TWMS J. App. Eng. Math. V.6, N.2, 2016, pp. 213-223

EXISTENCE AND UNIQUENESS OF TRIPLED FIXED POINTS FOR MIXED MONOTONE OPERATORS WITH PERTURBATIONS AND APPLICATION

HOJJAT AFSHARI [1], ALIREZA KHEIRYAN [2], §

ABSTRACT. In this article, we get the existence and uniqueness of tripled fixed points without assuming the operator to be compact or continuous, which extends the existing corresponding results. As applications, we utilize the results obtained in this paper to study the existence and uniqueness of positive solutions for a fractional differential equation boundary value problem.

Keywords: tripled fixed point, mixed monotone operator, positive solution, fractional differential equation.

AMS Subject Classification: 83-02, 99A00

1. Introduction

In recent years, boundary value problems of nonlinear fractional differential equations with a variety of boundary conditions have been investigated by many researchers. Fractional differential equations appear naturally in various fields of science and engineering and constitute an important field of research . It should be noted that most papers dealing with the existence of solutions of nonlinear initial value problems of fractional differential equations mainly use the techniques of nonlinear analysis such as fixed point results, the Leary-Schauder theorem, stability, etc. (see for example [3]-[4]).

In 2012, V. Berinde and M. Borcut in [1] introduced the concept of tripled fixed point for nonlinear mappings in partially ordered complete metric spaces and obtained its existence. Recently, Y. Sang [6, 7] proved some results on a class of mixed monotone operators with perturbations. Following the paper of Sang, we will study tripled fixed point theorems for a class of mixed monotone operators with perturbations on ordered Banach spaces, and then we will get the existence and uniqueness of tripled fixed points without assuming the operator to be compact or continuous.

Suppose $(E, \| \,.\, \|)$ is a Banach space which is partially ordered by a cone $P \subseteq E$, that is, $x \leq y$ if and only if $y - x \in P$. If $x \neq y$, then we denote $x < y$ *or* $x > y$. We denote the zero element of E by θ. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies (i) $x \in P,\ \lambda \geq 0 \Longrightarrow \lambda x \in P$; (ii) $x \in P,\ -x \in P \Longrightarrow x = \theta$. A cone P is called normal if there exists a constant $N > 0$ such that $\theta \leq x \leq y$ implies $\| x \| \leq N \| y \|$.

[1] Faculty of Basic Sciences, University of Bonab, Bonab 5551761167, Iran.
e-mail: hojat.afshari@yahoo.com, hojat.afshari@bonabu.ac.ir;

[2] Department of Mathematics, Shahid Madani University of Azerbaidjan 5551761167, Iran.
e-mail: alireza kheiryan@yahoo.com;

§ Manuscript received: June 21, 2015; accepted: December 15, 2015.
TWMS Journal of Applied and Engineering Mathematics, Vol.6, No.2;

Also we define the order interval $[x_1, x_2] = \{x \in E | x_1 \leq x \leq x_2\}$ for all $x_1, x_2 \in E$. We say that and operator $A : E \to E$ is increasing whenever $x \leq y$ implies $Ax \leq Ay$. For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$, such that $\lambda x \leq y \leq \mu x$. Clearly, $\sim$ is an equivalent relation. Given $h > \theta$, we denote by P_h the set $P_h = \{x \in E | x \sim h\}$. It is easy to see that $P_h \subset P$ is convex and $\lambda P_h = P_h$ for all $\lambda > 0$. If $P \neq \phi$ and $h \in P$, it is clear that $P_h = P$. Also recall that an operator $A : E \to E$ is sub-linear, if for any $x, y \in E$, $A(x + y) \leq A(x) + A(y)$.

Definition 1.1. [1] *Let $(X, \leq)$ be a partially ordered set and $F : X \times X \times X \to X$. We say F has the mixed monotone property if for any $x, y, z \in X$,*

$$x_1, x_2 \in X, x_1 \leq x_2 \text{ implies } F(x_1, y, z) \leq F(x_2, y, z), \tag{1}$$

$$y_1, y_2 \in X, y_1 \leq y_2 \text{ implies } F(x, y_1, z) \geq F(x, y_2, z), \tag{2}$$

$$\text{and } z_1, z_2 \in X, z_1 \leq z_2 \text{ implies } F(x, y, z_1) \leq F(x, y, z_2). \tag{3}$$

Definition 1.2. [1] *An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of a mapping $F : X \times X \times X \to X$ if $F(x, y, z) = x, F(y, x, y) = y$ and $F(z, y, x) = z$.*

Theorem 1.1. [6] *Let P be a normal cone in E, and $A : P \times P \to P$ a mixed monotone operator. Let $B : E \to E$ be sublinear. Assume that for all $a < t < b$, there exist two positive-valued functions $\tau(t), \varphi(t, x, y)$ on interval (a, b) such that*
(H_1) $\tau : (a, b) \to (0, 1)$ is surjection;
(H_2) $\varphi(t, x, y) > \tau(t)$ for all $t \in (a, b)$, $x, y \in P$;
(H_3) $A(\tau(t)x, \frac{1}{\tau(t)}y) \geq \varphi(t, x, y)A(x, y)$ for all $t \in (a, b)$, $x, y \in P$; item[(H_4)]
$(I - B)^{-1} : E \to E$ exists and is an increasing operator.
Furthermore, for any $t \in (a, b), \phi(t, x, y)$ is non-increasing in x for fixed y, and nondecreasing in y for fixed x. In addition, suppose that there exist $h \in P - \{\theta\}$ and $t_0 \in (a, b)$ such that

$$\tau(t_0)h \leq (I - B)^{-1}A(h, h) \leq \frac{\varphi(t_0, \frac{h}{\tau(t_0)}, \tau(t_0)h)}{\tau(t_0)}h.$$

