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A WEIGHTED GOAL PROGRAMMING APPROACH TO FUZZY LINEAR REGRESSION WITH QUASI TYPE-2 FUZZY INPUT-OUTPUT DATA

E.HOSSEINZADEH[1], H.HASSANPOUR[1], M.AREFI[2], M.AMAN[1], §

Abstract. This study attempts to develop a regression model when both input data and output data are quasi type-2 fuzzy numbers. To estimate the crisp parameters of the regression model, a linear programming model is proposed based on goal programming. To handle the outlier problem, an omission approach is proposed. This approach examines the behavior of value changes in the objective function of proposed model when observations are omitted. In order to illustrate the proposed model, some numerical examples are presented. The applicability of the proposed method is tested on a real data set on soil science. The predictive performance of the model is examined by cross-validation.

Keywords: fuzzy linear regression, goal programming, type-2 fuzzy set, uncertainty, quasi type-2 fuzzy number.

AMS Subject Classification: 03E72, 90C05, 62J86, 62J05.

1. Introduction

Type-2 fuzzy logic is increasingly being advocated as a methodology for reasoning in situations where high uncertainties are present. Type-2 fuzzy sets were first presented by Zadeh in 1975 [42]. Mizumoto and Tanaka [27] further explored the logical operations of type-2 fuzzy sets. In the later 1990s and into the new millennium the definitions of logical operations for generalized type-2 fuzzy sets were completed [18, 24] and the first type-2 fuzzy logic textbook [23] published. Currently, the number of reported applications of interval type-2 fuzzy logic is growing year on year. The concept of type-2 fuzzy sets (T2FSs) was introduced by Zadeh [42] as an extension of the concept of ordinary fuzzy sets (henceforth called type-1 fuzzy sets). T2FSs are characterized by fuzzy membership functions (MF), i.e., the membership grades themselves are fuzzy sets in $[0, 1]$. They are very useful in circumstances where it is difficult to determine an exact membership function for a fuzzy set; hence, they are useful for incorporating linguistic uncertainties, e.g., the words that are used in linguistic knowledge which can mean different things to different people [22]. As a way of illustration, suppose a number of people are asked

[1] Faculty of Mathematical Science and Statistics, Department of Mathematics, University of Birjand, Birjand, I.R of Iran.
e-mail: e.hosseinzade@birjand.ac.ir, hhassanpour@birjand.ac.ir, mamann@birjand.ac.ir;
[2] Faculty of Mathematical Science and Statistics, Department of Statistics, University of Birjand, Birjand, I.R of Iran.
e-mail: Arefi@birjand.ac.ir;
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about the temperature of the room where they are present. All the subjects mention " approximately 70° Fahrenheit ". Nonetheless, if each individual subject is asked to show the " approximately 70° Fahrenheit MF ", different MFs are likely to be presented, even if the MFs are all of the same kind (e.g., triangular). This implies the statement that " Words can mean different things to different people "[24]. Accordingly, T2FSs prove helpful in cases where an exact form of an MF cannot be determined. Karnik and Mendel [18] introduced the centroid and generalized centroid of a type-II fuzzy set and explained how to calculate them. Furthermore, they showed how to compute the centroid of interval and Gaussian type-II fuzzy sets. Karnik and Mendel [25] discussed set theoretic operations for type-II sets, properties of membership grades of type-II fuzzy sets, and type-II relations and other compositions, and cartesian products under minimum and product t-norms. Mendel and John [25] defined a new representation theorem of type-II fuzzy sets and introduced formulas for the union, intersection, and complement for type-II fuzzy sets without applying Extension Principle, by using this new representation. Mendel [22] examined questions such as "What is a type-II fuzzy set?", "the importance of definition of type-II fuzzy sets", "How and why are type-II fuzzy sets used in rule-based systems?". Mendel [22] described the important advances that have been made during the past five years both general and interval type-II fuzzy sets and systems. Recently, Liu and Mendel [26] have presented the concept of quasi type-2 fuzzy system. This kind of system is a step forward to a more general kind of type-2 fuzzy system. The idea is developed on the basis of the alpha-level representation for general type-2 fuzzy sets. Quasi type-2 fuzzy systems become interesting since they should be more robust to uncertainty than their interval type-2 counterparts.

Regression analysis is one of the most widely used statistical techniques for determining the relationship between variables in order to describe or predict some stochastic phenomena. Research on fuzzy regression analysis began by Celmins [2], Diamond [5], and Tanaka et al. [37], and was continued by some authors, e.g., Kacprzyk and Fedrizzi [19]; Wang and Tsaur [38]; Kao and Chyu [17]; D'Urso [6]; Wang et al. [39]; Nasrabadi et al. [30]; Hojati et al. [14]; Modarres et al. [28]; Guo and Tanaka [7]; Coppi et al. [3]; Yao and Yu [41]; Hassanpour et al. [10, 12, 11, 13]; Ramli et al. [35]; Kelkinnama and Taheri [20]; Kocadagli [21]. But, based on our best knowledge, there has not been any research on regression analysis for quasi type-2 fuzzy data. Recently, Rabiei et al. [34] have proposed a Least-squares approach to regression modeling for interval-valued fuzzy data. Also, Poleshchuk and Komarov [33] have presented a regression model for interval type-2 fuzzy sets based on the least squares estimation technique.

The authors [15] have proposed a goal programming approach to calculate the regression coefficients of fuzzy linear regression model, when the inputs are crisp, but the outputs and regression coefficients are QT2FNs. In this paper, a similar method is applied to calculate the coefficients of model when the independent variables (inputs), as well as the response variable (output), are QT2FNs and regression coefficient are assumed to be crisp numbers. To accomplish this, we introduce a distance on the space of quasi type-2 fuzzy numbers and a goal programming method to obtain the coefficients of regression model.

Existing outliers in the data set, causes the result of fuzzy linear regression be incorrect. Chen [4], to reduce the effect of outliers, considered the minimization model of Tanaka [36] with n additional constraints. It is important for a data analyst to be able to identify outliers and assess their effect on various aspects of the analysis. Some methods have been presented to detect outliers in FLR [4, 32]. However, they have some drawbacks, e.g, they must pre-assign some values to the parameters without any instructive suggestion to chose them, and they cannot conduct a formal test for the outliers. To overcome the mentioned

shortcomings, we use an omission approach based on Hung and Yang method [16], which examines the value change in the objective function when some of the observations are omitted from the data set in QT2FLR model. Moreover, to define the cutoffs for outliers, we use the box plot procedure as a visual display tool.
The remainder of the paper is organized as follows: Section 2, contains some preliminaries of quasi type-2 fuzzy set theory. Also, a distance between quasi type-2 fuzzy numbers is introduced. In Section 3, our method is explained. In Section 4, an omission approach for outlier detection is proposed. Also, the effect of outliers in input and output data is considered. In Section 5, a real world data sets are used to illustrate how the proposed method is implemented. The predictive performance of the model is examined by cross-validation in Section 6. Finally, a brief conclusion is given in Section 7.

2. . Preliminaries

In this section, a review of the basic terminology used in this paper is presented[25, 15, 8].

2.1. **Type-2 fuzzy sets.**

A type-2 fuzzy set(T2FS), denoted by $\tilde{A}$, in a crisp set X is characterised by a type-2 membership function $\mu_{\tilde{A}}(x,u)$, i.e.,

$$\tilde{A} = \{((x,u), \mu_{\tilde{A}}(x,u))|x \in X,\ u \in J_x \subseteq [0,1]\}, \tag{1}$$

where $0 \leq \mu_{\tilde{A}}(x,u) \leq 1$, u is a primary grade and $\mu_{\tilde{A}}(x,u)$ is a secondary grade. A T2FS can be graphically shown in three dimensional space(3D). At each value of x, say x', the two dimensional (2D) plane whose axes are u and $\mu_{\tilde{A}}(x,u)$ is called a vertical Slice (VS) of $\tilde{A}$, i.e.,

$$VS(x') = \mu_{\tilde{A}}(x',u) \equiv \mu_{\tilde{A}}(x') = \{(u, f_{x'}(u))|u \in J_{x'}\} \tag{2}$$

where $f_{x'}(u) : J_x \to [0,1]$ is a function that assigns a secondary grade to each primary grade u for some fixed x. The VS is a type-1 fuzzy set(T1FS) in $[0,1]$.
The Footprint Of Uncertainty (FOU) is derived from the union of all primary memberships. The FOU is bounded by two membership functions, a lower one, $\underline{\mu_{\tilde{A}}(x)}$ and an upper one, $\overline{\mu_{\tilde{A}}(x)}$. The FOU can be described in terms of its upper and lower membership functions which themselves are T1FSs:

$$FOU(\tilde{A}) = \{J_x|x \in X\} = [\underline{FOU(\tilde{A})}, \overline{FOU(\tilde{A})}] \tag{3}$$

The principal membership function (PrMF) defined as the union of all the primary memberships having secondary grades equal to 1

$$Pr(\tilde{A}) = \{(x,u)|x \in X,\ f_x(u) = 1\} \tag{4}$$

An interval type-2 fuzzy set (IT2FS) is defined as a T2FS whose all secondary grades are of unity, i.e., for all x, $f_x(u) = 1$. An IT2FS can be completely determined by its FOU given by Equation (3).

Let $\tilde{A}$ be a T2FS satisfying the following assumptions: [8]

A1: All the VSs of the T2FS are fuzzy numbers, i.e., $\forall x$, $h(\tilde{A}) = \sup \mu_{\tilde{A}}(x) = 1$.

A2: All the VSs of the T2FS are piecewise functions of the same type (e.g., linear).

The first assumption assures that the T2FS contains an FOU and a Pr. This fact is clear since all the VSs are normal which makes it clear that for all the domain values there is at least one primary grade with secondary grade at unity. The second property assures that only a set parameters are needed to define a special kind of T2FS which is directly related to FOU and Pr. These assumptions allows the kind of T2FS be completely determined using its FOU and Pr, just like a T1FS can be completely determined by its core and support, which based on certain assumptions.

Definition 2.1. [8] *A T2FS is called a quasi type-2 fuzzy number (QT2FN) if it is completely determined by its FOU and Pr. The set of all QT2FNs is denoted by* $QT2F(\mathbb{R})$.

The Extension Principle has been used by Zadeh [42] and Mizumoto and Tanaka [27] to derive the intersection and union of T2FSs. Karnik and Mendel [18] provided an in-depth investigation on these operations.

Theorem 2.1. [25] *(QT2 Extension Principle) Let* $X = X_1 \times \cdots \times X_n$ *be the Cartesian product of universes, and* $\tilde{A}_1, \cdots, \tilde{A}_n$ *be QT2FSs in each respective universe. Also let* Y *be another universe and* $\tilde{B} \in Y$ *be a QT2FS such that* $\tilde{B} = f(\tilde{A}_1, \cdots, \tilde{A}_n)$*, where* $f : X \longrightarrow Y$ *is a monotone mapping. Then application of Extension Principle to QT2FSs (QT2 Extension Principle) leads to the following:*

$$\tilde{B}(y) = \sup_{(x_1,\cdots,x_n)\in f^{-1}(y)} \inf(\tilde{A}_1(x_1), \cdots, \tilde{A}_n(x_n))$$

where $y = f(x_1, \cdots, x_n)$.

