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FIXED POINTS OF CONTRACTIVE SET VALUED MAPPINGS WITH SET VALUED DOMAINS ON A METRIC SPACE WITH GRAPH

P. DEBNATH[1], §

Abstract. In this article we consider general contractive mappings of the form $F : CB(X) \to CB(X)$, where $CB(X)$ is the set of all nonempty closed and bounded subsets of a complete metric space X endowed with a graph G. We prove some fixed point results for F and discuss how the connectivity of the graph G is related to the fixed points of F.

Keywords: Fixed point; Connected graph; G-contraction; Metric space.

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1. Introduction

Some important works on fixed point theorems for set valued and multivalued contraction mappings were carried out by Nadler [8] and Assad and Kirk [2]. The concepts of fixed point theory and graph theory were combined by Espinola and Kirk [4] to prove some interesting fixed point theorems in R-trees. Jachymski [5, 7] used similar type of combination to extend the works of Ran and Reurings [12], Nieto and Rodríguez-López [10], Petruşel and Rus [11], Nieto, Pouso and Rodríguez-López [9] on fixed point theory. Some fixed point results on a metric space with a graph has been recently investigated by Beg et al. [3] and Aleomraninejad et al. [1].

The results on set-valued and multi-valued contractions on a metric space with graph are generalizations of their single-valued analogues. An example in this direction is the work of Beg et al. [3]. But to obtain similar results as that of single-valued contractions, the definition of a contraction map $F : X \to CB(X)$ (see [3]) was slightly modified than its single-valued counterpart given by Jachymski [7].

Our motivation and objective for considering mappings $F : CB(X) \to CB(X)$ instead of $F : X \to X$ or $F : X \to CB(X)$ is to investigate the necessary changes that have to be made in the definition of a contraction map in this new setting in order to obtain similar results. Also we investigate how the proofs of our results are influenced due to these changes. Knowledge of such generalized results can help us to study the convergence of some iterative scheme to the unique fixed point in this context.

For graph theoretic notations and terminologies, the readers are referred to Hararay [9].

In a metric space (X, d), two sequences $\{x_n\}$ and $\{y_n\}$ are said to be equivalent if $d(x_n, y_n) \to 0$ and they are said to be Cauchy equivalent if they are equivalent as well as Cauchy.

[1] Department of Mathematics, National Institute of Technology Silchar, Assam-788010, India.
e-mail: debnath.pradip@yahoo.com;

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Let (X, d) be a complete metric space and $CB(X)$ be the class of all nonempty closed and bounded subsets of X. For $A, B \in X$, let

$$D(A,B) = \max\{\sup_{b\in B} d(b,A), \sup_{a\in A} d(a,B)\},$$

where $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(CB(X), D)$ is a metric space and D is said to be a Hausdorff metric induced by d.

Let (X, d) be a metric space and $\Delta = \{(x,x) : x \in X\}$ denote the diagonal of the Cartesian product $X \times X$. Consider a directed graph G such that the set of its vertices coincides with X (i.e., $V(G) = X$) and the set of its edges $E(G)$ is such that $\Delta \subseteq E(G)$. We assume G has no parallel edges and thus we identify G with the pair $(V(G), E(G))$.

G^{-1} denotes the conversion of a graph G, the graph obtained from G by reversing the direction of edges of G. $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the directions of the edges of G. We consider G as a directed graph whose set of edges is symmetric, thus we have

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

If $x, y \in V(G)$, then a path in G from x to y is a sequence $\{x_i\}_{i=0}^n$ of vertices such that $x_0 = x$, $x_n = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, 2, \ldots, n$.

A graph G is connected if there is a path between any two vertices of G. G is said to be weakly connected if $\widetilde{G}$ is connected. We call a subset $A \subset X$ locally connected if there exists a path between any two points of A. By convention we say that every singleton set in X is locally connected because $\Delta \subset E(G)$.

If G is such that $E(G)$ is symmetric and x is a vertex in G, then the subgraph G_x consisting of all edges and vertices which are contained in some path beginning at x is called the component of G containing x. In this case $V(G_x) = [x]_G$, where $[x]_G$ is the equivalence class of the following relation $\boldsymbol{P}$ defined on $V(G)$ by the rule:

$$y\boldsymbol{P}z \text{ if there is a path in } G \text{ from } y \text{ to } z.$$

2. Fixed points of contractive set valued mappings with set valued domains

Let $A, B \subseteq X$. By the statement 'there is an edge between A and B', we mean there is an edge between some $x \in A$ and $y \in B$. Again the meaning of the statement 'there is a path between A and B' is that there is a path between some $x \in A$ and $y \in B$.