Then
(i) there are $u_0, v_0 \in P_h$ and $r \in (0, 1)$ such that $rv_0 \leq u_0 \leq v_0$, $u_0 \leq (I - B)^{-1}A(u_0, v_0) \leq (I - B)^{-1}A(v_0, u_0) \leq v_0$;
(ii) equation (4) has a unique solution x^* in $[u_0, v_0]$;
(iii) for any initial $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = (I - B)^{-1}A(x_{n-1}, y_{n-1}),\ y_n = (I - B)^{-1}A(y_{n-1}, x_{n-1}),\ n = 1, 2, \ldots,$$

we have $\| x_n - x^* \| \to 0$ and $\| y_n - x^* \| \to 0$ as $n \to \infty$.

2. Main results

Theorem 2.1. *Let P be a normal cone in E, and $A : P \times P \times P \to P$ a mixed monotone operator. Let $B : E \to E$ be sublinear. Assume that for all $a < t < b$, there exist two positive-valued functions $\tau(t), \varphi(t, x, y, z)$ on an interval (a, b) such that*
(H1) *$\tau : (a, b) \longrightarrow (0, 1)$ is a surjective;*
(H2) *$\varphi(t, x, y, z) > \tau(t)$ for all $t \in (a, b)$, $x, y, z \in P$;*
(H3) *$A(\tau(t)x, \frac{1}{\tau(t)}y, \tau(t)z) \geq \varphi(t, x, y, z)A(x, y, z)$ for all $t \in (a, b)$, $x, y, z \in P$;*
(H4) *$(I - B)^{-1} : E \to E$ exists and is an increasing operator.*
For any $t \in (a, b)$, $\varphi(t, x, y, z)$ is nondecreasing in x, z for fixed y and non-increasing in y

for fixed x, z. In addition, suppose that there exist $h \in P\backslash\{\theta\}$ and $t_0 \in (a, b)$ such that

$$\frac{\tau(t_0)}{\varphi(t_0, h, h, h)} h \leq (I - B)^{-1} A(h, h, h) \leq \frac{1}{\tau(t_0)} h. \tag{4}$$

Then

(i) *there are $u_0, v_0, \xi_0 \in P_h$ and $r \in (0, 1)$ such that $rv_0 \leq u_0 \leq v_0$, $rv_0 \leq \xi_0 \leq v_0$, and*

$$u_0 \leq (I - B)^{-1} A(u_0, v_0, u_0) \leq (I - B)^{-1} A(v_0, u_0, v_0) \leq v_0,$$
$$\xi_0 \leq (I - B)^{-1} A(\xi_0, v_0, \xi_0) \leq (I - B)^{-1} A(v_0, \xi_0, v_0) \leq v_0;$$

(ii) *equation $A(x, x, x) + B(x) = x$, $x \in E$ has a unique solution x^* in $[\bar{u_0}, v_0]$ that $\bar{u_0} = \max\{u_0, \xi_0\}$;*

(iii) *for any initial $x_0, y_0, z_0 \in P_h$, constructing successively the sequences*

$$x_n = (I - B)^{-1} A(x_{n-1}, y_{n-1}, z_{n-1}), \quad y_n = (I - B)^{-1} A(y_{n-1}, x_{n-1}, y_{n-1}),$$

$$z_n = (I - B)^{-1} A(z_{n-1}, y_{n-1}, x_{n-1}), \quad n = 1, 2, ...,$$

we have $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0$ and $\|z_n - x^*\| \to 0$ as $n \to \infty$.

Proof. For convenience, we denote $C = (I - B)^{-1} A$. By the fact that operator B is sub-linear, we have $B\theta \geq \theta$, which together with $(H4)$ imply, $\theta \leq (I - B)^{-1}\theta \leq (I - B)^{-1} x$, $x \in P$. Consequently, $(I - B)^{-1}$ is a positive operator. Hence we have that $C : P \times P \times P \longrightarrow P$. According to $(H4)$, we have that C is mixed monotone.
Since B is sub-linear, we know that for any $x \in P$ and $\beta \in (0, 1)$, we obtain $(I - B)(\beta x) \leq \beta(I - B)x$. Thus $(I - B)(\beta(I - B)^{-1} x) \leq \beta(I - B)(I - B)^{-1} x = \beta x; i.e., (I - B)(\beta(I - B)^{-1} x) \leq \beta x$. Therefore, we have

$$\beta(I - B)^{-1} x \leq (I - B)^{-1}(\beta x). \tag{5}$$

For any $t \in (a, b)$, it follows from $(H1) - (H4)$ and (5) that

$$\begin{aligned} C(\tau(t)x, \tfrac{1}{\tau(t)} y, \tau(t)z) &= (I - B)^{-1} A(\tau(t)x, \tfrac{1}{\tau(t)} y, \tau(t)z) \\ &\geq (I - B)^{-1} \varphi(t, x, y, z) A(x, y, z) \\ &\geq \varphi(t, x, y, z)(I - B)^{-1} A(x, y, z) = \varphi(t, x, y, z) C(x, y, z). \end{aligned} \tag{6}$$

Since $\tau(t_0) < \varphi(t_0, h, h, h)$, we can take a positive integer k such that

$$\left(\frac{\varphi(t_0, h, h, h)}{\tau(t_0)}\right)^k \geq \frac{1}{\tau(t_0)}. \tag{7}$$