Definition 2.2. [9] *Let* $\tilde{A}$ *and* $\tilde{B}$ *be two QT2FSs on the universal set* X*. Then,* $\tilde{A}$ *is called a subset of* $\tilde{B}$*, denoted by* $\tilde{A} \subseteq \tilde{B}$*, if*

$$\underline{A}(x) \leq \underline{B}(x) \quad , \quad A(x) \leq B(x) \quad and \quad \overline{A}(x) \leq \overline{B}(x), \qquad \forall\ x \in X \tag{5}$$

where $\underline{A}(x) = \underline{FOU}(\tilde{A})$, $A(x) = Pr(\tilde{A})$ *and* $\overline{A}(x) = \overline{FOU}(\tilde{A})$.
In addition, $\tilde{A}$ *is called equal to* $\tilde{B}$*, denoted by* $\tilde{A} = \tilde{B}$*, if*

$$\underline{A}(x) = \underline{B}(x) \quad , \quad A(x) = B(x) \quad and \quad \overline{A}(x) = \overline{B}(x) \qquad \forall\ x \in X. \tag{6}$$

Definition 2.3. [8] $\tilde{A} \in QT2F(\mathbb{R})$ *is called a positive QT2FN* ($\tilde{A} > 0$)*, if* $\overline{A}(x) = A(x) = \underline{A}(x) = 0$ *whenever* $x < 0$*; and* $\tilde{A}$ *is called a negative QT2FN* ($\tilde{A} < 0$)*, if* $\overline{A}(x) = A(x) = \underline{A}(x) = 0$ *whenever* $x > 0$.

Definition 2.4. *A QT2FN is called triangular if all of the membership functions* $\underline{FOU}(\tilde{A})$, $\overline{FOU}(\tilde{A})$, $Pr(\tilde{A})$ *and vertical slices are triangular fuzzy numbers.*
A triangular QT2FN, say $\tilde{A}$*, is denoted by* $\tilde{A} = \langle c; l_1, r_1; l, r; l_2, r_2\rangle$*. The parameters used here are* $Pr(\tilde{A}) = \langle c; l, r\rangle$ *and* $FOU(\tilde{A}) = \langle c; l_1, r_1; l_2, r_2\rangle$ *in which* $\langle c; l_1, r_1\rangle = \overline{FOU}(\tilde{A})$ *and* $\langle c; l_2, r_2\rangle = \underline{FOU}(\tilde{A})$.

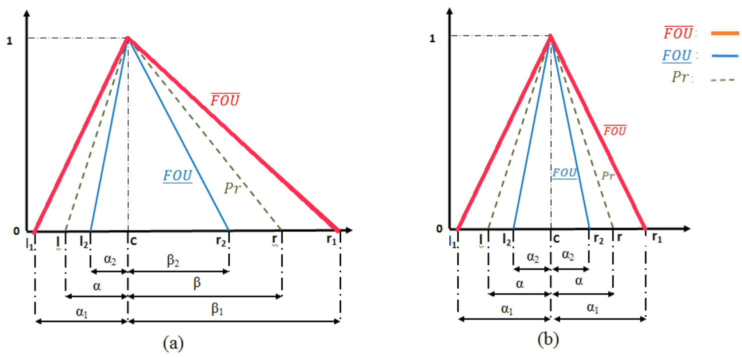

FIGURE 1. (a) Non-symmetric triangular QT2FN; (b) Symmetric triangular QT2FN

A triangular QT2FN, say $\tilde{A}$ can be denoted by its left and right spreads as follows:

$$\tilde{A} = \langle c; \alpha_1, \beta_1; \alpha, \beta; \alpha_2, \beta_2 \rangle$$

where $\alpha_1 = c - l_1,\ \alpha = c - l,\ \alpha_2 = c - l_2$ and $\beta_1 = r_1 - c,\ \beta = r - c,\ \beta_2 = r_2 - c$ are the left and right spreads of $\underline{FOU}$, Pr and $\overline{FOU}$, respectively, and $0 \le \alpha_2 \le \alpha \le \alpha_1$, $0 \le \beta_2 \le \beta \le \beta_1$ (Fig 1). Specially, $\tilde{A}$ is called symmetric if $\alpha_1 = \beta_1$, $\alpha = \beta$ and $\alpha_2 = \beta_2$. In such a case $\tilde{A}$ is denoted by $\tilde{A} = \langle c; \alpha_1; \alpha; \alpha_2 \rangle$.
An ordinary triangular fuzzy number can be considered as a degenerated QT2FN in which all of the spreads of $\underline{FOU}(\tilde{A})$, $\overline{FOU}(\tilde{A})$ and $Pr(\tilde{A})$ are equal and their secondary membership functions are zero. Also, a real number can be considered as a degenerated QT2FN whose spreads and secondary membership functions are zero. The following formulas for addition of two triangular QT2FNs and multiplication of a triangular QT2FN by a scaler are drawn from extension principle of zadeh [43].

Proposition 2.1. *If $\tilde{A} = \langle c; \alpha_1, \beta_1; \alpha, \beta; \alpha_2, \beta_2 \rangle$ and $\tilde{B} = \langle c'; \alpha'_1, \beta'_1; \alpha', \beta'; \alpha'_2, \beta'_2 \rangle$ be in $QT2F(\mathbb{R})$ and $\lambda \in \mathbb{R}$, then*

$$\tilde{A}_1 + \tilde{A}_2 = \langle c + c'; \alpha_1 + \alpha'_1, \beta_1 + \beta'_1; \alpha + \alpha', \beta + \beta'; \alpha_2 + \alpha'_2, \beta_2 + \beta'_2 \rangle, \tag{7}$$

$$\lambda \tilde{A}_1 = \begin{cases} \langle \lambda c; \lambda\alpha_1, \lambda\beta_1; \lambda\alpha, \lambda\beta; \lambda\alpha_2, \lambda\beta_2 \rangle & \lambda \ge 0, \\ \langle \lambda c; -\lambda\beta_1, -\lambda\alpha_1; -\lambda\beta, -\lambda\alpha; -\lambda\beta_2, -\lambda\alpha_2 \rangle & \lambda < 0. \end{cases} \tag{8}$$

Proposition 2.2. *Let r and s be two real numbers and $\tilde{A}$ be a quasi type-2 fuzzy number. Then*

$$(r + s)\tilde{A} \subseteq r\tilde{A} + s\tilde{A}. \tag{9}$$

Proof. According to Definition 2.2, Relation (9) is satisfied if and only if this relation is satisfied for lower, principal and upper fuzzy numbers which define $\tilde{A}$. In other words

$$(r + s)\tilde{A} \subseteq r\tilde{A} + s\tilde{A} \iff (r + s)\underline{A}(x) \le r\underline{A}(x) + s\underline{A}(x), \tag{10}$$

$$(r + s)A(x) \le rA(x) + sA(x),$$
$$(r + s)\overline{A}(x) \le r\overline{A}(x) + s\overline{A}(x).$$

However, the right hand side relations are hold for type-1 fuzzy numbers (see [12], Proposition 2.7), and the proof is complete. □

2.2. A distance between quasi type-2 fuzzy numbers.

The observed data in our study are assumed to be QT2FN. So, we try to close the membership functions of observed and estimated responses from fuzzy linear regression model by closing their corresponding parameters. Since QT2FNs are completely characterized by the parameters of their FOU and Pr, closing the parameteres of two QTFNs is enough (in fact necessary and sufficient) to close their membership functions, which is the purpose of this paper. To do this, based on the distance defined between two triangular fuzzy numbers by Hassanpour et al. [11], we propose the following weighted distance between QT2FNs in which different weights (w_i) are used to show different importance of the parameters [15].

Definition 2.5. *Let $\tilde{A} = \langle c; \alpha_1, \beta_1; \alpha, \beta; \alpha_2, \beta_2 \rangle$ and $\tilde{B} = \langle c'; \alpha'_1, \beta'_1; \alpha', \beta'; \alpha'_2, \beta'_2 \rangle$ be in $QT2F(\mathbb{R})$ and $w_i > 0$ for $i = 1, \cdots, 7$. The distance between $\tilde{A}$ and $\tilde{B}$ as follows:*

$$d_w(\tilde{A}, \tilde{B}) = w_1|\alpha_1 - \alpha'_1| + w_2|\beta_1 - \beta'_1| + w_3|\alpha_2 - \alpha'_2| + w_4|\beta_2 - \beta'_2| + \\ w_5|\alpha - \alpha'| + w_6|c - c'| + w_7|\beta - \beta'|. \tag{11}$$

Among advantages of using d_w, we can refer to its ease in both theory and application. Furthermore, the special formula of the proposed distance helps us to convert the nonlinear programming model proposed to calculate the regression coefficients, to a linear one. The advantages of this conversion are that solving LP problems is very easy, and their exact solution can be obtained by the Simplex method. However, most of available algorithms for solving nonlinear programming problems yield approximate solutions. In addition, depending on the relative importance of the parameters of quasi type-2 fuzzy numbers, we can assign different values to the weights w_i and obtain different solutions. For example, one can set $w_i = 1$ for all i, if the parameters of Pr and FOU have the same importance, and can set $w_5 = w_6 = w_7 = 2$, $w_1 = w_2 = w_3 = w_4 = 1$ if the importance of Pr is twice the importance of FOU.

Proposition 2.3. *The function d_w defined in Definition 2.5 is a metric on $QT2F(\mathbb{R})$, i.e., for each $\tilde{A}, \tilde{B}, \tilde{C} \in QT2F(\mathbb{R})$ we have:*

1. $d_w(\tilde{A}, \tilde{B}) \geq 0$ *and* $d_w(\tilde{A}, \tilde{A}) = 0$.
2. $d_w(\tilde{A}, \tilde{B}) = d_w(\tilde{B}, \tilde{A})$.
3. $d_w(\tilde{A}, \tilde{C}) \leq d_w(\tilde{A}, \tilde{B}) + d_w(\tilde{B}, \tilde{C})$.

Proof. Straightforward. □

Proposition 2.4. *The metric space $(QT2F(\mathbb{R}), d_w)$ is complete.*

Proof. $(\mathbb{R}, L^1)$ is a complete metric space, and the metric space $(QT2F(\mathbb{R}), L^1)$ inherits properties from $(\mathbb{R}, L^1)$. Therefore $(QT2F(\mathbb{R}), d_w)$ is a complete metric space too. □

Remark 2.1. *In practice, we set $\sum_{i=1}^{7} w_i = 1$, to avoid repeated weights. Furtheremore, if we set $w_i = 1 \;\; \forall\, i$, for two crisp numbers, the above distance is reduced to the absolute difference between them.*

Remark 2.2. *For two triangular fuzzy numbers, the above distance is reduced to the wighted sum of absolute difference between their middle point and their spreads (in fact, it is an extension of the metric introduced in* [12] *on T1FRs).*

3. . The proposed regression model

Consider a set of QT2F data $\{(\tilde{x}_{i1}, \tilde{x}_{i2}, \cdots, \tilde{x}_{ip}, \tilde{y}_i) | i = 1, \cdots, n\}$, in which $\tilde{x}_{ij}$ $(i = 1, \cdots, n,\ j = 1, 2, \cdots, p)$ is the value of jth independent variable $(\tilde{x}_j)$ and $\tilde{y}_i$ $(i = 1, \cdots, n)$ is the corresponding value of dependent variable $\tilde{y}$ in the ith case. The purpose of quasi type-2 fuzzy linear regression (QT2FLR) is to fit a fuzzy linear model to the given fuzzy data. This model can be considered as follows:

$$\tilde{Y} = a_0 + a_1\tilde{x}_1 + a_2\tilde{x}_2 + \cdots + a_p\tilde{x}_p. \tag{12}$$

In model (12), the coefficients $a_0, a_1, \cdots, a_p$ are assumed to be crisp numbers. These parameters must be calculated such that the estimated responses $\tilde{Y}_i$,

$$\tilde{Y}_i = a_0 + a_1\tilde{x}_{i1} + a_2\tilde{x}_{i2} + \cdots + a_p\tilde{x}_{ip} \quad i = 1, \cdots, n, \tag{13}$$

be close to the corresponding fuzzy observed responses $\tilde{y}_i$ $(i = 1, \cdots, n)$, as much as possible.

