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ON A NEW SUBCLASS OF HARMONIC MEROMORPHIC FUNCTIONS WITH FIXED RESIDUE ξ

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Abstract. We use the differential operator $D^{n,\mu}_{\lambda,\delta,\varphi}$ to introduce a new class $SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}(w,k,\alpha)$ of meromorphic harmonic functions with fixed residue ξ in U_w. Then we give the coefficient estimates, distortion theorem and extreme points of classes $SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}(w,k,\alpha)$ and $SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}[w,k,\alpha]$.

Keywords: Harmonic functions, Meromorphic functions, Differential operator.

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1. Introduction

$f = u + iv$ is a complex harmonic function in a domain D if both u and v are real continuous harmonic functions in D. In any simply connected domain $D \subset \mathbb{C}$, f is written in the form of $f = h + \overline{g}$, where both h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'| > |g'|$ [3].There are many papers on harmonic functions defined on the domain $U = \{z : |z| < 1\}$ [1,4,5,6].

For $0 \le w < 1$, we let $SH(w)$ denote the class of functions harmonic univalent, orientation preserving and meromorphic in U, with $\lim_{z\to w} f(z) = \infty$ which are the representation

$$f(z) = h(z) + \overline{g(z)} + Alog|z - w| \tag{1}$$

where

$$h(z) = \frac{\xi}{z-w} + \sum_{k=1}^{\infty} c_k z^k \qquad and \qquad g(z) = \sum_{k=1}^{\infty} d_k z^k \tag{2}$$

and $\xi = Res(f, w)$ with $0 < \xi \le 1$, $z \in U \setminus \{w\}$ or we may set for $z \in U_w = \{z : 0 < |z - w| < 1 - w\}$

$$h(z) = \frac{\xi}{z-w} + \sum_{k=1}^{\infty} a_k (z-w)^k \qquad and \qquad g(z) = \sum_{k=1}^{\infty} b_k (z-w)^k. \tag{3}$$

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We further remove the logarithmic singularity by letting $A = 0$ and focus the subclass $SH[w]$ of all harmonic, orientation preserving, and meromorphic mappings which have the development

$$f(z) = h(z) + \overline{g(z)} \tag{4}$$

where

$$h(z) = \frac{\xi}{z-w} + \sum_{k=1}^{\infty} c_k z^k \qquad and \qquad g(z) = \sum_{k=1}^{\infty} d_k z^k, \qquad c_k, d_k \geq 0; z \in U \setminus \{w\} \tag{5}$$

or we may set for $z \in U_w = \{z : 0 < |z-w| < 1-w\}$

$$h(z) = \frac{\xi}{z-w} + \sum_{k=1}^{\infty} a_k (z-w)^k \qquad and \qquad g(z) = \sum_{k=1}^{\infty} b_k (z-w)^k, \qquad a_k, b_k \geq 0 \tag{6}$$

where $h(z)$ has a simple pole at the point w with residue ξ . For $\xi = 1$ and $w = 0$ the function f was studied by Bostancı, Yalçın and Öztürk [2].

For the function f in the class $SH(w)$, we define the following $D^{n,\mu}_{\lambda,\delta,\varphi}$ operator, for $0 \leq \alpha < 1; \lambda, \delta, \varphi, \mu \geq 0; \lambda > \delta; \varphi > \mu$ and $0 \leq w < 1$ where $\xi = Res(f,w)$ with $0 < \xi \leq 1$, $z \in U_w$.

$$D^{0,\mu}_{\lambda,\delta,\varphi} f(z) = f(z)$$

$$D^{n,\mu}_{\lambda,\delta,\varphi} f(z) = D^{n,\mu}_{\lambda,\delta,\varphi} h(z) + \overline{D^{n,\mu}_{\lambda,\delta,\varphi} g(z)}, \qquad n = 1, 2, 3, ... \tag{7}$$

where

$$D^{n,\mu}_{\lambda,\delta,\varphi} h(z) = [1-(\lambda-\delta)(\varphi-\mu)](D^{n-1,\mu}_{\lambda,\delta,\varphi} h(z)) + (\lambda-\delta)(\varphi-\mu)(z-w)(D^{n-1,\mu}_{\lambda,\delta,\varphi} h(z))' + \frac{2\xi(\lambda-\delta)(\varphi-\mu)}{z-w}$$

$$= \frac{\xi}{z-w} + \sum_{k=1}^{\infty} [1 + (\lambda-\delta)(\varphi-\mu)(k-1)]^n a_k (z-w)^k$$

and

$$D^{n,\mu}_{\lambda,\delta,\varphi} g(z) = (z-w)(D^{n-1,\mu}_{\lambda,\delta,\varphi} g(z))' = \sum_{k=1}^{\infty} [1+(\lambda-\delta)(\varphi-\mu)(k-1)]^n b_k (z-w)^k.$$