Let $\xi_0 = u_0 = [\tau(t_0)]^k h$, $v_0 = \frac{1}{[\tau(t_0)]^k} h$, and construct successively the sequences $u_n = C(u_{n-1}, v_{n-1}, \xi_{n-1}), v_n = C(v_{n-1}, u_{n-1}, v_{n-1}), \xi_n = C(\xi_{n-1}, v_{n-1}, u_{n-1}), n = 1, 2, ...$ It is clear that $u_0, v_0, \xi_0 \in P$ and $u_0 < v_0$, $\xi_0 < v_0$, $u_1 = C(u_0, v_0, \xi_0) \leq C(v_0, u_0, v_0) = v_1$ and $\xi_1 = C(\xi_0, v_0, u_0) \leq C(v_0, u_0, v_0) = v_1$. In general, we obtain $u_n \leq v_n$, $\xi_n \leq v_n$, $n = 1, 2, ...$. Note that $\varphi(t, x, y, z) > \tau(t)$ for all $t \in (a, b)$, $x, y, z \in P$. Combining (4) with (11), we have

$$\begin{aligned}
u_1 &= C(u_0, v_0, \xi_0) = C([\tau(t_0)]^k h, \frac{1}{[\tau(t_0)]^k} h, [\tau(t_0)]^k h) \\
&= C(\tau(t_0)[\tau(t_0)]^{k-1} h, \frac{1}{\tau(t_0)} \frac{1}{[\tau(t_0)]^{k-1}} h, \tau(t_0)[\tau(t_0)]^{k-1} h) \\
&\geq \varphi(t_0, [\tau(t_0)]^{k-1} h, \frac{h}{[\tau(t_0)]^{k-1}}, [\tau(t_0)]^{k-1} h) C([\tau(t_0)]^{k-1} h, \frac{h}{[\tau(t_0)]^{k-1}}, [\tau(t_0)]^{k-1} h) \\
&= \varphi(t_0, [\tau(t_0)]^{k-1} h, \frac{h}{[\tau(t_0)]^{k-1}}, [\tau(t_0)]^{k-1} h) C(\tau(t_0)[\tau(t_0)]^{k-2} h, \frac{1}{\tau(t_0)} \frac{1}{[\tau(t_0)]^{k-2}} h, \\
&\tau(t_0)[\tau(t_0)]^{k-2} h) \\
&\geq \varphi(t_0, [\tau(t_0)]^{k-1} h, \frac{h}{[\tau(t_0)]^{k-1}}, [\tau(t_0)]^{k-1} h) \varphi(t_0, [\tau(t_0)]^{k-2} h, \frac{h}{[\tau(t_0)]^{k-2}}, [\tau(t_0)]^{k-2} h) \\
&\quad \times C(\tau(t_0)[\tau(t_0)]^{k-2} h, \frac{1}{\tau(t_0)} \frac{1}{[\tau(t_0)]^{k-2}} h, \tau(t_0)[\tau(t_0)]^{k-2} h) \\
&\geq \ldots \geq \varphi(t_0, [\tau(t_0)]^{k-1} h, \frac{h}{[\tau(t_0)]^{k-1}}, [\tau(t_0)]^{k-1} h) \varphi(t_0, [\tau(t_0)]^{k-2} h, \frac{h}{[\tau(t_0)]^{k-2}}, [\tau(t_0)]^{k-2} h) \\
&\quad \times \ldots \varphi(t_0, h, h, h) C(h, h, h) \\
&\geq [\tau(t_0)]^{k-1} \varphi(t_0, h, h, h) C(h, h, h) \geq [\tau(t_0)]^k h = u_0
\end{aligned}$$

and, similarly is proved that $\xi_1 \geq \xi_0$.
From (11), we have

$$C(\frac{x}{\tau(t)}, \tau(t) y, \frac{z}{\tau(t))}) \leq \frac{1}{\varphi(t, \frac{x}{\tau(t)}, \tau(t) y, \frac{z}{\tau(t))})} C(x, y, z), \forall t \in (a, b), x, y, z \in P. \qquad (8)$$

Note that $\varphi(t, x, y, z)$ is non-decreasing in x, z and non-increasing in y, it follows from (4), (12) and (2.1) that

$$\begin{aligned}
v_1 &= C(v_0, u_0, v_0) = C(\frac{h}{[\tau(t_0)]^k}, [\tau(t_0)]^k h, \frac{h}{[\tau(t_0)]^k}) \\
&= C(\frac{1}{\tau(t)_0} \frac{h}{[\tau(t_0)]^{k-1}}, \tau(t_0)[\tau(t_0)]^{k-1} h, \frac{1}{\tau(t)_0} \frac{h}{[\tau(t_0)]^{k-1}}) \\
&\leq \frac{1}{\varphi(t_0, \frac{h}{[\tau(t_0)]^k}, [\tau(t_0)]^k h, \frac{h}{[\tau(t_0)]^k})} C(\frac{h}{[\tau(t_0)]^{k-1}}, [\tau(t_0)]^{k-1} h, \frac{h}{[\tau(t_0)]^{k-1}}) \\
&= \frac{1}{\varphi(t_0, \frac{h}{[\tau(t_0)]^k}, [\tau(t_0)]^k h, \frac{h}{[\tau(t_0)]^k})} C(\frac{1}{\tau(t_0)} \frac{h}{[\tau(t_0)]^{k-2}}, \tau(t_0)[\tau(t_0)]^{k-2} h, \frac{1}{\tau(t_0)} \frac{h}{[\tau(t_0)]^{k-2}}) \\
&\leq \frac{1}{\varphi(t_0, \frac{h}{[\tau(t_0)]^k}, [\tau(t_0)]^k h, \frac{h}{[\tau(t_0)]^k})} \frac{1}{\varphi(t_0, \frac{h}{[\tau(t_0)]^{k-1}}, [\tau(t_0)]^{k-1} h, \frac{h}{[\tau(t_0)]^{k-1}})} \\
&\quad \times C(\frac{h}{[\tau(t_0)]^{k-2}}, [\tau(t_0)]^{k-2} h, \frac{h}{[\tau(t_0)]^{k-2}}) \\
&\leq \ldots \leq \frac{1}{\varphi(t_0, \frac{h}{[\tau(t_0)]^k}, [\tau(t_0)]^k h, \frac{h}{[\tau(t_0)]^k})} \frac{1}{\varphi(t_0, \frac{h}{[\tau(t_0)]^{k-1}}, [\tau(t_0)]^{k-1} h, \frac{h}{[\tau(t_0)]^{k-1}})} \times \\
&\ldots \times \frac{1}{\varphi(t_0, \frac{h}{\tau(t_0)}, \tau(t_0) h, \frac{h}{\tau(t_0)})} C(h, h, h) < \frac{1}{[\varphi(t_0, h, h, h)]^k} \frac{h}{\tau(t_0)} \leq \frac{1}{[\tau(t_0)]^k} h = v_0.
\end{aligned}$$