In this paper, given inputs $\tilde{x}_{ij} = \langle c_{\tilde{x}_{ij}}; \alpha_{1_{\tilde{x}_{ij}}}, \beta_{1\tilde{x}_{ij}}; \alpha_{\tilde{x}_{ij}}, \beta_{\tilde{x}_{ij}}; \alpha_{2\tilde{x}_{ij}}, \beta_{2\tilde{x}_{ij}} \rangle,\ i = 1, \cdots, n,\ j = 1, \cdots, p$, are supposed to be positive non-symmetric QT2FNs, by a simple translation of all data, if necessary. Also, suppose that the observed responses are non-symmetric QT2FNs $\tilde{y}_i = \langle c_{\tilde{y}_i}; \alpha_{1\tilde{y}_i}, \beta_{1\tilde{y}_i}; \alpha_{\tilde{y}_i}, \beta_{\tilde{y}_i}; \alpha_{2\tilde{y}_i}, \beta_{2\tilde{y}_i} \rangle,\ \ i = 1, \cdots, n$.

To calculate the coefficients of model (12), we need to multiply the QT2FN $\tilde{x}_j$ by the

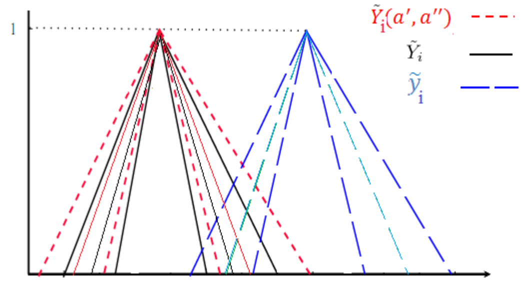

FIGURE 2. Approximation of $\tilde{Y}_i$ by $\tilde{Y}_i(\mathbf{a}', \mathbf{a}'')$ as a prediction of $\tilde{y}_i$

scaler a_j. This multiplication depends on the sign of the scaler, as seen in (8). Therefore, for different states of the signs of regression coefficients, different models (LP models, least-squares models, ...) must be formulated and solved to estimate the coefficients. To avoid this, we propose a goal programming model to estimate the regression coefficients which is independent of the sign of them.
Note that a real number a has infinite representations in the form of $a = a' - a''$, where a' and a'' are nonnegative real numbers. This matter leads to the following lemma.

Lemma 3.1. *Let $a_j \in \mathbb{R}$, $j = 0, \cdots, p$ and $\tilde{x}_{ij} = \langle c_{\tilde{x}_{ij}}; \alpha_{1\tilde{x}_{ij}}, \beta_{1\tilde{x}_{ij}}; \alpha_{\tilde{x}_{ij}}, \beta_{\tilde{x}_{ij}}; \alpha_{2\tilde{x}_{ij}}, \beta_{2\tilde{x}_{ij}}\rangle$, $i = 1, \cdots, n$, $j = 1, \cdots, p$ be in $QT2F(\mathbb{R})$. Set $a_j = a'_j - a''_j$ for $j = 1, \cdots, p$ where $a'_j, a''_j \geq 0$. Then for each choice of a'_j and a''_j we have:*

$$\tilde{Y}_i \subseteq \langle a_0 + \sum_{j=1}^{p}(a'_j - a''_j)c_{\tilde{x}_{ij}}; \sum_{j=1}^{p}(a'_j\alpha_{1\tilde{x}_{ij}} + a''_j\beta_{1\tilde{x}_{ij}}), \sum_{j=1}^{p}(a'_j\beta_{1\tilde{x}_{ij}} + a''_j\alpha_{1\tilde{x}_{ij}}); \sum_{j=1}^{p}(a'_j\alpha_{\tilde{x}_{ij}} + a''_j\beta_{\tilde{x}_{ij}}),$$
$$\sum_{j=1}^{p}(a'_j\beta_{\tilde{x}_{ij}} + a''_j\alpha_{\tilde{x}_{ij}}); \sum_{j=1}^{p}(a'_j\alpha_{2\tilde{x}_{ij}} + a''_j\beta_{2\tilde{x}_{ij}}), \sum_{j=1}^{p}(a'_j\beta_{2\tilde{x}_{ij}} + a''_j\alpha_{2\tilde{x}_{ij}})\rangle \tag{14}$$

where $\tilde{Y}_i$ is given by (13).

Proof. By using Proposition 2.2 and Equations (7) and (8) we have:

$$\begin{aligned}
\tilde{Y}_i &= a_0 + \textstyle\sum_{j=1}^{p} a_j\tilde{x}_{ij} = a_0 + \sum_{j=1}^{p}(a'_j - a''_j)\tilde{x}_{ij} \subseteq a_0 + \sum_{j=1}^{p}(a'_j\tilde{x}_{ij} - a''_j\tilde{x}_{ij}) \\
&= a_0 + \textstyle\sum_{j=1}^{p}(a'_j\langle c_{\tilde{x}_{ij}}; \alpha_{1\tilde{x}_{ij}}, \beta_{1\tilde{x}_{ij}}; \alpha_{\tilde{x}_{ij}}, \beta_{\tilde{x}_{ij}}; \alpha_{2\tilde{x}_{ij}}, \beta_{2\tilde{x}_{ij}}\rangle \\
&\qquad\qquad -a''_j\langle c_{\tilde{x}_{ij}}; \alpha_{1\tilde{x}_{ij}}, \beta_{1\tilde{x}_{ij}}; \alpha_{\tilde{x}_{ij}}, \beta_{\tilde{x}_{ij}}; \alpha_{2\tilde{x}_{ij}}, \beta_{2\tilde{x}_{ij}}\rangle) \\
&= a_0 + \textstyle\sum_{j=1}^{p}(\langle a'_jc_{\tilde{x}_{ij}}; a'_j\alpha_{1\tilde{x}_{ij}}, a'_j\beta_{1\tilde{x}_{ij}}; a'_j\alpha_{\tilde{x}_{ij}}, a'_j\beta_{\tilde{x}_{ij}}; a'_j\alpha_{2\tilde{x}_{ij}}, a'_j\beta_{2\tilde{x}_{ij}}\rangle + \\
&\qquad \langle -a''_jc_{\tilde{x}_{ij}}; a''_j\beta_{1\tilde{x}_{ij}}, a''_j\alpha_{1\tilde{x}_{ij}}; a''_j\beta_{\tilde{x}_{ij}}, a''_j\alpha_{\tilde{x}_{ij}}; a''_j\beta_{2\tilde{x}_{ij}}, a''_j\alpha_{2\tilde{x}_{ij}}\rangle) \\
&= \langle a_0 + \textstyle\sum_{j=1}^{p}(a'_j - a''_j)c_{\tilde{x}_{ij}}; \sum_{j=1}^{p}(a'_j\alpha_{1\tilde{x}_{ij}} + a''_j\beta_{1\tilde{x}_{ij}}), \sum_{j=1}^{p}(a'_j\beta_{1\tilde{x}_{ij}} + a''_j\alpha_{1\tilde{x}_{ij}}); \\
&\quad \textstyle\sum_{j=1}^{p}(a'_j\alpha_{\tilde{x}_{ij}} + a''_j\beta_{\tilde{x}_{ij}}), \sum_{j=1}^{p}(a'_j\beta_{\tilde{x}_{ij}} + a''_j\alpha_{\tilde{x}_{ij}}); \sum_{j=1}^{p}(a'_j\alpha_{2\tilde{x}_{ij}} + a''_j\beta_{2\tilde{x}_{ij}}), \\
&\quad \textstyle\sum_{j=1}^{p}(a'_j\beta_{2\tilde{x}_{ij}} + a''_j\alpha_{2\tilde{x}_{ij}})\rangle
\end{aligned}$$

and the proof is completed. □

Let us denote the right hand side of relation (14) by $\tilde{Y}_i(\mathbf{a}', \mathbf{a}'')$ where $\mathbf{a}'$ and $\mathbf{a}''$ are nonnegative p-dimensional vectors with jth element a'_j and a''_j, respectively. Clearly, for each choice of $\mathbf{a}'$ and $\mathbf{a}''$, $\tilde{Y}_i(\mathbf{a}', \mathbf{a}'')$ is a triangular QT2FN, and we consider it as an approximation for $\tilde{Y}_i$ (Fig 2). Indeed, there are many approximations for $\tilde{Y}_i$, among which, we try to choose the best approximation. We have to find $\tilde{Y}_i$ as close as possible to $\tilde{y}_i$. Instead, first we approximate $\tilde{Y}_i$ to its supersets $\tilde{Y}_i(\mathbf{a}', \mathbf{a}'')$ for different choices of $\mathbf{a}'$ and $\mathbf{a}''$, as seen in Lemma 3.1. Then, we try to find appropriate values of $\mathbf{a}'$ and $\mathbf{a}''$ so that $\tilde{Y}_i(\mathbf{a}', \mathbf{a}'')$ be close to $\tilde{y}_i$ as much as possible. To this end, we attempt to close the membership function of each approximated response $\tilde{Y}_i(\mathbf{a}', \mathbf{a}'')$ to that of corresponding observed response $\tilde{y}_i$ as much as possible. Therefore, we introduce the following mathematical programming problem, which finds the best choices of $\mathbf{a}'$ and $\mathbf{a}''$ for the coefficients of model (13):

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} d_w(\tilde{Y}_i(\mathbf{a}', \mathbf{a}''), \tilde{y}_i) \\ s.t. \quad & a_0 \in \mathbb{R}, \quad a'_j, a''_j \geq 0 \quad j = 1, 2, \cdots, p, \end{aligned} \tag{15}$$

where d was introduced in Definition 2.5. The model (15) can be converted to a GP model by choosing appropriate deviation variables. To this end, set

$$\tilde{Y}_i(\mathbf{a}', \mathbf{a}'') = \langle c_{\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')}; \alpha_{1\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')}, \beta_{1\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')}; \alpha_{\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')}, \beta_{\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')}; \alpha_{2\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')}, \beta_{2\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')} \rangle$$

for $i = 1 \cdots n$ and define:

$$n_{ik} = \frac{1}{2}\{|k_{\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')} - k_{y_i}| - (k_{\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')} - k_{y_i})\} \tag{16}$$

$$p_{ik} = \frac{1}{2}\{|k_{\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')} - k_{y_i}| + (k_{\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')} - k_{y_i})\}, \tag{17}$$

for $k = c, \alpha_1, \beta_1, \alpha, \beta, \alpha_2, \beta_2$. In fact, n_{ik} and p_{ik} ($k \in \{c, \alpha_1, \beta_1, \alpha, \beta, \alpha_2, \beta_2\}$) are the negative and positive deviations between the parameters of the ith estimated and observed response, respectively. It can be easily seen that

$$n_{i\alpha_1} = \begin{cases} \alpha_{1\tilde{y}_i} - \alpha_{1\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')} & \alpha_{1\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')} \leq \alpha_{1\tilde{y}_i}, \\ 0 & \alpha_{1\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')} > \alpha_{1\tilde{y}_i}, \end{cases} \tag{18}$$

$$p_{i\alpha_1} = \begin{cases} \alpha_{1\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')} - \alpha_{1\tilde{y}_i} & \alpha_{1\tilde{y}_i} \leq \alpha_{1\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')}, \\ 0 & \alpha_{1\tilde{y}_i} > \alpha_{1\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')}, \end{cases} \tag{19}$$

for $i = 1 \cdots n$. Similar relations are hold true for other deviation variables. By using the above deviation variables, the model (15) converts to the following GP model:

$$\begin{aligned} (WGP): \min \quad z = & \textstyle\sum_{i=1}^{n}(w_1(n_{ic} + p_{ic}) + w_2(n_{i\alpha_1} + p_{i\alpha_1}) + w_3(n_{i\beta_1} + p_{i\beta_1}) + w_4(n_{i\alpha} + p_{i\alpha}) \\ & + w_5(n_{i\beta} + p_{i\beta}) + w_6(n_{i\beta_2} + p_{i\beta_2}) + w_7(n_{i\alpha_2} + p_{i\alpha_2})) \end{aligned}$$

$$s.t. \quad \sum_{j=1}^{p}(a'_j \alpha_{1\tilde{x}_{ij}} + a''_j \beta_{1\tilde{x}_{ij}}) + n_{i\alpha_1} - p_{i\alpha_1} = \alpha_{1\tilde{y}_i} \qquad i = 1, \cdots, n \tag{20}$$