So, we can define the class $SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}(w,k,\alpha)$ with the help of the differential operator $D^{n,\mu}_{\lambda,\delta,\varphi}$ as follows:

A function f in $SH(w)$ is in the class $SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}(w,k,\alpha)$ if it satisfies the following inequality

$$\left| \frac{(z-w)^2 (D^{n,\mu}_{\lambda,\delta,\varphi} f(z))' + 1}{(2\gamma-1)(z-w)^2 (D^{n,\mu}_{\lambda,\delta,\varphi} f(z))' + (2\gamma\alpha-1)} \right| < \beta \tag{8}$$

where $0 \leq \alpha < 1; \frac{1}{2} \leq \gamma \leq 1; 0 < \beta \leq 1; \lambda, \delta, \varphi, \mu \geq 0; \lambda > \delta; \varphi > \mu$ and $0 \leq w < 1$ where $\xi = Res(f,w)$ with $0 < \xi \leq 1$, $z \in U \setminus \{w\}$.

Let us write

$$SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}[w,k,\alpha] = SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}(w,k,\alpha) \cap SH[w] \tag{9}$$

where $SH[w]$ is the class of functions of the form (4) and (6) that are meromorphic and harmonic in U_w.

In the present paper, we give some important results as coefficient estimates, distortion bounds, extreme points for the classes $SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}(w,k,\alpha)$ and $SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}[w,k,\alpha]$.

2. Coefficients Estimates

Now, we obtain coefficient inequalities for a function in the class $SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}(w,k,\alpha)$.

Theorem 2.1. *A function $f(z) = h(z) + \overline{g(z)}$ where $h(z)$ and $g(z)$ are defined by (3) is in the class $SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}(w,k,\alpha)$ if and only if*

$$\sum_{k=1}^{\infty} k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(1+2\beta\gamma-\beta)(|a_k|+|b_k|) \le 2\beta\gamma(\xi-\alpha)-(1-\xi)(1-\beta) \quad (10)$$

for $0 \le \alpha < 1$, $0 < \beta \le 1$, $\frac{1}{2} \le \gamma \le 1$.

Proof. Suppose (10) holds. Consider the expression

$$\left|(z-w)^2(D^{n,\mu}_{\lambda,\delta,\varphi}f(z))'+1\right| - \beta\left|(2\gamma-1)(z-w)^2(D^{n,\mu}_{\lambda,\delta,\varphi}f(z))'+(2\gamma\alpha-1)\right| < 0$$

provided

$$\left|(1-\xi)+\sum_{k=1}^{\infty} k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(a_k+b_k)(z-w)^{k+1}\right|$$

$$-\beta\left|-\xi(2\gamma-1)+(2\gamma\alpha-1)+\sum_{k=1}^{\infty} k(2\gamma-1)[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(a_k+b_k)(z-w)^{k+1}\right| < 0$$

for $|z-w| = r < 1-w$

$$< (1-\xi)+\sum_{k=1}^{\infty} k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(|a_k|+|b_k|)r^{k+1} - 2\xi\beta\gamma+\xi\beta+2\beta\gamma\alpha-\beta$$

$$+\beta\sum_{k=1}^{\infty} k(2\gamma-1)[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(|a_k|+|b_k|)r^{k+1}$$

$$= \sum_{k=1}^{\infty} k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(1+2\beta\gamma-\beta)(|a_k|+|b_k|)r^{k+1} - 2\beta\gamma(\xi-\alpha)+(1-\xi)(1-\beta) \le 0. \quad (11)$$

The inequality in (11) holds true for all $|z-w| = r < 1-w < 1$. Therefore, letting $r \to 1$ in (11), we obtain

$$\sum_{k=1}^{\infty} k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(1+2\beta\gamma-\beta)(|a_k|+|b_k|) \le 2\beta\gamma(\xi-\alpha)-(1-\xi)(1-\beta).$$

Hence $f(z) \in SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}(w,k,\alpha)$.