Thus, we obtain

$$u_0 \leq u_1 \leq v_1 \leq v_0, \quad \xi_0 \leq \xi_1 \leq v_1 \leq v_0. \tag{9}$$

By induction, it is easy to obtain that

$$u_0 \leq u_1 \leq ... \leq u_n \leq ... \leq v_n \leq ... \leq v_1 \leq v_0,$$

$$\xi_0 \leq \xi_1 \leq ... \leq \xi_n \leq ... \leq v_n \leq ... \leq v_1 \leq v_0.$$

Take any $r \in (0, [\tau(t_0)]^{2k})$, then $r \in (0,1)$ and $u_0 \geq rv_0$, $\xi_0 \geq rv_0$. So we can know that $u_n \geq u_0 \geq rv_0 \geq rv_n, \xi_n \geq \xi_0 \geq rv_0 \geq rv_n, \quad n = 1, 2, ...$ Let $r_n = \sup\{r > 0 \mid u_n \geq rv_n, \quad \xi_n \geq rv_n\}$.
Thus, we have $u_n \geq r_n v_n$, $\xi_n \geq r_n v_n$, $n = 1, 2, ...,$ and then
$u_{n+1} \geq u_n \geq r_n v_n \geq r_n v_{n+1}, n = 1, 2, ..., \xi_{n+1} \geq \xi_n \geq r_n v_n \geq r_n v_{n+1}, n = 1, 2, ...$
Therefore, $r_{n+1} \geq r_n$; i.e., $0 < r_0 \leq r_1 \leq ... \leq r_n \leq ... \leq 1$.
Set $r^* = \lim_{n\to\infty} r_n$, we will show that $r^* = 1$. In fact, if $0 < r^* < 1$, by $(H1)$, there exists $t_1 \in (a, b)$ such that $\tau(t_1) = r^*$. Consider the following two cases:
Case i: There exists an integer N such that $r_N = r^*$. In this case, we have $r_n = r^*$ and $u_n \geq r^* v_n$ for all $n \geq N$ hold. Hence

$$\begin{aligned} u_{n+1} &= C(u_n, v_n, \xi_n) \geq C(r^* v_n, \frac{1}{r^*} u_n, r^* v_n) \\ &= C(\tau(t_1) v_n, \frac{1}{\tau(t_1)} u_n, \tau(t_1) v_n) \\ &\geq \varphi(t_1, v_n, u_n, v_n) C(v_n, u_n, v_n) \\ &\geq \varphi(t_1, v_0, u_0, v_0) C(v_0, u_0, v_0) = \varphi(t_1, v_0, u_0, v_0) v_{n+1}, \quad n \geq N. \end{aligned}$$

By the definition of r_n, we have, $r_{n+1} = r^* \geq \varphi(t_1, v_0, u_0, v_0) > \tau(t_1) = r^*, n \geq N$, which is a contradiction.
Case ii: For all integers n, $r_n < r^*$. Then we obtain $0 < \frac{r_n}{r^*} < 1$. By $(H1)$, there exists $z_n \in (a, b)$ such that $\tau(z_n) = \frac{r_n}{r^*}$. Hence

$$\begin{aligned} u_{n+1} &= C(u_n, v_n, \xi_n) \geq C(r_n v_n, \frac{1}{r_n} u_n, r_n v_n) \\ &= C(\frac{r_n}{r^*} r^* v_n, \frac{1}{\frac{r_n}{r^*} r^*} u_n, \frac{r_n}{r^*} r^* v_n) = C(\tau(z_n) r^* v_n, \frac{1}{\tau(z_n) r^*} u_n, \tau(z_n) r^* v_n) \\ &\geq \varphi(z_n, r^* v_n, \frac{1}{r^*} u_n, r^* v_n) C(r^* v_n, \frac{1}{r^*} u_n, r^* v_n) \\ &\geq \varphi(z_n, r^* v_0, \frac{1}{r^*} u_0, r^* v_0) C(\tau(t_1) v_n, \frac{1}{\tau(t_1)} u_n, \tau(t_1) v_n) \\ &\geq \varphi(z_n, r^* v_0, \frac{1}{r^*} u_0, r^* v_0) \varphi(t_1, v_n, u_n, v_n) C(v_n, u_n, v_n) \\ &\geq \varphi(z_n, r^* v_0, \frac{1}{r^*} u_0, r^* v_0) \varphi(t_1, u_0, v_0, u_0) v_{n+1}. \end{aligned}$$