$$\sum_{j=1}^{p}(a'_j \beta_{1\tilde{x}_{ij}} + a''_j \alpha_{1\tilde{x}_{ij}}) + n_{i\beta_1} - p_{i\beta_1} = \beta_{1\tilde{y}_i} \qquad i = 1, \cdots, n \tag{21}$$

$$\sum_{j=1}^{p}(a'_j \alpha_{\tilde{x}_{ij}} + a''_j \beta_{\tilde{x}_{ij}}) + n_{i\alpha} - p_{i\alpha} = \alpha_{\tilde{y}_i} \qquad i = 1, \cdots, n \tag{22}$$

$$\sum_{j=1}^{p}(a'_j\beta_{\tilde{x}_{ij}} + a''_j\alpha_{\tilde{x}_{ij}}) + n_{i\beta} - p_{i\beta} = \beta_{\tilde{y}_i} \qquad i = 1, \cdots, n \tag{23}$$

$$\sum_{j=1}^{p}(a'_j\alpha_{2\tilde{x}_{ij}} + a''_j\beta_{2\tilde{x}_{ij}}) + n_{i\alpha_2} - p_{i\alpha_2} = \alpha_{2\tilde{y}_i} \quad i = 1, \cdots, n \tag{24}$$

$$\sum_{j=1}^{p}(a'_j\beta_{2\tilde{x}_{ij}} + a''_j\alpha_{2\tilde{x}_{ij}}) + n_{i\beta_2} - p_{i\beta_2} = \beta_{2\tilde{y}_i} \quad i = 1, \cdots, n \tag{25}$$

$$n_{ik}p_{ik} = 0 \qquad i = 1, \cdots, n, \qquad k = c, \alpha_1, \beta_1, \alpha, \beta, \alpha_2, \beta_2 \tag{26}$$

$$n_{ik}, p_{ik} \geq 0 \qquad i = 1, \cdots, n, \qquad k = c, \alpha_1, \beta_1, \alpha, \beta, \alpha_2, \beta_2 \tag{27}$$

$$a_0 \in \mathbb{R}\ , \qquad a'_j, a''_j \geq 0, \qquad j = 1, \cdots, p. \tag{28}$$

It is clear that if $\alpha_{\tilde{y}_i} = \beta_{\tilde{y}_i}$ for all i, we can set $\alpha_{\tilde{x}_{ij}} = \beta_{\tilde{x}_{ij}}$ for each i and j. Accordingly, the constraints (22) and (23) will be equivalent, and one of them (in fact n constraints) can be removed. Similarly, if $\alpha_{1\tilde{y}_i} = \beta_{1\tilde{y}_i}$ ($\alpha_{2\tilde{y}_i} = \beta_{2\tilde{y}_i}$) for all i, we can set $\alpha_{1\tilde{x}_{ij}} = \beta_{1\tilde{x}_{ij}}$ ($\alpha_{1\tilde{x}_{ij}} = \beta_{1\tilde{x}_{ij}}$) for each i and j. Therefore, the constraints (20) and (21) ((24) and (25)) will be be equivalent, and the constraints (21) ((25)) can be removed, then we obtain a smaller model. In addition, one can remove the constraints (26) and solve the obtained LP model by the Simplex method [1]. Another feature of WGP is that, it does not produce negative spreads for QT2FNs. This fact has been shown in the following proposition.

Proposition 3.1. *The estimated responses from WGP have non-negative spreads.*

Proof. The estimated responses from WGP are:
$\tilde{Y}_i(\mathbf{a}', \mathbf{a}'') = \langle c_{\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')}; \alpha_{1\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')}, \beta_{1\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')}; \alpha_{\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')}, \beta_{\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')}; \alpha_{2\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')}, \beta_{2\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')}\rangle$
We have to show that $k_{\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')} \geq 0$ for $k \in \{\alpha_1, \beta_1, \alpha, \beta, \alpha_2, \beta_2\}$. According to the proof of Lemma (3.1) we have:

$\alpha_{1\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')} = \sum_{j=1}^{p}(a'_j\alpha_{1\tilde{x}_{ij}} + a''_j\beta_{1\tilde{x}_{ij}})$, $\beta_{1\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')} = \sum_{j=1}^{p}(a'_j\beta_{1\tilde{x}_{ij}} + a''_j\alpha_{1\tilde{x}_{ij}})$
$\alpha_{\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')} = \sum_{j=1}^{p}(a'_j\alpha_{\tilde{x}_{ij}} + a''_j\beta_{\tilde{x}_{ij}})$, $\beta_{\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')} = \sum_{j=1}^{p}(a'_j\beta_{\tilde{x}_{ij}} + a''_j\alpha_{\tilde{x}_{ij}})$
$\alpha_{2\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')} = \sum_{j=1}^{p}(a'_j\alpha_{2\tilde{x}_{ij}} + a''_j\beta_{2\tilde{x}_{ij}})$, $\beta_{2\tilde{Y}_i(\mathbf{a}',\mathbf{a}'')} = \sum_{j=1}^{p}(a'_j\beta_{2\tilde{x}_{ij}} + a''_j\alpha_{2\tilde{x}_{ij}})$

which are all non-negative. Because $a'_j, a''_j, \alpha_{1\tilde{x}_{ij}}, \beta_{1\tilde{x}_{ij}}, \alpha_{\tilde{x}_{ij}}, \beta_{\tilde{x}_{ij}}, \alpha_{2\tilde{x}_{ij}}$ and $\beta_{2\tilde{x}_{ij}}$ are all non-negative. □

Although the estimated responses $\tilde{Y}_i$, $i = 1, \cdots, n$ have been approximated by $\tilde{Y}_i(\mathbf{a}', \mathbf{a}'')$, $i = 1, \cdots, n$, the following theorem shows that WGP yields the exact regression coefficients and estimated responses if the given quasi type-2 fuzzy input-output data satisfy in a quasi type-2 fuzzy linear model.

Theorem 3.1. *Suppose that the observed quasi type-2 fuzzy input-output data satisfy in the quasi type-2 fuzzy linear model*

$$\tilde{y} = \bar{a}_0 + \bar{a}_1\tilde{x}_1 + \bar{a}_2\tilde{x}_2 + \cdots + \bar{a}_p\tilde{x}_p.$$

Then:
(i) The optimal solution of WGP contains $\bar{a}_0, \cdots, \bar{a}_p$.
(ii) The estimated responses from WGP and the observed responses are exactly the same (i.e., $\tilde{Y}_i(\boldsymbol{a}', \boldsymbol{a}'') = \tilde{y}_i$*).*

Proof. (i). Set $a_0 = \bar{a}_0$,

$$a'_j = \begin{cases} \bar{a}_j & \text{if} \quad \bar{a}_j \geq 0 \\ 0 & \text{if} \quad \bar{a}_j < 0 \end{cases} \quad \text{and} \quad a''_j = \begin{cases} 0 & \text{if} \quad \bar{a}_j \geq 0 \\ -\bar{a}_j & \text{if} \quad \bar{a}_j < 0 \end{cases}$$

for $j = 1, \cdots, p$, and $n_{ik} = p_{ik} = 0$, for $i = 1, 2, \cdots, n$ and $k = c, \alpha_1, \beta_1, \alpha, \beta, \alpha_2, \beta_2$. First, we show that these values yield a feasible solution for WGP. Clearly, they satisfy in the constraints (26)-(28). Without loss of generality, for simplicity, suppose that $\bar{a}_j \geq 0,\ j = 1, \cdots, p$. By the assumption we have:

$$a_0 + \sum_{j=1}^{p}(a'_j - a''_j)c_{\tilde{x}_{ij}} + n_{ic} - p_{ic} = \bar{a}_0 + \sum_{j=1}^{p}\bar{a}_j c_{\tilde{x}_{ij}} = c_{\tilde{y}_i}, \tag{29}$$

$$\sum_{j=1}^{p}(a'_j\alpha_{1\tilde{x}_{ij}} + a''_j\beta_{1\tilde{x}_{ij}}) + n_{i\alpha_1} - p_{i\alpha_1} = \sum_{j=1}^{p}\bar{a}_j\alpha_{1\tilde{x}_{ij}} = \alpha_{1\tilde{y}_i}, \tag{30}$$

$$\sum_{j=1}^{p}(a'_j\beta_{1\tilde{x}_{ij}} + a''_j\alpha_{1\tilde{x}_{ij}}) + n_{i\beta_1} - p_{i\beta_1} = \sum_{j=1}^{p}\bar{a}_j\beta_{1\tilde{x}_{ij}} = \beta_{1\tilde{y}_i}, \tag{31}$$

$$\sum_{j=1}^{p}(a'_j\alpha_{\tilde{x}_{ij}} + a''_j\beta_{\tilde{x}_{ij}}) + n_{i\alpha} - p_{i\alpha} = \sum_{j=1}^{p}\bar{a}_j\alpha_{\tilde{x}_{ij}} = \alpha_{\tilde{y}_i} \tag{32}$$

$$\sum_{j=1}^{p}(a'_j\beta_{\tilde{x}_{ij}} + a''_j\alpha_{\tilde{x}_{ij}}) + n_{i\beta} - p_{i\beta} = \sum_{j=1}^{p}\bar{a}_j\beta_{\tilde{x}_{ij}} = \beta_{\tilde{y}_i} \tag{33}$$

$$\sum_{j=1}^{p}(a'_j\alpha_{2\tilde{x}_{ij}} + a''_j\beta_{2\tilde{x}_{ij}}) + n_{i\alpha_2} - p_{i\alpha_2} = \sum_{j=1}^{p}\bar{a}_j\alpha_{2\tilde{x}_{ij}} = \alpha_{2\tilde{y}_i} \tag{34}$$

$$\sum_{j=1}^{p}(a'_j\beta_{2\tilde{x}_{ij}} + a''_j\alpha_{2\tilde{x}_{ij}}) + n_{i\beta_2} - p_{i\beta_2} = \sum_{j=1}^{p}\bar{a}_j\beta_{2\tilde{x}_{ij}} = \beta_{2\tilde{y}_i} \tag{35}$$

The Equations (29)-(35) show that the suggested values satisfy in the constraints (20)-(25). (i.e., we have a feasible solution for WGP). To show the optimality, note that the objective function value of WGP for this feasible solution is 0, which is the least possible value for a non-negative function. Therefore, the suggested values yield the optimal solution of WGP. Indeed, in the optimal solution we have $a'_j - a''_j = \bar{a}_j$.

(ii). The second part of the theorem immediately follows from (29)-(33). In fact, from (29)-(33) we have:

$$c_{\tilde{Y}_i}(\mathbf{a}', \mathbf{a}'') = c_{\tilde{y}_i}, \qquad \alpha_{1\tilde{Y}_i}(\mathbf{a}', \mathbf{a}'') = \alpha_{1\tilde{y}_i}, \qquad \beta_{1\tilde{Y}_i}(\mathbf{a}', \mathbf{a}'') = \beta_{1\tilde{y}_i}, \qquad \alpha_{\tilde{Y}_i}(\mathbf{a}', \mathbf{a}'') = \alpha_{\tilde{y}_i}$$

$$\beta_{\tilde{Y}_i}(\mathbf{a}', \mathbf{a}'') = \beta_{\tilde{y}_i}, \qquad \alpha_{2\tilde{Y}_i}(\mathbf{a}', \mathbf{a}'') = \alpha_{2\tilde{y}_i}, \qquad \beta_{2\tilde{Y}_i}(\mathbf{a}', \mathbf{a}'') = \beta_{2\tilde{y}_i}.$$

and the proof is completed. □

Remark 3.1. *The explained method can be applied to QT2FLR with triangular IT2FN outputs easily. Since, IT2FN completely determined using its FOU, it is enough to modify the distance introduced in Definition 2.5 for two triangular IT2FN $\tilde{A} = \langle c; \alpha_1, \beta_1; \alpha_2, \beta_2\rangle$ and $\tilde{B} = \langle c'; \alpha'_1, \beta'_1; \alpha'_2, \beta'_2\rangle$ as follows:*

$$d_w(\tilde{A}, \tilde{B}) = w_1|c - c'| + w_2|\alpha_1 - \alpha'_1| + w_3|\beta_1 - \beta'_1| + w_4|\alpha_2 - \alpha'_2| + w_5|\beta_2 - \beta'_2|. \tag{36}$$

In which $w_1, \cdots, w_5 > 0$. The WGP model in this case is obtained from WGP by removing the constraints (22) and (23).