□

Next, we give a necessary and sufficient condition for a function $f(z) \in SH(w)$ to be in the class $SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}[w,k,\alpha]$.

Theorem 2.2. *Let $f(z) \in SH(w)$ be a function defined by (4) and (6). Then $f(z) \in SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}[w,k,\alpha]$ if and only if the inequality*

$$\sum_{k=1}^{\infty} k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(1+2\beta\gamma-\beta)(a_k+b_k) \leq 2\beta\gamma(\xi-\alpha)-(1-\xi)(1-\beta)$$

is satisfied.

Proof. In view of Theorem 2.1, we only need to prove the "only if part" of the theorem. Assume that $f(z) \in SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}[w,k,\alpha]$. Then

$$\left| \frac{(z-w)^2(D^{n,\mu}_{\lambda,\delta,\varphi} f(z))' + 1}{(2\gamma-1)(z-w)^2(D^{n,\mu}_{\lambda,\delta,\varphi} f(z))' + (2\gamma\alpha-1)} \right| < \beta$$

$$= \left| \frac{(1-\xi)+\sum_{k=1}^{\infty} k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(a_k+b_k)(z-w)^{k+1}}{\xi(2\gamma-1)-\sum_{k=1}^{\infty} k(2\gamma-1)[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(a_k+b_k)(z-w)^{k+1}-(2\gamma\alpha-1)} \right| \leq \beta, (z \in U_w).$$

Using the fact that $Re(z) \leq |z|$ for all z, we obtain

$$= Re\left\{ \frac{(1-\xi)+\sum_{k=1}^{\infty} k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(a_k+b_k)(z-w)^{k+1}}{2\gamma(\xi-\alpha)+(1-\xi)-\sum_{k=1}^{\infty} k(2\gamma-1)[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(a_k+b_k)(z-w)^{k+1}} \right\} < \beta. \tag{12}$$

Now choose the values of z on the real axis. Upon clearing the denominator in (12) and letting $(z-w) \to 1^-$ through positive values, we obtain

$$\sum_{k=1}^{\infty} k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(1+2\beta\gamma-\beta)(a_k+b_k) \leq 2\beta\gamma(\xi-\alpha)-(1-\xi)(1-\beta)$$

So, the proof is completed. □

3. Distortion Theorem

Distortion property for function f to be in the class $SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}[w,k,\alpha]$ is given as follows.

Theorem 3.1. *If f be of the form (4) and (6) is in the class, $SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}[w,k,\alpha]$ then, for $|z-w| = r < 1-w$*

$$\frac{\xi}{r} - \frac{2\beta\gamma(\xi-\alpha)-(1-\xi)(1-\beta)}{k(1+2\beta\gamma-\beta)} r^2 \leq |f(z)| \leq \frac{\xi}{r} + \frac{2\beta\gamma(\xi-\alpha)-(1-\xi)(1-\beta)}{k(1+2\beta\gamma-\beta)} r^2 \tag{13}$$

Proof. Let $SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}[w,k,\alpha]$. We obtain

$$|f(z)| = \left| \frac{\xi}{z-w} + \sum_{k=1}^{\infty} a_k(z-w)^k + \overline{\sum_{k=1}^{\infty} b_k(z-w)^k} \right|$$

$$\geq \frac{1}{|z-w|}\left[\xi - |z-w| \sum_{k=1}^{\infty}(a_k+b_k)|z-w|^k \right]$$

$$\geq \frac{1}{r}\left[\xi - r^2 \sum_{k=1}^{\infty}(a_k+b_k) \right]$$

$$\geq \frac{\xi}{r} - \frac{2\beta\gamma(\xi-\alpha)-(1-\xi)(1-\beta)}{k(1+2\beta\gamma-\beta)} \sum_{k=1}^{\infty} \frac{k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(1+2\beta\gamma-\beta)}{2\beta\gamma(\xi-\alpha)-(1-\xi)(1-\beta)}(a_k+b_k)r^2$$

$$\geq \frac{\xi}{r} - \frac{2\beta\gamma(\xi-\alpha)-(1-\xi)(1-\beta)}{k(1+2\beta\gamma-\beta)} r^2.$$

The other side is similar. The bound (13) is attained for the function $f(z)$ given by

$$\begin{cases} f(z) &= \frac{\xi}{z-w} + \frac{2\beta\gamma(\xi-\alpha)-(1-\xi)(1-\beta)}{k(1+2\beta\gamma-\beta)}(z-w)^2 \\ f(z) &= \frac{\xi}{z-w} + \frac{2\beta\gamma(\xi-\alpha)-(1-\xi)(1-\beta)}{k(1+2\beta\gamma-\beta)}\overline{(z-w)^2} \end{cases} \tag{14}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, $\frac{1}{2} \leq \gamma \leq 1$ and $0 \leq w < 1$ where $\xi = Res(f, w)$ with $0 < \xi \leq 1$, $z \in U_w$.