By the definition of r_n, we have,

$$r_{n+1} \geq \varphi(z_n, r^* u_0, \tfrac{1}{r^*} v_0, r^* u_0) \varphi(t_1, u_0, v_0, u_0) > \tau(z_n) \varphi(t_1, u_0, v_0, u_0) = \tfrac{r_n}{r^*} \varphi(t_1, u_0, v_0, u_0).$$

Let $n \to \infty$, we have, $r^* \geq \varphi(t_1, u_0, v_0, u_0) > \tau(t_1) = r^*$, which is also a contradiction. Thus, $\lim_{n\to\infty} r_n = 1$.
Furthermore, as in the proof of [[2], Theorem 2.1], there exists $x^* \in [\bar{u}_0, v_0]$ such that $\bar{u}_0 = \max\{u_0, \xi_0\}$ and $\lim_{n\to\infty} \bar{u}_n = \lim_{n\to\infty} v_n = x^*$, and x^* is the fixed point of operator C. Let

$$c_1 = \sup\{0 < c \leq 1 \mid cx_* \leq x^* \leq \frac{1}{c} x_*\}.$$

Clearly, $0 < c_1 \leq 1$ and $c_1 x_* \leq x^* \leq \frac{1}{c_1} x_*$. If $0 < c_1 < 1$, according to $(H1)$, there exists $t_2 \in (a, b)$ such that $\tau(t_2) = c_1$. Then

$$\begin{aligned} x^* = C(x^*, x^*, x^*) &\geq C(c_1 x_*, \frac{1}{c_1} x_*, c_1 x_*) \\ &= C(\tau(t_2) x_*, \frac{1}{\tau(t_2)} x_*, \tau(t_2) x_*) \\ &\geq \varphi(t_2, x_*, x_*, x_*) C(x_*, x_*, x_*) = \varphi(t_2, x_*, x_*, x_*) x_*, \end{aligned}$$

and

$$\begin{aligned} x^* &= C(x^*, x^*, x^*) \leq C(\frac{1}{c_1} x_*, c_1 x_*, \frac{1}{c_1} x_*) \\ &= C(\frac{1}{\tau(t_2)} x_*, \tau(t_2) x_*, \frac{1}{\tau(t_2)} x_*) \\ &\leq \frac{1}{\varphi(t_2, \frac{x_*}{\tau(t_2)}, \tau(t_2) x_*, \frac{x_*}{\tau(t_2)})} C(x_*, x_*, x_*) = \frac{1}{\varphi(t_2, \frac{x_*}{\tau(t_2)}, \tau(t_2) x_*, \frac{x_*}{\tau(t_2)})} x_*. \end{aligned}$$

Since

$$\varphi(t_2, x_*, x_*, x_*) \leq \varphi(t_2, \frac{x_*}{\tau(t_2)}, \tau(t_2) x_*, \frac{x_*}{\tau(t_2)}),$$

we have

$$\varphi(t_2, x_*, x_*, x_*) x_* \leq x^* \leq \frac{1}{\varphi(t_2, x_*, x_*, x_*)} x_*.$$

Hence, $c_1 \geq \varphi(t_2, x_*, x_*, x_*) > \tau(t_2) = c_1$, which is a contradiction. Thus we have $c_1 = 1$; i.e., $x^* = x_*$. Therefore, C has a unique fixed point x^* in P_h. Note that $[\bar{u}_0, v_0] \subset P_h$, so we know that x^* is the unique fixed point of C in $[\bar{u}_0, v_0]$. For any initial $x_0, y_0, z_0 \in P_h$, we can choose a small number $\bar{e} \in (0, 1)$ such that

$$\bar{e} h \leq x_0 \leq \frac{1}{\bar{e}} h, \quad \bar{e} h \leq y_0 \leq \frac{1}{\bar{e}} h, \quad \bar{e} h \leq z_0 \leq \frac{1}{\bar{e}} h.$$

From $(H1)$, there is $t_3 \in (a, b)$ such that $\tau(t_3) = \bar{e}$, thus

$$\tau(t_3) h \leq x_0 \leq \frac{1}{\tau(t_3)} h, \quad \tau(t_3) h \leq y_0 \leq \frac{1}{\tau(t_3)} h, \quad \tau(t_3) h \leq z_0 \leq \frac{1}{\tau(t_3)} h.$$

We can choose a sufficiently large positive integer q such that, $\left(\frac{\varphi(t_3, h, h, h)}{\tau(t_3)}\right)^q \geq \frac{1}{\tau(t_3)}$. Take $\hat{\xi}_0 = \hat{u}_0 = [\tau(t_3)]^q h$, $\hat{v}_0 = \frac{1}{[\tau(t_3)]^q} h$. We can find that

$$\hat{u}_0 \leq x_0 \leq \hat{v}_0, \quad \hat{u}_0 \leq y_0 \leq \hat{v}_0, \quad \hat{u}_0 \leq z_0 \leq \hat{v}_0.$$

Constructing successively the sequences

$$x_n = C(x_{n-1}, y_{n-1}, z_{n-1}), \quad y_n = C(y_{n-1}, x_{n-1}, y_{n-1}), \quad z_n = C(z_{n-1}, y_{n-1}, x_{n-1}),$$
$$n = 1, 2, ...$$
$$\hat{u}_n = C(\hat{u}_{n-1}, \hat{v}_{n-1}, \hat{\xi}_{n-1}), \quad \hat{v}_n = C(\hat{v}_{n-1}, \hat{u}_{n-1}, \hat{v}_{n-1}), \quad \hat{\xi}_n = C(\hat{\xi}_{n-1}, \hat{v}_{n-1}, \hat{u}_{n-1}),$$
$$n = 1, 2, ...$$