4. . An omission approach for outlier detection

The existence of outliers in a set of experimental data can cause incorrect interpretation of the fuzzy linear regression results. To handle the outlier problem, Hung and Yang [16] proposed an omission approach for Tanaka's linear programming method. Their approach has the capability to examine the behavior of value changes in the objective function of the related optimization models when observations are omitted. In this section, we use this omission approach for our GP method to detect outliers in input and output data in QT2FLR models. Some methods have been presented to detect outliers in FLR [4, 32]. However, there are some drawbacks to the existing methods. In fact, they must pre-assign some values to parameters and they can not conduct a formal test for the outliers. To overcome the drawbacks in the existing methods, we use an omission approach which examine the value change in the objective function's behaviour when some of the observations are omitted from the data set in QT2FLR model. Moreover, to define the cutoffs for outliers, we use the box plot procedure as a visual display tool.
This approach measures the influence of the ith observation on the value of the objective function in our WGP model when the ith observation is omitted. Based on this idea, we develop an omission approach for detecting a single outlier in a data set as follows. The procedure is to first delete the t th observation. We then apply GP approach to the remaining $(n-1)$ observations and obtain the minimum value of the objective function which is denoted by $J_M^{(t)}$. After deleting the t th observation, WGP approach becomes

$$\begin{aligned} J_M^{(t)} = min \ \textstyle\sum_{i=1,i\neq t}^{n} & (w_1(n_{ic}+p_{ic}) + w_2(n_{i\alpha_1}+p_{i\alpha_1}) + w_3(n_{i\beta_1}+p_{i\beta_1}) + w_4(n_{i\alpha}+p_{i\alpha}) \\ & +w_5(n_{i\beta}+p_{i\beta}) + w_6(n_{i\beta_2}+p_{i\beta_2}) + w_7(n_{i\alpha_2}+p_{i\alpha_2})) \\ & s.t. \qquad (20) \ - \ (28) \qquad i\neq t. \end{aligned} \tag{37}$$

Let J_M be the minimum value of the objective function obtained from WGP for all observations. Denote the absolute difference between J_M and $J_M^{(t)}$ by d_t,

$$d_t = |J_M - J_M^{(t)}|, \qquad t = 1, \cdots n.$$

The ratio of d_t to J_M is called the normalized absolute difference and denoted by r_t,

$$r_t = \frac{d_t}{J_M} \qquad t = 1, \cdots n. \tag{38}$$

The r_t value shows the size of the absolute difference relative to J_M. A large value of r_t indicates a large impact of the tth observation on the value of the objective function. We usually assume that there is at most one outlier in a given data set and require that the label of the outlying observation is unknown. We therefore use

$$r_{max} = \max\{r_t \mid 1 \leq t \leq n\}$$

to detect a single outlier in a QT2FLR model. But determination of the critical value for r_{max} is difficult. Therefore, we use a box plot method to compare the values of r_t to each other to determine of outliers.
By similar arguments of [3,4,14], we use the interquartile range (IQR), that is the difference between the first and third quartiles, to define the outlier cutoffs as follows. In a box plot, inner fences are constructed to the left and right of the box at a distance of 1.5 times the IQR. Outer fences are constructed in the same way at a distance of 3 times the IQR. It is well known that the median and the first and third quartiles are insensitive to outlying data values. Since a box plot itself contains these values, a box plot will automatically guard against undue inuence of outlying cases. On the other hand, the outliers cutoffs are

TABLE 1. Observed and predicted quasi type-2 fuzzy valued and their distance (for $w = 1$)

i	$\tilde{x}_i$	$\tilde{y}_i$	$\tilde{Y}_i$	d_w
1	$\langle 2; 1.25, 1.35, 0.5; 0.6, 0.25, 0.35\rangle$	$\langle 4; 1.55, 1.5; 0.8, 0.75; 0.55, 0.5\rangle$	$\langle 4.83; 1.22, 1.27; 0.50, 0.55; 0.26, 0.31\rangle$	*2.34*
2	$\langle 3.5; 1.59, 1.62; 0.84, 0.87; 0.59, 0.62\rangle$	$\langle 5.5; 1.41, 1.55; 0.66, 0.8; 0.41, 0.55\rangle$	$\langle 5.5; 1.53, 1.55; 0.81, 0.82; 0.57, 0.58\rangle$	*0.51*
3	$\langle 5.5; 1.45, 1.75; 0.7, 1; 0.45, 0.75\rangle$	$\langle 7.5; 1.25, 1.75; 0.5, 1; 0.25, 0.75\rangle$	$\langle 6.38; 1.47, 1.60; 0.75, 0.88; 0.51, 0.64\rangle$	*2.21*
4	$\langle 7; 1.75, 1.42; 1, 0.67; 0.75, 0.42\rangle$	$\langle 6.5; 1.47, 1.28; 0.72, 0.53; 0.47, 0.28\rangle$	$\langle 7.05; 1.59, 1.45; 0.87, 0.72; 0.63, 0.48\rangle$	*1.58*
5	$\langle 8.5; 1.45, 1.59; 0.7, 0.84; 0.45; 0.59\rangle$	$\langle 8.5; 1.46, 1.39; 0.71, 0.64; 0.46, 0.39\rangle$	$\langle 7.72; 1.43, 1.49; 0.70, 0.77; 0.46, 0.53\rangle$	*1.19*
6	$\langle 10.5; 1.56, 1.39; 0.81, 0.64; 0.56, 0.39\rangle$	$\langle 8; 1.42, 1.4; 0.67, 0.65; 0.42, 0.4\rangle$	$\langle 8.61; 1.45, 1.38; 0.73, 0.65; 0.49, 0.41\rangle$	*0.83*
7	$\langle 11; 1.32, 1.51; 0.57, 0.76; 0.32, 0.51\rangle$	$\langle 10.5; 1.52, 1.57; 0.77, 0.82; 0.52, 0.57\rangle$	$\langle 8.83; 1.31, 1.40; 0.59, 0.68; 0.35, 0.44\rangle$	*2.63*
8	$\langle 12.5; 1.44, 1.65; 0.69, 0.9; 0.44, 0.65$	$\langle 9.5; 1.75, 1.72; 1, 0.97; 0.75, 0.72\rangle$	$\langle 9.5; 1.43, 1.53; 0.71, 0.81; 0.47, 0.57\rangle$	*1.36*
		Mean of distance		*1.58*

determined by the first and third quartiles.They can dampen the inuence of even a single wild data value. Therefore, we use a box plot to determine whether r_{max} is an outlier or not. The data points that lie between the inner and outer fences are denoted by a circle "$\circ$" called mild outliers. The data points that lie beyond the outer fences are denoted by an asterisk "$*$", called extreme outliers.

4.1. . **Cross validation.** To further investigation of the performance of the model, we apply an index based on the cross validation method [40] to examine the predictive ability of the models. To this end, each time, the ith observation is left out from the data set, while the remaining observations are used to develop a quasi type-2 fuzzy regression model. Then the obtained model is used to predict the response value of the i th observation (denoted by $\tilde{Y}_{(-i)}(x_i)$). Finally, to compare the i th observed response $\tilde{y}_i$ and the predicted value $\tilde{Y}_{(-i)}(x_i)$, we calculate the mean of distances d_w (Eq. 2.5) between y_i and $\tilde{Y}_{(-i)}(x_i)$ which we call it MDC.

Definition 4.1. *For QT2F regression model (12), the MDC index is defined by*

$$MDC = \frac{1}{n}\sum_{i=1}^{n} d_w(\tilde{Y}_{(-i)}(x_i), \tilde{y}_i), \tag{39}$$

where $\tilde{Y}_{(-i)}(x_i)$ is the QT2F response predicted by omitting the ith observation or the ith input-output data.

Definition 4.2. *For QT2F regression model (15), the mean of distances between estimated and observed values is defined by*

$$MD^* = \frac{1}{n}\sum_{i=1}^{n} d_w(\tilde{Y}_i, \tilde{y}_i) \tag{40}$$

Now, by using the above indices, the relative error of the estimated responses can be defined as

$$RE = \frac{|MDC - MD^*|}{MD^*}. \tag{41}$$

5. . Numerical Results

Since examples containing quasi type-2 fuzzy data did not exist in previous works (note that almost all of the previous studies concenterated on type-1 fuzzy data), we change the data of examples given in [28, 12] to non-symmetric triangular QT2FNs.

Example 5.1. *This example considers 8 non-symmetric quasi type-2 fuzzy input-output data. The observed inputs and responses are given in Table 1. The regression model obtained by solving LP model related to WGP for the data of Table 1 with $w_1 = w_2 = \cdots = w_7 = 1$ is as follows:*

$$\tilde{Y} = 3.9444 + 0.4444\tilde{x}.$$

An outlier may be outlying with respect to its spreads, its center, or both. In the following example, we consider different cases to see the effect of outliers on our results.

Example 5.2. *To see the effect of outliers in above example, we consider*
(a) *the outlier with respect to the center of response variable,*
(b) *the outlier with respect to the center of independent variable,*
(c) *the outlier with respect to the spreads of response variables,*
(d) *the outlier with respect to the spreads and center values.*
This data set is derived from Table 1 by changing
(a) $c_{\tilde{y}_4}$ *from* 6.5 *to* 16*,* ***(b)*** $c_{\tilde{x}_3}$ *from* 5.5 *to* 17*,*
(c) *the data No. 1 from* $\langle 4; 1.55, 1.5; 0.8, 0.75; 0.55, 0.5\rangle$ *to* $\langle 4; 3.55, 3.5; 2.8, 2.75; 2.55, 2.5\rangle$ *and* ***(d)*** *the data No. 1 to* $\langle 14; 3.55, 3.5; 2.8, 2.75; 2.55, 2.5\rangle$*.*
The results of ${J_M}^{(i)}$ *and* r_i *obtained by the proposed approach are presented in Table 2.*

TABLE 2. The values of ${J_M}^{(i)}$ *and* r_i *for Example 2.*

	a		**b**		**c**		**d**	
i	${J_M}^{(i)}$	r_i	${J_M}^{(i)}$	r_i	${J_M}^{(i)}$	r_i	${J_M}^{(i)}$	r_i
1	17.63	0.138	13.01	0.161	10.31	0.58*	10.31	0.67*
2	19.64	0.04	14.25	0.08	23.60	0.043	30.08	0.062
3	18.70	0.086	10.23	0.34*	22.23	0.099	30.8	0.039
4	10.91	0.46*	13.93	0.103	22.91	0.071	29.59	0.077
5	19.87	0.028	13.79	0.111	23.38	0.052	31.54	0.016
6	18.39	0.101	14.73	0.051	23.17	0.06	30.95	0.035
7	18.4	0.1	12.19	0.215	22.01	0.01	29.62	0.076
8	17.52	0.143	13.32	0.142	22.34	0.094	30.59	0.046

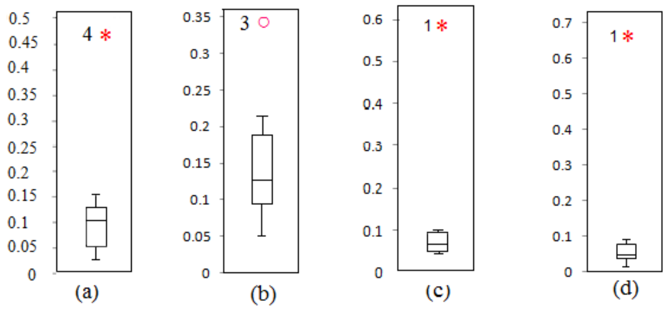

FIGURE 3. Box plots of the values of the normalized absolute difference.