In $CB(X)$ we define a relation $\boldsymbol{R}$ in the following way:

For $A, B \in CB(X)$, $A\boldsymbol{R}B \Leftrightarrow$ there is a path between A and B.

For $A \in CB(X)$, the equivalence class of $\boldsymbol{R}$ is denoted and defined by

$$[A]_G = \{B \subseteq X : A\boldsymbol{R}B\}.$$

Remark 2.1. *To study the mappings $F : CB(X) \to CB(X)$ there can be two approaches: (i) We can equip $CB(X)$ with a graph and (ii) First we equip X with a graph and then consider the induced graph structure on $CB(X)$. In the present paper we have adopted the second approach. And in this case, in fact, both the approaches are equivalent because of the definition of the relation R on $CB(X)$. Thus if we equip $CB(X)$ with a graph, this in turn equips X with the same graph and conversely.*

For the rest of this paper, by X we mean a complete metric space (X, d) unless otherwise stated.

Definition 2.1. *Let $F : CB(X) \to CB(X)$ be a set valued mapping with set valued domain. The mapping F is said to be a G-contraction if the following conditions hold:*

(*i*) *There is an edge between A and $B \Rightarrow$ there is an edge between $F(A)$ and $F(B)$ for all $A, B \in CB(X)$.*

(*ii*) *There is a path between A and $B \Rightarrow$ there is a path between $F(A)$ and $F(B)$ for all $A, B \in CB(X)$.*

(*iii*) *There exists $k \in (0,1)$ such that there is an edge between A and $B \Rightarrow D(F(A), F(B)) \leq kD(A, B)$ for all $A, B \in CB(X)$.*

Definition 2.2. *Let $F : CB(X) \to CB(X)$ be a set valued mapping with set valued domain. $A \in CB(X)$ is said to be a fixed point of F if $F(A) = A$.*

The set of all fixed points of F is denoted by $Fix\ F$.

Example 2.1. (*i*) *Any constant function $F : CB(X) \to CB(X)$ is a G-contraction for $\Delta \subset E(G)$.*

(*ii*) *Any G-contraction is a G_0-contraction, where the graph G_0 is defined by $E(G_0) = X \times X$.*

The following proposition follows immediately from the symmetry of D and the definition of $\widetilde{G}$.

Proposition 2.1. *If $F : CB(X) \to CB(X)$ is a G-contraction, then F is both a $\widetilde{G}$-contraction and a G^{-1}-contraction.*

Lemma 2.1. *Let $F : CB(X) \to CB(X)$ be a G-contraction with constant $k \in (0,1)$. Then given $A \in CB(X)$ and $B \in [A]_{\tilde{G}}$, there exists $r(A, B) > 0$ such that*

$$D(F^n(A), F^n(B)) \leq k^n r(A, B) \text{ for all } n \in \mathbb{N}.$$

Proof. Let $A \in CB(X)$ and $B \in [A]_{\tilde{G}}$. So, there exists a path $(x_i)_{i=0}^n$ from x to y for some $x \in A$ and $y \in B$, i.e., $x_0 = x$ and $x_n = y$ and $(x_{i-1}, x_i) \in E(\widetilde{G})$ for $i = 1, 2, \ldots, n$ such that $x_0 \in A_0 = A$, $x_1 \in A_1$, $\ldots$, $x_n \in A_n = B$, where each $A_i \in CB(X)$. By Proposition 2.1, F is a $\widetilde{G}$-contraction.

Thus for $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} D(F(A_{i-1}), F(A_i)) &\leq kD(A_{i-1}, A_i); \\ D(F^2(A_{i-1}), F^2(A_i)) &\leq kD(F(A_{i-1}), F(A_i) \\ &\leq k^2 D(A_{i-1}, A_i); \end{aligned}$$

and continuing this way, we obtain

$$D(F^n(A_{i-1}), F^n(A_i)) \leq k^n D(A_{i-1}, A_i).$$

Now by triangle inequality, we have

$$\begin{aligned} D(F^n(A), F^n(B)) &\leq \sum_{i=1}^{n} D(F^n(A_{i-1}), F^n(A_i)) \\ &\leq k^n \sum_{i=1}^{n} D(A_{i-1}, A_i) \\ &= k^n r(A, B), \text{ where } r = \sum_{i=1}^{n} D(A_{i-1}, A_i). \end{aligned}$$

□

The next theorem illustrates how the connectivity of the graph G is related to the fixed points of F.