These bounds are sharp. □

4. Extreme Points

Now, we determine the extreme points of the closed convex hull of $SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}[w,k,\alpha]$ denoted by $clcoSH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}[w,k,\alpha]$.

Theorem 4.1. *$f \in SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}[w,k,\alpha]$ if and only if f can be expressed as*

$$f(z) = \sum_{k=0}^{\infty}[X_k h_k(z) + Y_k g_k(z)] \tag{15}$$

where

$$h_0(z) = \frac{\xi}{z-w}, \qquad g_0(z) = 0,$$

$$h_k(z) = \frac{\xi}{z-w} + \frac{2\beta\gamma(\xi-\alpha)-(1-\xi)(1-\beta)}{k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(1+2\beta\gamma-\beta)}(z-w)^k \quad for \quad k = 1,2,3,..., \quad and$$

$$g_k(z) = \frac{2\beta\gamma(\xi-\alpha)-(1-\xi)(1-\beta)}{k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(1+2\beta\gamma-\beta)}\overline{(z-w)^k} \quad for \quad k = 1,2,3,...,$$

$$X_k \geq 0, \qquad Y_k \geq 0 \qquad and \qquad \sum_{k=0}^{\infty}(X_k+Y_k) = 1.$$

In particular, the extreme points of $SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}[w,k,\alpha]$ are $\{h_k\}$ and $\{g_k\}$, $(k = 0,1,2,...)$.

Proof. Note that, for the functions f of the form (15), we can write

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty}[X_k h_k(z)+Y_k g_k(z)] \\ &= \sum_{k=0}^{\infty}(X_k+Y_k)\frac{\xi}{z-w} + \sum_{k=1}^{\infty}\frac{2\beta\gamma(\xi-\alpha)-(1-\xi)(1-\beta)}{k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(1+2\beta\gamma-\beta)}X_k(z-w)^k \\ &\quad + \sum_{k=0}^{\infty}\frac{2\beta\gamma(\xi-\alpha)-(1-\xi)(1-\beta)}{k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(1+2\beta\gamma-\beta)}Y_k\overline{(z-w)^k}. \end{aligned}$$

Then

$$\sum_{k=1}^{\infty} k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(1+2\beta\gamma-\beta)\frac{X_k}{k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(1+2\beta\gamma-\beta)}$$

$$+\sum_{k=0}^{\infty} k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(1+2\beta\gamma-\beta)\frac{Y_k}{k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(1+2\beta\gamma-\beta)}$$

$$= \sum_{k=1}^{\infty}(X_k + Y_k) - X_0 = 1 - X_0 \leq 1$$

So, $f \in SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}[w,k,\alpha]$ and completed the first part of proof. Conversely, suppose that $f \in SH^{n,\gamma,\beta,\xi}_{\lambda,\delta,\varphi,\mu}[w,k,\alpha]$. Set

$$X_k = \frac{k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(1+2\beta\gamma-\beta)}{2\beta\gamma(\xi-\alpha)-(1-\xi)(1-\beta)}a_k, \qquad k \geq 1, and$$

$$Y_k = \frac{k[1+(k-1)(\lambda-\delta)(\varphi-\mu)]^n(1+2\beta\gamma-\beta)}{2\beta\gamma(\xi-\alpha)-(1-\xi)(1-\beta)}b_k, \qquad k \geq 0,$$

$0 \leq X_k \leq 1$ $(k \geq 1)$ and $0 \leq Y_k \leq 1$ $(k \geq 0)$. We define

$$X_0 = 1 - \sum_{k=1}^{\infty} X_k - \sum_{k=0}^{\infty} Y_k \qquad and \qquad X_0 \geq 0.$$

Consequently we obtain equality as follows,

$$f(z) = \sum_{k=0}^{\infty}[X_k h_k(z) + Y_k g_k(z)]$$

and hence this completes the proof of Theorem 4.1. □

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