The values of median, first and third quartiles of r_i*, which have been denoted by* Me*,* Q_1*,* Q_3*, respectively, are presented in Table 3. Hence, in part (**a**) the data point No. 4 is an extreme outlier, in part (**b**) the data No. 3 is a mild outlier and in parts (**c**) and (**d**) the data No. 1 is an extereme outlier. The obtained regression models are as follows:*

$$\tilde{Y}_{(a)} = 3.4 + 0.6\tilde{x} \quad , \quad \tilde{Y}_{(b)} = 2.9524 + 0.5238\tilde{x}$$
$$\tilde{Y}_{(c)} = 3.94 + 0.44\tilde{x} \quad , \quad \tilde{Y}_{(d)} = 5.928 + 0.285\tilde{x}$$

6. . Application to soil science

In soil science studies, sometimes, problems arise in measurement of physical, chemical and/or biological soil properties. The problem results from the difficulty, time and cost of direct measurements. Pedomodels (derived from Greek root of pedo as soil) have become

TABLE 3. The values of Me, Q_1, Q_3, IQR and cutoffs for Table 2

	(a)	*(b)*	*(c)*	*(d)*
Me	*0.1005*	*0.126*	*0.065*	*0.054*
Q_1	*0.06*	*0.091*	*0.047*	*0.037*
Q_3	*0.14*	*0.185*	*0.096*	*0.076*
IQR	*0.04*	*0.094*	*0.049*	*0.039*
$Q_1 - 1.5IQR$	*0.003*	*-0.049*	*-0.02*	*-0.02*
$Q_1 - 3IQR$	*-0.057*	*-0.189*	*-0.1*	*-0.08*
$Q_3 + 1.5IQR$	*0.2*	*0.32*	*0.169*	*0.134*
$Q_3 + 3QR$	*0.26*	*0.46*	*0.243*	*0.193*

a popular topic in soil science and environmental research. They are predictive functions of certain soil properties based on other easily or cheaply measured properties [31]. In this article, two pedomodels including one and two independent variables are studied to develop the relationships between different chemical and physical soil properties by means of quasi type-2 fuzzy regression technique. Based on a study in a part of Silakhor plain (situated in a province located in the west of Iran), a total of 24 core samples were obtained from 0.0 to $25 - cm$ depth [29].

Table 4. Observed and predicted quasi type-2 fuzzy values of SAR-ESP and their distances (for $w = 1$)

i	SAR ($\tilde{x}_i$)	ESP ($\tilde{y}_i$)	Predicted ESP ($\tilde{Y}_i$)	d_w
1	$\langle 0.78; 0.78, 0.55; 0.39, 0.43; 0.01, 0.32\rangle$	$\langle 3.08; 3.06, 2.92; 1.55, 2.37; 0.04, 1.82\rangle$	$\langle 5.58; 5.47, 4.3; 2.95, 3.15; 0.42, 2\rangle$	9.06
2	$\langle 0.64; 0.57, 0.24; 0.31, 0.17; 0.06, 0.11\rangle$	$\langle 2.86; 2.55, 2.03; 1.40, 1.93; 0.26, 1.83\rangle$	$\langle 4.87; 3.81, 2.13; 2.15, 1.44; 0.49, 0.75\rangle$	5.94
3	$\langle 0.62; 0.57, 0.22; 0.32, 0.21; 0.07, 0.2\rangle$	$\langle 6.25; 5.74, 3.6; 3.23, 2.37; 0.72, 1.14\rangle$	$\langle 4.77; 3.79, 2.01; 2.22, 1.66; 0.66, 1.32\rangle$	6.94
4	$\langle 0.49; 0.38, 0.14; 0.27, 0.1; 0.16, 0.07\rangle$	$\langle 4.11; 3.17, 3.81; 2.24, 2.74; 1.32, 1.67\rangle$	$\langle 4.11; 2.52, 1.3; 1.79, 0.95; 1.07, 0.61\rangle$	6.67
5	$\langle 1.1; 0.83, 0.52; 0.64, 0.42; 0.45, 0.32\rangle$	$\langle 1.04; 0.78, 1.01; 0.6, 0.71; 0.43, 0.42\rangle$	$\langle 7.21; 5.75, 4.17; 4.45, 3.33; 3.16, 2.5\rangle$	25.61
6	$\langle 0.61; 0.36, 0.36; 0.31, 0.19; 0.26, 0.02\rangle$	$\langle 2.71; 1.62, 2.43; 1.39, 1.58; 1.17, 0.73\rangle$	$\langle 4.72; 2.64, 2.64; 2.14, 1.53; 1.64, 0.41\rangle$	4.83
7	$\langle 0.74; 0.31, 0.25; 0.25, 0.23; 0.2, 0.22\rangle$	$\langle 4.45; 1.87, 1.9; 1.54, 1.16; 1.22, 0.43\rangle$	$\langle 5.38; 2.21, 1.9; 1.85, 1.75; 1.49, 1.59\rangle$	3.61
8	$\langle 1.15; 0.64, 1.05; 0.53, 0.76; 0.42, 0.47\rangle$	$\langle 6.92; 3.86, 4.31; 3.2, 3.43; 2.54, 2.55\rangle$	$\langle 7.47; 5.17, 7.26; 4.15, 5.33; 3.14, 3.4\rangle$	9.13
9	$\langle 1.08; 0.87, 0.91; 0.59, 0.64; 0.32, 0.38\rangle$	$\langle 7.41; 6, 1.63; 4.09, 1.4; 2.18, 1.17\rangle$	$\langle 7.11; 6.44, 6.64; 4.43, 4.68; 2.42, 2.72\rangle$	11.19
10	$\langle 0.38; 0.37, 0.16; 0.2, 0.12; 0.03, 0.08\rangle$	$\langle 9.08; 8.74, 9.03; 4.68, 8.13; 0.63, 7.23\rangle$	$\langle 3.55; 2.48, 1.41; 1.38, 0.97; 0.27, 0.53\rangle$	36.91
11	$\langle 0.61; 0.58, 0.24; 0.31, 0.2; 0.04, 0.16\rangle$	$\langle 6.56; 6.19, 3.45; 3.33, 3.26; 0.47, 3.08\rangle$	$\langle 4.72; 3.88, 2.15; 2.15, 1.59; 0.43, 1.04\rangle$	10.36
12	$\langle 0.98; 0.62, 0.25; 0.4, 0.14; 0.18, 0.03\rangle$	$\langle 5.05; 3.17, 3.24; 2.06, 2.86; 0.95, 2.48\rangle$	$\langle 6.6; 4.14, 2.25; 2.64, 1.32; 1.15, 0.39\rangle$	7.92
13	$\langle 0.71; 0.62, 0.61; 0.59, 0.59; 0.57, 0.58\rangle$	$\langle 5.23; 4.55, 4.1; 4.35, 2.27; 4.16, 0.44\rangle$	$\langle 5.23; 4.55, 4.49; 4.37, 4.37; 4.2, 4.25\rangle$	6.38
14	$\langle 0.5; 0.44, 0.37; 0.35, 0.355; 0.26, 0.34\rangle$	$\langle 5.16; 4.59, 2.35; 3.63, 1.66; 2.67, 0.98\rangle$	$\langle 4.16; 3.15, 2.8; 2.58, 2.6; 2, 2.41\rangle$	6.97
15	$\langle 0.77; 0.4, 0.59; 0.27, 0.49; 0.14, 0.39\rangle$	$\langle 11.1; 5.84, 10.57; 3.9, 7.18; 1.97, 3.79\rangle$	$\langle 5.53; 3.15, 4.12; 2.23, 3.35; 1.31, 2.58\rangle$	22.04
16	$\langle 0.99; 0.7, 0.78; 0.44, 0.58; 0.19, 0.39\rangle$	$\langle 4.47; 3.16, 0.86; 2, 0.55; 0.85, 0.24\rangle$	$\langle 6.65; 5.24, 5.64; 3.43, 4.14; 1.62, 2.64\rangle$	17.25
17	$\langle 3.6; 3.42, 2.17; 2.84, 1.72; 2.27, 1.28\rangle$	$\langle 28.84; 27.5, 24.56; 22.8, 23; 18.13, 21.7\rangle$	$\langle 19.9; 23.74, 17.4; 19.6, 13.9; 15.6, 10.5\rangle$	45.84
18	$\langle 0.86; 0.57, 0.4; 0.29, 0.31; 0.02, 0.22\rangle$	$\langle 9.43; 6.28, 7.75; 3.24, 5.06; 0.2, 2.37\rangle$	$\langle 5.99; 4, 3.13; 2.18, 2.26; 0.37, 1.39\rangle$	15.33
19	$\langle 0.61; 0.37, 0.47; 0.31, 0.27; 0.26, 0.07\rangle$	$\langle 4.5; 2.7, 2.31; 2.3, 1.75; 1.9, 1.2\rangle$	$\langle 4.72; 2.83, 3.34; 2.26, 2.03; 1.69, 0.73\rangle$	2.37
20	$\langle 0.64; 0.4, 0.48; 0.37, 0.38; 0.34, 0.28\rangle$	$\langle 9.3; 5.79, 8.81; 5.35, 4.56; 4.91, 0.31\rangle$	$\langle 4.87; 3.03, 3.44; 2.73, 2.78; 2.43, 2.12\rangle$	21.23
21	$\langle 0.71; 0.65, 0.47; 0.48, 0.425; 0.32, 0.38\rangle$	$\langle 9.48; 8.7, 7.68; 6.5, 4.88; 4.31, 2.09\rangle$	$\langle 5.23; 4.57, 3.66; 3.5, 3.19; 2.42, 2.72\rangle$	19.61
22	$\langle 0.61; 0.47, 0.19; 0.27, 0.15; 0.08, 0.11\rangle$	$\langle 3.65; 2.83, 1.03; 1.66, 1.01; 0.49, 0.99\rangle$	$\langle 4.72; 3.14, 1.71; 1.88, 1.24; 0.62, 0.77\rangle$	2.87
23	$\langle 0.63; 0.14, 0.19; 0.09, 0.19; 0.04, 0.19\rangle$	$\langle 10.14; 2.27, 9.35; 1.49, 7.22; 0.71, 5.09\rangle$	$\langle 4.82; 1.08, 1.34; 0.77, 1.28; 0.46, 1.22\rangle$	25.26
24	$\langle 1.13; 0.99, 0.99; 0.63, 0.57; 0.28, 0.16\rangle$	$\langle 3; 2.64, 2.45; 1.69, 2.28; 0.75, 2.12\rangle$	$\langle 7.36; 7.28, 7.28; 4.6, 4.29; 1.92, 1.31\rangle$	20.74
		Mean of distance		14.34

6.1. . **Pedomodel of ESP-SAR..** We first wish to provide a relationship between exchangeable sodium percentage (ESP), as the dependent variable, and sodium absorption ratio (SAR), as an independent variable. The exchange sodium percentage, ESP, governs the source/sink phenomenon for ionic constituents, i.e., sodium, as a contaminant in sodic soils, is calculated from the ratio of exchangeable sodium, Na_x, to cation exchangeable capacity, CEC. All these soil parameters, measured on soil colloidal surface, are time consuming and costly. Due to close relationship between the distribution of cations in the exchange and solution phases, it is preferred to estimate ESP from sodium absorption ratio, SAR, in soil solution [29]. In this case, ESP is considered as cost and time variable, therefore the need for less expensive indirect measurement is emphasized. Measurements of SAR have been related to ESP due to low cost, simplicity, and the possibility of relating measurements to the quantity and quality parameters. But, due to some impreciseness in related experimental environment, the observations of response variable (ESP) are given in fuzzy form. Thus, we may use a fuzzy method for modeling such a data set [29].