Theorem 2.1. *The following statements are equivalent:*

(i) *G is weakly connected.*

(ii) *For any G-contraction $F : CB(X) \to CB(X)$, given $A, B \in CB(X)$, the sequences $\{F^n(A)\}$ and $\{F^n(B)\}$ are Cauchy equivalent.*

(iii) *For any G-contraction $F : CB(X) \to CB(X)$, $card(Fix\ F) \leq 1$.*

Proof. (i) $\Rightarrow$ (ii): Let F be a G-contraction and $A, B \in CB(X)$. By the hypothesis, $CB(X) \subseteq [A]_{\widetilde{G}} = P(X)$, where $P(X)$ denotes the power set of X and so, $F(A) \in [A]_{\widetilde{G}}$. By Lemma 2.1, we have

$$D(F^n(A), F^{n+1}(A)) \leq k^n r(A, F(A)) \text{ for all } n \in \mathbb{N}.$$

Hence $\sum_{n=0}^{\infty} D(F^n(A), F^{n+1}(A)) < \infty$ and as such, $D(F^n(A), F^{n+1}(A)) \to 0$. Thus $\{F^n(A)\}$ is a Cauchy sequence. Also since $B \in [A]_{\widetilde{G}}$, again by Lemma 2.1, we have

$$D(F^n(A), F^n(B)) \leq k^n r(A, B) \text{ for all } n \in \mathbb{N}.$$

Therefore, $\sum_{n=0}^{\infty} D(F^n(A), F^n(B)) < \infty$ and so, $D(F^n(A), F^n(B)) \to 0$. Thus the sequences $\{F^n(A)\}$ and $\{F^n(B)\}$ are equivalent. Since $\{F^n(A)\}$ is Cauchy, it is obvious that $\{F^n(B)\}$ is Cauchy as well.

(ii) $\Rightarrow$ (iii): Let F be a G-contraction and $A, B \in Fix\ F$. By (ii) we have, $D(F^n(A), F^n(B)) \to 0$. But $F(A) = A$ and $F(B) = B$ and therefore, we must have $A = B$.

(iii) $\Rightarrow$ (i): Let $\widetilde{G}$ be not connected. Let $A_0 \in CB(X)$ such that $A_0 = [x_0]_{\widetilde{G}}$ for some fixed $x_0 \in X$. Then both $[A_0]_{\widetilde{G}}$ and $CB(X) \setminus [A_0]_{\widetilde{G}}$ are nonempty.

Let $B_0 \in CB(X) \setminus [A_0]_{\widetilde{G}}$ such that there is a path between any two points of B_0 (if B_0 is a singleton set, we can use the fact that $\Delta \subseteq E(G)$). Define $H : CB(X) \to CB(X)$ by

$$H(U) = \begin{cases} A_0 & \text{if } U \in [A_0]_{\widetilde{G}} \\ B_0 & \text{if } U \in CB(X) \setminus [A_0]_{\widetilde{G}}. \end{cases}$$

Clearly, $Fix\ H = \{A_0, B_0\}$. We show that H is a G-contraction. Let $A, B \in [A_0]_{\widetilde{G}}$ such that there is an edge (path) between A and B. Then $[A]_{\widetilde{G}} = [B]_{\widetilde{G}}$, so either $A, B \in [A_0]_{\widetilde{G}}$ or $A, B \in CB(X) \setminus [A_0]_{\widetilde{G}}$. In both the cases we have $F(A) = F(B)$. Thus there exists an edge (path) between $H(A)$ and $H(B)$ for A_0 and B_0 both are locally connected.

Also we have $D(F(A), F(B)) = 0 \leq \frac{1}{2} D(A, B)$. Thus H is a G-contraction having two fixed points which violates (iii). $\square$

The following is an immediate consequence of Theorem 2.1.