By using the mixed monotone properties of operator C, $\hat{\eta}_n \leq x_n \leq \hat{v}_n, \hat{\eta}_n \leq y_n \leq \hat{v}_n, \hat{\eta}_n \leq z_n \leq \hat{v}_n, \quad n = 1, 2, ...$ which $\hat{\eta}_n = \max\{\hat{u}_n, \hat{\xi}_n\}$.
Similarly to the above proof, we can know that there exists $y^* \in P_h$ such that, $C(y^*, y^*, y^*) = y^*, \lim_{n\to\infty} \hat{\eta}_n = \lim_{n\to\infty} \hat{v}_n = y^*$. By the uniqueness of fixed points of operator C in P_h, we have $y^* = x^*$. Taking into account that P is normal, we deduce that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = x^*$. This completes the proof. □

Theorem 2.2. *Let P be a normal cone in E, and $A : P \times P \times P \to P$ a mixed monotone operator. Let $B : E \to E$ be sublinear. Assume that for all $a < t < b$, there exist two positive-valued functions $\tau(t)$, $\varphi(t,x,y,z)$ on interval (a,b) such that the properties $(H1)-(H4)$ in Theorem (2.1) are satisfied. Furthermore, for any $t \in (a,b)$, $\varphi(t,x,y,z)$ is non-increasing in x and z for fixed y, and non-decreasing in y for fixed x or z. In addition, suppose that there exist $h \in P \setminus \{\theta\}$ and $t_0 \in (a,b)$ such that*

$$\tau(t_0)h \leq (I-B)^{-1}A(h,h,h) \leq \frac{\varphi(t_0, \frac{h}{\tau(t_0)}, \tau(t_0)h, \frac{h}{\tau(t_0)})}{\tau(t_0)}h. \tag{10}$$

Then the conclusions (i), (ii), (iii) in Theorem (2.1) hold.

Proof. As in the proof of Theorem (2.1), it suffices to verify that (2.3) holds. For any $t \in (a,b)$, note that $\varphi(t,x,y,z)$ is non-increasing in x, z and non-decreasing in y, it follows from (14), (11) and (12) that

$$\begin{aligned}
u_1 &= C(u_0, v_0, \xi_0) = C([\tau(t_0)]^k h, \frac{h}{[\tau(t_0)]^k}, [\tau(t_0)]^k h) \\
&\geq \varphi(t_0, [\tau(t_0)]^{k-1}h, \frac{h}{[\tau(t_0)]^{k-1}}, [\tau(t_0)]^{k-1}h)\varphi(t_0, [\tau(t_0)]^{k-2}h, \frac{h}{[\tau(t_0)]^{k-2}}, [\tau(t_0)]^{k-2}h)... \\
&\varphi(t_0,h,h,h)C(h,h,h) \geq [\varphi(t_0,h,h,h)]^k \tau(t_0)h \geq [\tau(t_0)]^k h = u_0.
\end{aligned}$$

Similarly, is proved that $\xi_1 \geq \xi_0$.
Note that $\varphi(t,x,y,z) > \tau(t)$ for all $t \in (a,b)$, $x,y,z \in P$. Combining (14) with (2.1), we obtain

$$\begin{aligned}
v_1 &= C(v_0, u_0, v_0) = C(\frac{h}{[\tau(t_0)]^k}, [\tau(t_0)]^k, \frac{h}{[\tau(t_0)]^k}) \\
&\leq \frac{1}{\varphi(t_0, \frac{h}{[\tau(t_0)]^k}, [\tau(t_0)]^k, \frac{h}{[\tau(t_0)]^k})} \frac{1}{\varphi(t_0, \frac{h}{[\tau(t_0)]^{k-1}}, [\tau(t_0)]^{k-1}, \frac{h}{[\tau(t_0)]^{k-1}})} \cdots \\
&\times \frac{1}{\varphi(t_0, \frac{h}{\tau(t_0)}, \tau(t_0), \frac{h}{\tau(t_0)})} C(h,h,h) < \frac{1}{[\tau(t_0)]^{k-1}} \frac{1}{\varphi(t_0, \frac{h}{\tau(t_0)}, \tau(t_0), \frac{h}{\tau(t_0)})} C(h,h,h) \\
&\leq \frac{1}{[\tau(t_0)]^k} h = v_0.
\end{aligned}$$

Thus, we know that (2.3) holds. The rest proof is similar to that of Theorem (2.1). □

2.1. **Application.** We study the existence and uniqueness of a positive solution for the fractional differential equation

$$\frac{D^\alpha}{Dt} u(t) = f(t, u(t)) \; t \in [0,1], \; 3 < \alpha \leq 4. \tag{11}$$

Subject to conditions;

$$u(0) = u'(0) = u(1) = u'(1) = 0, \tag{12}$$

where D^α is the Riemann-Liouville fractional derivative of order α., consider the Banach space of continuous functions on $[0,1]$ with sup norm and set $P = \{y \in C[0,1] : \min_{t \in [0,1]} y(t) \geq 0\}$. Then P is normal cone. From [5], we have the following lemma.

