}_r\mathcal{G}_s(a_i; b_j; q, z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \Upsilon_n c_n z^n, \end{aligned} \tag{3} $$

where

$$ \Upsilon_n = \frac{(a_1, q)_{n-1} ... (a_r, q)_{n-1}}{(q, q)_{n-1} (b_1, q)_{n-1} ... (b_s, q)_{n-1}}, (|q| < 1) $$

It should be remarked that the linear operator (3) is a generalization of many operators considered earlier. For $a_i = q^{\alpha_i}, b_j = q^{\beta_j}, \alpha_i, \beta_j \in \mathbb{C}, \beta_j \neq 0, -1, -2, ..., (i = 1, ..., r, j = 1, .., s)$ and $q \to 1$, we obtain the Dziok-Srivastava linear operator [2] (for $r = s + 1$), so that it includes (as its special cases) various other linear operators introduced and studied by Ruscheweyh [6], Carlson-Shaffer [9] and Bernardi-Libera-Livingston operators ([10]-[11]-[12]).

The q-difference operator is defined by

$$ d_q h(z) = \frac{h(qz) - h(z)}{(q - 1)z}, q \neq 1, z \neq 0, \tag{4} $$

and

$$ \lim_{q \to 1} d_q h(z) = h'(z), $$

where $h'(z)$ is the ordinary derivative. For more properties of d_q see ([3]-[4]).

2. Findings

The following lemma is crucial in the studies.

Lemma 2.1. (see [7]). *Let $f \in \mathcal{A}$, then*

- **i:** *For $r = 1, s = 0$ and $a_1 = q$, we have $\mathcal{M}(q, -; q)f(z) = f(z)$.*
- **ii:** *For $r = 1, s = 0$, and $a_1 = q^2$, we have $\mathcal{M}(q^2, -; q)f(z) = z d_q f(z)$ and $\lim_{q \to 1} \mathcal{M}(q^2, -; q)f(z) = z f'(z)$, where d_q is the q-derivative defined by (4)*

Aldweby and Darus [13] stated the following:

Definition 2.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathfrak{B}_s^r(a_i, b_j; q; \mu)$ if it is satisfying the condition

$$\left| \frac{z^2(\mathcal{M}(a_i, b_j; q)f(z))'}{[\mathcal{M}(a_i, b_j; q)f(z)]^2} - 1 \right| < 1 - \mu \quad (z \in \mathcal{U}; 0 \leq \mu < 1), \tag{5}$$

where $\mathcal{M}(a_i, b_j; q)f$ is the operator defined by (3).

Note that $\mathfrak{B}_0^1(q, -; q; \mu) = \mathfrak{B}(\mu)$, where the class $\mathfrak{B}(\mu)$ of analytic and univalent functions was introduced and studied by Frasin and Darus [5].

Using the operator $\mathcal{M}(a_i, b_j; q)f(z)f$, Aldweby and Darus [13] stated the following new integral operator:

For $m \in \mathbb{N} \cup \{0\}$, $\gamma_1, \gamma_2, ..., \gamma_m, \delta \in \mathbb{C} \backslash \{0, -1, -2, ...\}$, and $|q| < 1$ we define the integral operator $I_{\gamma_k,\delta}(a_i, b_j; q; z) : \mathcal{A}^n \longrightarrow \mathcal{A}^n$ by

$$I_{\gamma_k,\delta}(a_i, b_j; q; z) = \left(\delta \int_0^z t^{\delta-1} \prod_{k=1}^m \left(\frac{\mathcal{M}(a_i, b_j; q)f(z)f_k(t)}{t} \right)^{\frac{1}{\gamma_k}} dt \right)^{\frac{1}{\delta}}, \tag{6}$$

where $f_k \in \mathcal{A}$.

It is interesting to note that the integral operator $I_{\gamma_k,\delta}(a_i, b_j; q; z)$ generalizes many operators introduced and studied by several authors (see in [13]).

An example of a result obtained for this integral operator can be read in [13] as follows:

Theorem 2.1. *Let $f_k \in \mathcal{A}$ for all $k = 1, ..., m, \gamma_k \in \mathcal{C}$ and $M \geq 1$ with*

$$\frac{1}{Re(\delta)} \sum_{k=1}^m \frac{[(2 - \mu_k)M + 1]}{|\gamma_k|} \leq 1.$$

If for all $k = 1, ..., m, \quad f_k \in \mathfrak{B}_s^r(a_i, b_j, q, \mu_k), 0 \leq \mu_k < 1$, and

$$|\mathcal{M}(a_i, b_j; q)f(z)f_k(z)| \leq M, \quad (z \in \mathcal{U})$$

then the integral operator $I_{\gamma_k,\delta}(a_i, b_j; q; z)$ defined by (6) is analytic and univalent in $\mathcal{U}$.

Other studies that we are looking at regarding this matter include introducing new classes (starlike-convex-close-to-convex) in the space of analytic functions by using new q-operators and investigating their properties.

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