Corollary 2.1. *Let $(CB(X), D)$ be complete. The following statements are equivalent:*

(i) *G is weakly connected.*

(ii) *For any G-contraction $F : CB(X) \to CB(X)$, there is $U \in CB(X)$ such that $\lim_{n\to\infty} F^n(Y) = U$ for all $Y \in CB(X)$.*

Theorem 2.2. *Let $F : CB(X) \to CB(X)$ be a G-contraction and $A_0 \in CB(X)$ such that $x_0 \in A_0$ for some $x_0 \in X$. Let $F(A_0) \in [A_0]_{\widetilde{G}}$ and $\widetilde{G}_{x_0}$ be the component of $CB(X)$ containing x_0. Then $[A_0]_{\widetilde{G}}$ is F-invariant and $F|_{[A_0]_{\widetilde{G}}}$ is a $\widetilde{G}_{x_0}$-contraction. Also if $U, V \in [A_0]_{\widetilde{G}}$, then the sequences $(F^n(U))$ and $(F^n(V))$ are Cauchy equivalent in the metric space $(CB(X), D)$.*

Proof. Let $A \in [A_0]_{\widetilde{G}}$. Then there exists a path $(x_i)_{i=0}^n$ in $\widetilde{G}$ from A_0 to A such that $x_0 \in A_0$, $x_1 \in A_1, \ldots, x_n \in A_n = A$, where each $A_i \in CB(X)$. Since F is a $\widetilde{G}$-contraction (by Proposition 2.1), clearly, there is a path from $F(A_0)$ to $F(A)$. Thus $F(A) \in [F(A_0)]_{\widetilde{G}}$. Since by hypothesis, $F(A_0) \in [A_0]_{\widetilde{G}}$, i.e., $[F(A_0)]_{\widetilde{G}} = [A_0]_{\widetilde{G}}$, we have that $F(A) \in [A_0]_{\widetilde{G}}$. Thus $[A_0]_{\widetilde{G}}$ is F-invariant.

Let $V(\widetilde{G}_{x_0}) = Y \subset X$. Clearly, $A_0 \in CB(Y)$. Consider $A, B \in CB(Y)$ such that there is an edge (path) between A and B. This implies that $A, B \in [A_0]_{\widetilde{G}}$. Since $[A_0]_{\widetilde{G}}$ is F-invariant, we have $F(A), F(B) \in [A_0]_{\widetilde{G}}$. Again using the fact that F is a $\widetilde{G}$-contraction, we have that there is an edge (path) between $F(A)$ and $F(B)$.

Moreover, since $E(\widetilde{G}_{x_0}) \subseteq E(\widetilde{G})$ and F is a $\widetilde{G}$-contraction, we must get a $k \in (0,1)$ such that

$$D(F(A), F(B)) \leq k(D(A, B)).$$

Thus $F|_{[A_0]_{\widetilde{G}}}$ is a $\widetilde{G}_{x_0}$-contraction.

Finally, since $\widetilde{G}_{x_0}$ is connected, from Theorem 2.1, it follows that if $U, V \in [A_0]_{\widetilde{G}}$, then the sequences $(F^n(U))$ and $(F^n(V))$ are Cauchy equivalent. □

Theorem 2.3. *Let $(CB(X), D)$ be complete with the following properties:*

(a) For any sequence $\{U_n\}$ in $CB(X)$, if $U_n \to U$ such that there is an edge between U_n and U_{n+1} for $n \in \mathbb{N}$, then there is a subsequence $\{U_{k_n}\}$ with an edge between U_{k_n} and U for $n \in \mathbb{N}$.

(b) The relation R on $CB(X)$ is transitive. In terms of the graph G it means that if there is a path between A & B and there is a path between B & C, then there is a path between A & C as well.

Let $F : CB(X) \to CB(X)$ be a G-contraction and

$$X_F = \{U \in CB(X) : \text{ there is an edge between } U \text{ and } F(U)\}.$$

Then the following statements hold:

(i) For any $U \in X_F$, $F|_{[U]_{\widetilde{G}}}$ has a unique fixed point.
(ii) If $X_F \neq \phi$ and G is weakly connected, then F has a unique fixed point.
(iii) If $X' = \{[U]_{\widetilde{G}} : U \in X_F\}$, then $F|_{X'}$ has a fixed point.
(iv) If $F \subseteq E(G)$ then F has a fixed point.
(v) $Fix\, F \neq \phi$ if and only if $X_F \neq \phi$.

Proof. (i) Let $U \in X_F$. Then $F(U) \in [U]_{\widetilde{G}}$. By Theorem 2.2, if $V \in [U]_{\widetilde{G}}$, then $\{F^n(U)\}$ and $\{F^n(V)\}$ are Cauchy equivalent. Since $(CB(X), D)$ is complete, we have $F^n(U) \to U^*$ for some $U^* \in CB(X)$. Clearly, $F^n(V) \to U^*$. Since there exists an edge between U and $F(U)$, the fact that F is a G-contraction yields there is an edge between $F^n(U)$ and $F^{n+1}(U)$ for all $n \in \mathbb{N}$.