Estimation of model parameters. To develop a relationship between ESP (as a dependent variable) and SAR (as an independent variable), using quasi type-2 fuzzy regression, first, the values of dependent variable $\tilde{y}_i$ and independent variables $\tilde{x}_{ij}$, were fuzzified by consulting an expert using asymmetric triangular QT2FN with right and left spreads proportional to $\tilde{y}_i$ and $\tilde{x}_{ij}$, respectively. The results are given in Table 4. These amounts of vagueness were based on expert opinion and might be considered as the acceptable levels of uncertainty.

In this study, for each choice of a_1' and a_1'', the regression model for the data of table 4 is as follows:

$$\tilde{Y} = a_0 + a_1\tilde{x}_1 = a_0 + (a_1' - a_1'')\tilde{x}_1 \tag{42}$$

In the above model, non-symmetric QT2FNs $\tilde{Y}$ and $\tilde{x}_1$ are cation sodium absorption ratio (SAR) and exchange sodium percentage (ESP), respectively.

According to the proposed method, the regression model is obtained as

$$\begin{gathered}\tilde{Y} = a_0 + (a_1' - a_1'')\tilde{x}_1 = 1.61 + (6.22 - 1.13)\tilde{x}_1 = \\ \langle 1.61 + 5.09c_{\tilde{x}_1}; 6.22\alpha_{1\tilde{x}_1} + 1.13\beta_{1\tilde{x}_1}, 6.22\beta_{1\tilde{x}_1} + 1.13\alpha_{1\tilde{x}_1}; 6.22\alpha_{\tilde{x}_1} + 1.13\beta_{\tilde{x}_1}, \\ 6.22\beta_{\tilde{x}_1} + 1.13\alpha_{\tilde{x}_1}; 6.22\alpha_{2\tilde{x}_1} + 1.13\beta_{2\tilde{x}_1}, 6.22\beta_{2\tilde{x}_1} + 1.13\alpha_{2\tilde{x}_1}\rangle\end{gathered} \tag{43}$$

The above QT2F regression model can be used to predict the ESP of a new case. For example, if for a new case, $SAR = \langle 1.08; 0.87, 0.91; 0.59, 0.64; 0.32, 0.38\rangle$ then by Eq. (43), we predict the ESP as $\tilde{Y} = \langle 7.10; 6.43, 6.64; 4.38, 4.64; 2.41, 2.72\rangle$.

6.2. . **Pedomodel of CEC-SAND-OM..** The second model provides a relationship between cation exchange capacity (CEC), as a function of two soil variables namely percentage of sand content (SAND) and organic matter content (OM)(Table 5). In the soil, organic matter can enhance the CEC, while the sand content has negative effect on the cation exchange capacity [29].

Estimation of model parameters. The regression model for the data of Table 5 is as follows:

$$\tilde{Y} = a_0 + a_1\tilde{x}_1 + a_2\tilde{x}_2 \tag{44}$$

where for each choice of a_j' and a_j'', $j = 1, 2$ Eq. (44) is obtained as following

$$\tilde{Y} = a_0 + (a_1' - a_1'')\tilde{x}_1 + (a_2' - a_2'')\tilde{x}_2 \tag{45}$$

In the above model, non-symmetric QT2FNs $\tilde{Y}$, $\tilde{x}_1$ and $\tilde{x}_2$ are cation exchange capacity (CEC), percentage of sand content (SAND) and organic matter content (OM), respectively. By solving the linear programming problem related to WGP, the regression model is obtained as

$$\begin{aligned}\tilde{Y} &= a_0 + (a_1' - a_1'')\tilde{x}_1 + (a_2' - a_2'')\tilde{x}_2 = 22.19 + (0.02 - 0.19)\tilde{x}_1 + (4.23 - 2.67)\tilde{x}_2 \\ &= 22.19 + (-0.17)\tilde{x}_1 + (1.56)\tilde{x}_2\end{aligned} \tag{46}$$

TABLE 6. Values of the independent variable ($\tilde{x}$), the dependent variable ($\tilde{y}$) and the corresponding predictions of $\tilde{Y}$ by cross-validation method and the distances between the observed and estimated values(for $w = 1$)

i	$\tilde{x}_i$	$\tilde{y}_i$	$\tilde{Y}_i$	d_w
1	$\langle 2; 1.25, 1.35, 0.5; 0.6, 0.25, 0.35\rangle$	$\langle 4; 1.55, 1.5; 0.8, 0.75; 0.55, 0.5\rangle$	$\langle 4.8, 1.2, 1.25, 0.49, 0.54, 0.26, 0.3\rangle$	2.42
2	$\langle 3.5; 1.59, 1.62; 0.84, 0.87; 0.59, 0.62\rangle$	$\langle 5.5; 1.41, 1.55; 0.66, 0.8; 0.41, 0.55\rangle$	$\langle 4.78, 1.57, 1.58, 0.83, 0.84, 0.58, 0.6\rangle$	1.37
3	$\langle 5.5; 1.45, 1.75; 0.7, 1; 0.45, 0.75\rangle$	$\langle 7.5; 1.25, 1.75; 0.5, 1; 0.25, 0.75\rangle$	$\langle 5.83, 1.45, 1.61, 0.73, 0.89, 0.49, 0.65\rangle$	2.69
4	$\langle 7; 1.75, 1.42; 1, 0.67; 0.75, 0.42\rangle$	$\langle 6.5; 1.47, 1.28; 0.72, 0.53; 0.47, 0.28\rangle$	$\langle 7.6, 1.66, 1.46, 0.92, 0.72, 0.67, 0.47\rangle$	2.28
5	$\langle 8.5; 1.45, 1.59; 0.7, 0.84; 0.45; 0.59\rangle$	$\langle 8.5; 1.46, 1.39; 0.71, 0.64; 0.46, 0.39\rangle$	$\langle 7.4, 1.45, 1.52, 0.71, 0.79, 0.47, 0.54\rangle$	1.56
6	$\langle 10.5; 1.56, 1.39; 0.81, 0.64; 0.56, 0.39\rangle$	$\langle 8; 1.42, 1.4; 0.67, 0.65; 0.42, 0.4\rangle$	$\langle 9.7, 1.5, 1.4, 0.76, 0.66, 0.52, 0.41\rangle$	2.02
7	$\langle 11; 1.32, 1.51; 0.57, 0.76; 0.32, 0.51\rangle$	$\langle 10.5; 1.52, 1.57; 0.77, 0.82; 0.52, 0.57\rangle$	$\langle 8.83, 1.29, 1.37, 0.58, 0.67, 0.34, 0.43\rangle$	2.72
8	$\langle 12.5; 1.44, 1.65; 0.69, 0.9; 0.44, 0.65$	$\langle 9.5; 1.75, 1.72; 1, 0.97; 0.75, 0.72\rangle$	$\langle 11.26, 1.4, 1.55, 0.68, 0.83, 0.45, 0.59\rangle$	3.14
		Mean of distance		2.27

The above QT2F regression model can be used to predict the CEC of a new case. For example, if for a new case,

$$SAND = \langle 38; 36.2, 17.29; 30.04, 12.26; 23.88, 7.24\rangle$$
$$OM = \langle 0.84; 0.24, 0.35; 0.235, 0.29; 0.23, 0.23\rangle$$

then by Eq. (46), the predicted CEC is

$$\tilde{Y} = \langle 17.04; 6.035, 9.33; 4.66, 7.78; 3.42, 6.25\rangle.$$

In order to evaluate the predictive ability of the above models, the MDC is calculated for each model. For example, the results of example 5.1 are given in Table 6. In this example the value of MDC was obtained to be 2.27 and the value of MD^* is 1.57, which the relative diffrence between them is equal to 0.3.

- The value of the MDC for the ESP-SAR regression was obtained to be 16.92, which is very close to the value of MD^*, i.e., 14.34. Note that the relative error between MDC and MD^* is RE=0.15.
- The value of the MDC for the CEC-OM-SAND regression was obtained to be 27.9, which is close to the value of MD^*, i.e., 27.21. Note that the relative error between MDC and MD^* is RE=0.02.

It is appeared that predictive ability to the SAND-OM-CEC model is much better than the ESP-SAR model.
Recently, Rabiei et al. [34] used a distance on the space of interval type-2 fuzzy numbers (which determinated using its FOU) and proposed a least-squares method (LS) to obtain coefficients of the proposed model. In soil science examples, if input and output variables are considered to be interval-valued fuzzy numbers, we use the distance introduced in (36). In the case of interval type-2 fuzzy input-output data, the regression models and the relative error between MDC and MD^* in two approachs are as follows:

- ESP-SAR regression model:
  $\tilde{Y}_{GP} = 1.172 + 5.742\tilde{x} \quad , \quad \tilde{Y}_{LS} = 0.835 + 6.879\tilde{x}$
  relative error between MDC and MD^* for GP model= 0.129
  relative error between MDC and MD^* for LS model= 0.273
- SAND-OM-CEC regression model:
  $\tilde{Y}_{GP} = 21.96 - 0.16\tilde{x}_1 + 1.57\tilde{x}_2 \quad , \quad \tilde{Y}_{LS} = 21.97 - 0.23\tilde{x}_1 + 2.57\tilde{x}_2.$
  relative error between MDC and MD^* for GP model= 0.04
  relative error between MDC and MD^* for LS model= 0.09

Since, the relative error between MDC and MD^* for GP model are less than LS model, the predictive ability of our model is much better than LS model in these examples.

Table 5. Observed and predicted quasi type-2 fuzzy values of SAND, OM and CEC (for w=1)