Lemma 2.1. *Given $y \in C[0,1]$ and $3 < \alpha \leq 4$, the unique solution of the fractional differential equation*

$$\frac{D^\alpha}{Dt}u(t) = f(t,u(t)) \ t \in [0,1], \ 3 < \alpha \leq 4 \qquad (13)$$
$$u(0) = u'(0) = u(1) = u'(1) = 0,$$

is given by
$u(t) = \int_0^1 G(t,s)f(s,y(s))ds$ *where*

$$G(t,s) = \begin{cases} \frac{(t-1)^{\alpha-1}+(1-s)^{\alpha-2}t^{\alpha-2}[(s-t)+(\alpha-2)(1-t)s]}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1 \\ \frac{(1-s)^{\alpha-2}t^{\alpha-2}[(s-t)+(\alpha-2)(1-t)s]}{\Gamma(\alpha)} & 0 \leq t \leq s \leq 1 \end{cases}$$

If $f(t,u(t)) = 1$ the unique solution of (2.1)is given by

$$u_0(t) = \int_0^1 G(t,s)ds = \frac{1}{\Gamma(\alpha+1)}t^{\alpha-2}(1-t)^2.$$

Lemma 2.2. [5] *The Green's function $G(t,s)$ has the following properties:*
(1) $G(t,s) > 0$ and $G(t,s)$ is continuous for $t,s \in [0,1]$;
(2) $\frac{(\alpha-2)h(t)k(s)}{\Gamma(\alpha)} \leq G(t,s) \leq \frac{M_0 k(s)}{\Gamma(\alpha)}$, where
$M_0 = \max\{\alpha-1,(\alpha-2)^2\}, h(t) = t^{\alpha-2}(1-t)^2, k(s) = s^2(1-s)^{\alpha-2}$.

Theorem 2.3. *Let $f(t,u(t),v(t),w(t)) \in C([0,1]\times[0,\infty)\times[0,\infty))$ is increasing in u,w and decreasing in v. Assume that for all $a < t < b$, there exist two positive-valued functions $\tau(t), \varphi(t,u,v,w)$ on an interval (a,b) such that*
(H_1) $\tau : (a,b) \to (0,1)$ is surjection;
(H_2) $\varphi(t,u,v,w) > \tau(t)$ for all $t \in (a,b)$, $u,v,w \in P$;
(H_3)
$\int_0^1 G(t,s)f(s,\tau(t)u(s),\frac{1}{\tau(t)}v(s),\tau(t)w(s)) \geq \varphi(t,u,v,w)\int_0^1 G(t,s)f(s,u(s),v(s),w(s))ds$
Furthermore, for any $t \in (a,b)$, $\varphi(t,u,v,w)$ is non-increasing in u for fixed v,w, nondecreasing in v for fixed u,w and non-increasing in w for fixed u,v. In addition, suppose that there exist $h \in P\setminus\{\theta\}$ and $t_0 \in (a,b)$ such that

$$\tau(t_0)h \leq \int_0^1 G(t,s)f(s,h(s),h(s),h(s))ds \leq \frac{\varphi(t_0,\frac{h}{\tau(t_0)},\tau(t_0)h,\frac{h}{\tau(t_0)})}{\tau(t_0)}h. \qquad (14)$$

Then
(i) there are $u_0, v_0, \xi_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leq u_0 \leq v_0, rv_0 \leq \xi_0 \leq v_0$,

$$\begin{aligned} u_0 &\leq \int_0^1 G(t,s)f(s,u_0(s),v_0(s),u_0(s))ds \\ &\leq \int_0^1 G(t,s)f(s,v_0(s),u_0(s),v_0(s))ds \leq v_0, \\ \xi_0 &\leq \int_0^1 G(t,s)f(s,\xi_0(s),v_0(s),\xi_0(s))ds \\ &\leq \int_0^1 G(t,s)f(s,v_0(s),\xi_0(s),v_0(s))ds \leq v_0; \end{aligned}$$

(ii) equation (4) has a unique solution x^* *in* $[\bar{u_0}, v_0]$ *that* $\bar{u_0} = \max\{u_0, \xi_0\}$*;*
(iii) for any initial $u_0, v_0, \xi_0 \in P_h$*, constructing successively the sequences*

$$\begin{aligned} u_n &= \int_0^1 G(t,s) f(s, u_{n-1}(s), v_{n-1}(s), \xi_{n-1}(s)) ds, \\ v_n &= \int_0^1 G(t,s) f(s, v_{n-1}(s), u_{n-1}(s), v_{n-1}(s)) ds, \\ \xi_n &= \int_0^1 G(t,s) f(s, \xi_{n-1}(s), v_{n-1}(s), u_{n-1}(s)) ds, \quad n = 1, 2, \ldots, \end{aligned}$$

we have $\| u_n - u^* \| \to 0$, $\| v_n - v^* \| \to 0$ *and* $\| \xi_n - \xi^* \| \to 0 as\ n \to \infty$.

Proof. By using Lemma (12), the problem is equivalent to the integral equation $u(t) = \int_0^1 G(t,s) f(s, y(s)) ds$ where

$$G(t,s) = \begin{cases} \frac{(t-1)^{\alpha-1} + (1-s)^{\alpha-2} t^{\alpha-2} [(s-t) + (\alpha-2)(1-t)s]}{\Gamma(\alpha)}, & 0 \le s \le t \le 1 \\ \frac{(1-s)^{\alpha-2} t^{\alpha-2} [(s-t) + (\alpha-2)(1-t)s]}{\Gamma(\alpha)} & 0 \le t \le s \le 1 \end{cases}$$

Define the operator $A : P \times P \times P \to E$ by the following,

$$A(u(t), v(t), \xi(t)) = \int_0^1 G(t,s) f(s, u(s), v(s), \xi(s)) ds.$$

Then u is solution for the problem if and only if $u = A(u, u, u)$. It is easy to see to check that the operator A is increasing in u, ξ and decreasing in v on P. By assumptions of theorem we have;

$$\begin{aligned} & A(\tau(t)u, \frac{1}{\tau(t)} v, \tau(t)\xi) \ge \varphi(t, u, v, \xi) A(u, v, \xi) \ \forall t \in (a, b), \ u, v, \xi \in P, \\ & \tau(t_0) h \le A(h, h, h) \le \frac{\varphi(t_0, \frac{h}{\tau(t_0)}, \tau(t_0) h, \frac{h}{\tau(t_0)})}{\tau(t_0)} h. \end{aligned}$$