By hypothesis, there exists a subsequence $\{F^{k_n}(U)\}$ such that there is an edge between $F^{k_n}(U)$ and U^* for every $n \in \mathbb{N}$. Using transitivity of the relation $\boldsymbol{R}$, we infer that there is a path in G (and hence also in $\widetilde{G}$) between U and U^*. Thus $U^* \in [U]_{\widetilde{G}}$.

Moreover, there exists $k \in (0,1)$ such that

$$D(F^{k_n+1}(U), F(U^*)) \leq kD(F^{k_n}(U), U^*).$$

But $F^{k_n}(U) \to U^*$ and so, $D(F^{k_n}(U), U^*) \to 0$. Thus $D(F^{k_n+1}(U), F(U^*)) \to 0$, i.e., $F^{k_n+1}(U) \to F(U^*) = U^*$. Clearly, U^* is the unique fixed point of $F|_{U_{\widetilde{G}}}$.

(ii) If G is weakly connected, then $[U]_{\widetilde{G}} = CB(X)$. Therefore, it follows from (i) that F has a unique fixed point.

(iii) It follows immediately from (i).

(iv) $F \subseteq E(G)$ means that all $U \in CB(X)$ are such that there exists an edge between U and $F(U)$. Therefore, $X' = CB(X)$. From (iii) it follows that F has a fixed point.

(v) Let $Fix\ F \neq \phi$. Then there exists $U \in CB(X)$ such that $F(U) = U$. Since $\Delta \subset E(G)$ and U is nonempty, we conclude that $X_F \neq \phi$. The converse follows from (ii) and (iii). □

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References

[1] Aleomraninejad, S. M. A., Rezapour, Sh., and Shahzad, N., (2012), Some fixed point results on a metric space with a graph, Topology Appl., 159, 659-663.
[2] Assad, N. A., and Kirk, W. A., (1972), Fixed point theorems for set-valued mappings of contractive type, Pacific J. Math. 43(3), 553-561.
[3] Beg, I., Butt, A. R., and Radojevic, S., (2010), The contraction principle for set valued mappings on a metric space with a graph, Comput. Math. Appl., 60, 1214-1219.
[4] Espinola, R., and Kirk, W. A., (2006), Fixed point theorems in R-trees with applications to graph theory, Topology Appl., 153, 1046-1055.
[5] Gwozdz-Lukawska, G., and Jachymski, J., (2009), IFS on a metric space with a graph structure and extensions of the Kelisky-Rivlin theorem, J. Math. Anal. Appl., 356, 453-463.
[6] Harary, F., (1972), Graph theory, 3rd Edition, Addison-Wesley, Reading, MA.
[7] Jachymski, J., (2007), The contraction principle for mappings on a metric space with a graph, Proc. Amer. Math. Soc., 136(4), 1359-1373.
[8] Nadler, S. B., (1969), Multi-valued contraction mappings, Pacific J. Math., 30(2), 475-488.
[9] Nieto, J. J., Pouso, R. L., and Rodriguez-Lopez, R., (2007), Fixed point theorems in ordered abstract spaces, Proc. Amer. Math. Soc., 135, 2505-2517.
[10] Nieto, J. J., and Rodriguez-Lopez, R., (2007), Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, Acta Math. Sinica English Ser., 2205-2212.
[11] Petrusel, A., and Rus, I. A., (2006), Fixed point theorems in ordered L-spaces, Proc. Amer. Math. Soc. 134, 411-418.
[12] Ran, A. C. M., and Reurings, M. C. B., (2004), A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc., 132, 1435-1443.

Pradip Debnath completed his Masters Degree in Mathematics from Assam University, Silchar, securing First Class First position, in the year 2007. In 2010, he obtained M.Phil. degree from Assam University, Silchar. He was awarded Ph.D. degree in Mathematics by NIT Silchar in the year 2013. He has published and reviewed several research articles for journals of international repute. His research interest includes Fuzzy Normed Linear Spaces, Fuzzy Graph Theory and Fixed Point Theory. Presently, he is working as an Assistant Professor in the Dept. of Mathematics, NIT Silchar, India.