i	SAND ($\tilde{x}_{i1}$)	OM ($\tilde{x}_{i2}$)	CEC ($\tilde{y}_i$)	Predicted CEC ($\tilde{Y}_i$)	d_w
1	$\langle 35; 26.44, 33.34; 20.9, 27.67; 15.36, 22\rangle$	$\langle 0.88; 0.84, 0.37; 0.68, 0.27; 0.52, 0.18\rangle$	$\langle 16.5; 11.54, 16.4; 9.2, 8.31; 6.87, 0.23\rangle$	$\langle 17.59; 11.54, 9.62; 9.39, 7.6; 7.25, 5.58\rangle$	14.5
2	$\langle 37; 24.31, 24.62; 21.99, 12.7; 19.68, 0.78\rangle$	$\langle 1.13; 0.8, 0.44; 0.76, 0.36; 0.72, 0.29\rangle$	$\langle 18.6; 6.91, 16.61; 5.1, 9.14; 3.3, 1.68\rangle$	$\langle 17.64; 9.83, 9.21; 7.11, 8.08; 4.4, 6.95\rangle$	20.72
3	$\langle 27; 8.59, 16.19; 6.7, 13.8; 4.81, 11.41\rangle$	$\langle 1.31; 0.75, 0.34; 0.49, 0.19; 0.24, 0.04\rangle$	$\langle 19.3; 6.73, 17.74; 6.41, 9.97; 6.09, 2.2\rangle$	$\langle 19.63; 7.38, 5.45; 5.4, 3.71; 3.42, 1.98\rangle$	23.41
4	$\langle 29; 8.96, 18.05; 8.8, 16.67; 8.64, 15.3\rangle$	$\langle 1.98; 1.83, 1.7; 1.32, 1.65; 0.81, 1.61\rangle$	$\langle 20.3; 5.88, 15.65; 4.5, 11.09; 3.1, 6.53\rangle$	$\langle 20.33; 15.93, 14.16; 13.4, 12.6; 10.86, 11\rangle$	34.18
5	$\langle 38; 33.2, 34.87; 19.31, 26.07; 5.42, 17.28\rangle$	$\langle 1.02; 0.99, 0.75; 0.7, 0.72; 0.41, 0.7\rangle$	$\langle 17.3; 8.13, 13; 6.58, 10.05; 5.03, 7.11\rangle$	$\langle 17.3; 13.63, 12.97; 10.34, 9.22; 7.05, 5.47\rangle$	13.75
6	$\langle 32; 31.81, 24.78; 16.13, 14.53; 0.45, 4.29\rangle$	$\langle 1.29; 1.16, 1; 0.75, 0.82; 0.35, 0.65\rangle$	$\langle 20.4; 12.13, 12.21; 6.45, 10.52; 0.77, 8.84\rangle$	$\langle 18.74; 13.04, 13.99; 8.55, 8.93; 4.05, 3.86\rangle$	16.31
7	$\langle 29; 25.9, 6.49; 14.25, 4.25; 2.61, 2.02\rangle$	$\langle 1.52; 0.65, 1.19; 0.4, 0.89; 0.15, 0.6\rangle$	$\langle 19.3; 6.5, 8.1; 6.08, 6.7; 5.66, 5.31\rangle$	$\langle 19.61; 7.74, 11.9; 5.21, 7.69; 2.68, 3.48\rangle$	12.02
8	$\langle 18; 16.54, 15.82; 9.3, 10.15; 2.06, 4.48\rangle$	$\langle 1.33; 0.83, 0.82; 0.66, 0.65; 0.49, 0.49\rangle$	$\langle 21.9; 19.94, 12.2; 14.42, 10.12; 8.91, 8.05\rangle$	$\langle 21.19; 9.11, 9.21; 6.7, 6.54; 4.29, 3.87\rangle$	34.59
9	$\langle 40; 30.84, 37.97; 21.85, 30.8; 12.87, 23.7\rangle$	$\langle 1.71; 0.38, 0.8; 0.32, 0.62; 0.27, 0.44\rangle$	$\langle 15.9; 13.43, 12.87; 9.54, 8.77; 5.66, 4.68\rangle$	$\langle 18.03; 11.72, 11.16; 9.44, 8.37; 7.16, 5.57\rangle$	8.44
10	$\langle 28; 21.02, 19.84; 16.26, 18.88; 11.5, 17.92\rangle$	$\langle 2; 2, 1.54; 1.79, 0.88; 1.59, 0.23\rangle$	$\langle 18.3; 7.72, 17.61; 5.7, 9.43; 3.71, 1.26\rangle$	$\langle 20.53; 16.81, 16.31; 13.93, 12.07; 11.04, 7.8\rangle$	37.39
11	$\langle 13; 7.78, 7.48; 6.7, 4.92; 5.63, 2.37\rangle$	$\langle 1.68; 0.88, 1.27; 0.83, 1; 0.79, 0.74\rangle$	$\langle 22.6; 8.73, 21.33; 7.26, 11.47; 5.79, 1.62\rangle$	$\langle 22.6; 8.73, 9.39; 7.31, 7.88; 5.9, 6.38\rangle$	20.45
12	$\langle 19; 7.98, 17.59; 6.6, 12.66; 5.23, 7.74\rangle$	$\langle 2.15; 1.38, 1.41; 1.215, 1.27; 1.05, 1.14\rangle$	$\langle 23.7; 6.08, 14.89; 3.44, 9.67; 0.8, 4.46\rangle$	$\langle 22.3; 13.17, 11.57; 11.13, 10.19; 9.09, 8.8\rangle$	32.65
13	$\langle 31; 17.27, 30.22; 14.3, 21.31; 11.4, 12.41\rangle$	$\langle 3.52; 2.76, 1.12; 1.53, 0.87; 0.3, 0.63\rangle$	$\langle 22.4; 18.82, 23.21; 18.3, 22.32; 17.8, 21.43\rangle$	$\langle 22.4; 20.87, 16.1; 13.23, 11.01; 5.6, 5.93\rangle$	53.27
14	$\langle 31; 25.1, 27.7; 17.11, 18.06; 9.13, 8.3\rangle$	$\langle 2.33; 1.06, 0.72; 0.75, 0.7; 0.44, 0.69\rangle$	$\langle 21.8; 16.04, 19.4; 15.5, 15.3; 14.98, 11.28\rangle$	$\langle 20.54; 12.3, 11.31; 8.9, 8.67; 5.5, 6.03\rangle$	41.03
15	$\langle 17; 16.36, 7.24; 8.76, 4.44; 1.17, 1.64\rangle$	$\langle 1.71; 1.63, 1.49; 1.105, 0.86; 0.58, 0.24\rangle$	$\langle 23.8; 18.4, 12.52; 15.16, 8.36; 11.95, 4.21\rangle$	$\langle 21.96; 12.63, 13.97; 8.03, 8.4; 3.43, 2.82\rangle$	26.09
16	$\langle 14; 13.21, 8.71; 7.1, 6.93; 1, 5.16\rangle$	$\langle 1.14; 0.22, 1.13; 0.14, 0.57; 0.06, 0.02\rangle$	$\langle 20.8; 16.32, 14.68; 12.265, 9.3; 8.21, 3.93\rangle$	$\langle 21.58; 5.91, 8.1; 3.61, 4.32; 1.32, 0.54\rangle$	41.65
17	$\langle 19; 11.94, 4.18; 7.75, 3.58; 3.57, 2.99\rangle$	$\langle 0.99; 0.84, 0.88; 0.79, 0.48; 0.74, 0.09\rangle$	$\langle 17.5; 10.83, 16.67; 8.63, 13.83; 6.43, 11\rangle$	$\langle 20.49; 6.97, 8.36; 5.5, 5.73; 4.02, 3.11\rangle$	36.68
18	$\langle 28; 24.34, 27.85; 23.31, 25.1; 22.3, 22.31\rangle$	$\langle 1.14; 0.94, 1.05; 0.61, 0.59; 0.3, 0.13\rangle$	$\langle 17.8; 8.34, 11.85; 6.44, 6.11; 4.55, 0.38\rangle$	$\langle 19.19; 12.67, 12.25; 9.51, 9.17; 6.35, 6.1\rangle$	19.79
19	$\langle 26; 23.13, 13.66; 18.3, 12.94; 13.45, 12.2\rangle$	$\langle 1.46; 0.75, 1.13; 0.57, 0.8; 0.39, 0.47\rangle$	$\langle 20.2; 15.6, 12.11; 8.98, 10.32; 2.37, 8.54\rangle$	$\langle 20.03; 9.32, 11.53; 7.44, 8.71; 5.55, 5.9\rangle$	15.99
20	$\langle 32; 16.83, 20.52; 11.25, 18.1; 5.67, 15.7\rangle$	$\langle 1.81; 1.71, 1.36; 0.88, 1.05; 0.06, 0.74\rangle$	$\langle 20; 15.11, 12.45; 11.94, 11.5; 8.78, 10.55\rangle$	$\langle 19.55; 15.19, 14.01; 10.28, 9.37; 5.37, 4.72\rangle$	15.1
21	$\langle 10; 7.06, 7.83; 4.47, 4.33; 1.89, 0.84\rangle$	$\langle 1.38; 1.12, 0.83; 0.71, 0.71; 0.3, 0.6\rangle$	$\langle 22.8; 15, 20.92; 13.55, 15.64; 12.13, 10.37\rangle$	$\langle 22.64; 8.62, 8.03; 5.84, 5.88; 3.07, 3.72\rangle$	52.56
22	$\langle 38; 36.2, 17.29; 30.04, 12.26; 23.88, 7.24\rangle$	$\langle 0.84; 0.24, 0.35; 0.23, 0.29; 0.23, 0.23\rangle$	$\langle 19.1; 6.08, 14.79; 4.74, 8.67; 3.4, 2.56\rangle$	$\langle 17.01; 6.06, 9.46; 4.78, 7.9; 3.5, 6.34\rangle$	12.11
23	$\langle 49; 32.6, 46.67; 16.82, 31.69; 1.04, 16.72\rangle$	$\langle 1.48; 1.36, 0.82; 1.05, 0.68; 0.74, 0.54\rangle$	$\langle 12.1; 3.74, 2.71; 3.67, 1.77; 3.6, 0.84\rangle$	$\langle 16.14; 17.64, 14.4; 12.73, 9.61; 7.81, 4.82\rangle$	54.74
24	$\langle 42; 25.19, 8.09; 21.47, 5.18; 17.75, 2.28\rangle$	$\langle 1.08; 0.88, 0.87; 0.82, 0.59; 0.76, 0.32\rangle$	$\langle 12.8; 11.18, 11.25; 6.5, 7.22; 1.83, 3.19\rangle$	$\langle 16.71; 8.15, 11.06; 6.52, 8.95; 4.89, 6.85\rangle$	15.61
	Mean of distance				27.21

7. Conclusion

This paper proposes a weighted goal programming approach (WGP) to estimate the coefficients of quasi type-2 fuzzy (QT2F) linear regression model with quasi type-2 fuzzy input-output data and crisp coefficients. To estimate the regression coefficients, a nonlinear programming problem has been presented and converted to a goal programming problem and then to a linear programming one. The advantage of this conversion is that linear programming problems can be solved exactly by available algorithms. Whereas, the available algorithms for solving nonlinear programming problems often give approximate solutions.

Since the existence of outliers in the data set causes incorrect results, the results of the proposed model illustrate that this model is less sensitive to outliers. To handle the outlier problem, an omission approach has been used to detect outliers. This approach examines how the values of the objective function of the linear programming problem used to obtain regression coefficients changes when some of the observations are omitted. The main disadvantage in some procedures for detecting outliers is lack of defining cutoffs for outliers. Also, they must pre-assign some values to some parameters and they cannot conduct a formal test for detecting outliers. The advantage of our approach is that, it does not need to select any parameters beforehand. Also, a simple display tool has been conducted by means of the boxplot procedure as a formal test to define the cutoffs for outliers.

The method was illustrated by some applied examples. The proposed method is quite general and can be used in many fields of application. The predict ability of the model evaluated by cross-validation method. In soil science examples, obtained results from comparison the WGP approach with least square method showed that, the predictive ability of our WGP model is better than least square model. Applying WGP approach to type-2 fuzzy linear regression model with type-2 fuzzy input-outputs and type-2 fuzzy coefficients is our future research.

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Elham Hosseinzadeh received her BSc in applied mathematics from Ferdowsi University of Mashhad, Mashhad, Iran, and MSc degree with first rank in applied mathematics/operations research from the University of Birjand, Birjand, Iran. She is a PhD student of applied mathematics (operations research) at the University of Birjand, Iran under direction of Dr. Hassan Hassanpour. Her research interests are in the field of multiobjective optimization, fuzzy goal programming, linear and nonlinear programming, fuzzy regression and network optimization.

Hassan Hassanpour received his BSc degree in pure mathematics from University of Birjand, Birjand, Iran, MSc degree in applied mathematics from Ferdowsi University of Mashhad, Mashhad, Iran and PhD degree in applied mathematics/operational research from Shahid Bahonar University of Kerman. He is an assistant professor in Department of Mathematics at the University of Birjand. His main research interests are in the field of fuzzy linear regression, multiple criteria optimization, goal programming and fuzzy goal programming, linear and nonlinear programming, and fuzzy regression.

Mohsen Arefi received his BSc and MSc degrees in mathematical statistics from University of Birjand, Birjand, Iran, in 2003 and 2005, respectively, and his Ph.D. degree in statistical inference from Isfahan University of Technology, Isfahan, Iran in 2010. His research interests include statistical inference in fuzzy and vague environments (specially in regression and testing statistical hypothesis), Bayesian statistics, intuitionistic fuzzy set theory, and pattern recognition

Massoud Aman received his BSc degree in mathematical and computer science from the Ferdowsi University of Mashhad in 1997 and his MSc degree in applied mathematics/operations research from AmirKabir University of Tehran in 1999 and his Ph.D. degree in applied mathematics/numerical analysis from Ferdowsi University of Mashhad in 2005. He is currently an assistant professor in the Department of Mathematical and Statistical Sciences at the University of Birjand. His research interests are in the areas of network flow problems, combinatorial optimization and numerical methods for solving partial differential equations.