Therefore A satisfies conditions of Theorem (5), now, by using Theorem (5), the operator A has a unique positive solution (u^*, u^*, u^*) such that $A(u^*, u^*, u^*) = u^*$, $u^* \in [\bar{u_0}, v_0]$ that $\bar{u_0} = \max\{u_0, \xi_0\}$. This completes the proof. □

Example 2.1. *Consider the periodic boundary value problem*

$$\begin{aligned} & D^{\frac{7}{2}} u(t) = f(t, u(t), v(t), w(t)) = \frac{1}{4}(1 + u(t) + \frac{1}{\sqrt{u(t)}} + \sqrt{u(t)}), \quad t \in [0, 1] \\ & u(0) = u'(0) = u(1) = u'(1) = 0, \end{aligned}$$

where, $u(t) = \frac{1}{4}(1 + u(t))$ *and* $v(t) = \frac{1}{4\sqrt{u(t)}}$, $w(t) = \frac{1}{4}\sqrt{u(t)}$.
For every $\lambda \in (0,1)$, $u, v, w \in P$, *we have*

$$\frac{1}{4}\int_0^1 G(t,s)[1 + \lambda u(s) + \frac{1}{\sqrt{\frac{1}{\lambda}u(s)}} + \sqrt{\lambda u(s)}]ds$$

$$= \frac{\lambda}{4}\int_0^1 G(t,s)[\frac{1}{\lambda} + u(s) + \frac{1}{\sqrt{\lambda u(s)}} + \sqrt{\frac{u(s)}{\lambda}}]ds$$

$$\geq \lambda \frac{\frac{1}{\lambda} + u(s) + \frac{1}{\sqrt{\lambda u(s)}} + \sqrt{\frac{u(s)}{\lambda}}}{1 + u(s) + \frac{1}{\sqrt{u(s)}} + \sqrt{u(s)}} \int_0^1 G(t,s)[\frac{1}{4} + \frac{u(s)}{4} + \frac{1}{4\sqrt{u(s)}} + \frac{1}{4}\sqrt{u(s)}]ds$$

we note that, $\lambda < \varphi(\lambda, u, v, w) = \lambda \frac{\frac{1}{\lambda}+u(s)+\frac{1}{\sqrt{\lambda u(s)}}+\sqrt{\frac{u(s)}{\lambda}}}{1+u(s)+\frac{1}{\sqrt{u(s)}}+\sqrt{u(s)}} < 1$. *For any* $\lambda \in (0,1)$, *by means of some calculations, we can obtain that* φ *is nonincreasing in* u *for fixed* v, w *, nonincreasing in* w *for fixed* u, v *and nondecreasing in* v *for fixed* u, w.
In the following, it suffices to verify that the condition (14) of Theorem (2.3) is satisfied. Put $u = v = w = h = 1$, $\tau(t_0) = t_0 = \lambda = 10^{-4}$. *From* $M_1 = \min_{t\in[0,1]} \int_0^1 G(t,s)ds = 0.00001$ *and* $M_2 = \max_{t\in[0,1]} \int_0^1 G(t,s)ds = 0.004$ *we can easily get*
$10^{-4} \leq 0.00001 \leq \int_0^1 G(t,s)[\frac{1}{4} + \frac{u(s)}{4} + \frac{1}{4\sqrt{u(s)}} + \frac{1}{4}\sqrt{u(s)}]ds$
$\leq 0.004 \leq \frac{10^4+10^4+10^4+10^4}{1+10^4+10^2+10^2} = \varphi(10^{-4}, 10^4, 10^{-4}, 10^4)$, $s \in [0,1]$.
This implies that (14) of Theorem (2.3) holds. The proof is complete.

References

[1] Berinde,V. and Borcut,M., (2012), Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces, Appl. Math. Comput., 218(10), pp. 5929-5936.

[2] Liang,K., Li,J. and Xiao,T.J., (2005), New existence and uniqueness theorems of positive fixed points for mixed monotone operators with perturbation, Comput. Math. Appl. 50, pp. 1569-1578. pp. 215-224.

[3] Sabatier,J., Agrawal,O.P. and Machado,J.A.T., (2007), Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering. Springer, Dordrecht.

[4] Samko,S.G., Kilbas,A.A. and Marichev,O.I., (1993), Fractional Integral and Derivative:Theory and Applications, Gordan & Breach, Switzerland.

[5] Xu,X.J, Jiang,D.Q. and Yuan,C.J., (2009), Multiple positive solutions for the boundary value problems of a nonlinear fractional differential equation, Nonlinear Anal. TMA. 71, pp. 4676-4688.

[6] Sang,Y., (2013), Existence and uniqueness of fixed points for mixed monotone operators with perturbations, Electronic Journal of Differential Equations, 233, pp. 1-16.

[7] Sang,Y., (2013), A class of φ-concave operators and applications, Fixed Point Theory and Applications, vol. 2013.

Hojjat Afshari received his Ph.D. degree in mathematics from Shahid Madani University of Azabaidjan in 2012. He has been an assistant professor in the Department of Mathematics at Bonab University in Iran since 2013. He is the author or co-author of more than 25 papers in the fields of functional analysis, fractional differential equations, absolute retractivity, and fixed point theory.

Alireza Kheiryan received his MSc. degree in mathematics from Shahid Madani University of Azabaidjan in 2013. He has been a mathematics teacher in Tabriz, Iran since 2011.