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HUB-INTEGRITY OF SPLITTING GRAPH AND DUPLICATION OF GRAPH ELEMENTS

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ABSTRACT. The hub-integrity of a graph $G = (V(G), E(G))$ is denoted as $HI(G)$ and defined by $HI(G) = \min\{|S| + m(G - S), S \textit{ is a hub set of } G\}$, where $m(G - S)$ is the order of a maximum component of $G - S$. In this paper, we discuss hub-integrity of splitting graph and duplication of an edge by vertex and duplication of vertex by an edge of some graphs.

Keywords: integrity, hub-integrity, splitting graph, duplication graph.

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1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph without loops or multiple edges. For graph theoretic terminology, we refer to Harary [4]. In the remaining portion of this section we will give brief summary of definitions and information related to the present work.

To model various systems like chemical, social systems, neural networks or the World Wide Web (www) and Internet; networks and complex systems are commonly used. Without any doubt, a very vital part of networks is the network topology which is the center of attention of mathematics and biological, computer, and physical sciences. While dealing with networks, the issue of an interconnection network is of great importance. It may have different architectural structures which need to be analyzed. For this purpose mathematics usually use graph theory as the most powerful tool. It is known that the underlying topology of an interconnection network is modeled by a graph $G = (V, E)$, where V and E stand for the set of processors and the set of communication links in the network, respectively.

While analyzing complex networks, stability which is a key aspect in designing computer networks, and vulnerability which can be defined as the measurement of the global power of its related graph, must be taken into account. If graph theoretical parameters are used to express the network requirements, the issue of analysis and design of networks become finding a graph G which satisfies certain pre-specified requirement. It is known that communication systems are often exposed to failures and attacks. In the literature various measures are suggested to measure the robustness of network and a variety of graph-theoretic parameters have been used to derive formula to calculate network reliability.

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In the analysis of the vulnerable communication network, two quantities play a vital role; namely (i) the number of elements that are not functioning, (ii) the size of the largest remaining (survived) sub network within which mutual communication can still occur. In adverse relationship, it is desirable that an opponent's network be such that the above referred two quantities can be made simultaneously small. Here the first parameter provides information about nodes which can be targeted for more disruption while the later gives the impact of damage after disruption. Based on these quantities, a number of graph parameters, such as connectivity, toughness, scattering number, integrity, tenacity, and their edge-analogues, have been proposed for measuring the vulnerability of networks. To estimate these quantities Barefoot et al. [1] (1987) introduced the concept of integrity of a graph as a new measure of vulnerability of network.

Definition 1.1. [1] *The integrity of a graph G is denoted by $I(G)$ and defined by $I(G) = \min\{|S| + m(G-S) : S \subseteq V(G)\}$, where $m(G-S)$ denotes the order of a maximum component of $G-S$.*

Suppose that $H \subseteq V(G)$ and let $x, y \in V(G)$. An H-path between x and y is a path where all intermediate vertices are from H. (This includes the degenerate cases where the path consists of the single edge xy or a single vertex x if $x = y$, call such an H-path trivial). A set $H \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G) - H$, there is an H-path in G between x and y. The smallest size of a hub set in G is called a hub number of G, and is denoted by $h(G)$ [10].

Sultan et al. [5] have introduced the concept of hub-integrity of a graph as a new measure of vulnerability which is defined as follows.

Definition 1.2. [5] *The hub-integrity of a graph G denoted by $HI(G)$ is defined by, $HI(G) = \min\{|S| + m(G-S)\}$, where S is a hub set and $m(G-S)$ is the order of a maximum component of $G-S$.*

Definition 1.3. [5] *A subset S of $V(G)$ is said to be an h-set, if $HI(G) = |S| + m(G-S)$.*

Vaidya and Kothari [8] have discussed domination integrity of a graph obtained by duplication of an edge by vertex and duplication of vertex by an edge in path and cycle. Also Vaidya and Kothari [9] have discussed domination integrity of splitting graph of path and cycle. In the present work, we investigate hub-integrity of splitting graphs and a graph obtained by duplication of an edge by vertex and duplication of vertex by an edge in some graphs.

Definition 1.4. [9] *For a graph G the splitting graph $S'(G)$ of graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$ where $N(v)$ and $N(v')$ are the neighborhood sets of v and v', respectively.*

Definition 1.5. [7] *Duplication of a vertex v_i by a new edge $e = v_i'v_i''$ in graph G produces a new graph G' such that $N(v_i') = \{v_i'', v_i\}$ and $N(v_i'') = \{v_i', v_i\}$.*

Definition 1.6. [7] *Duplication of an edge $e = uv$ by a new vertex w in graph G produces a new graph G' such that $N(w) = \{u, v\}$.*

2. Splitting Graph

Theorem 2.1. *For $p \geq 2$,*

$$h(S'(P_p)) = \begin{cases} 2 & \text{if } p = 2, 3, \\ p-2 & \text{if } p \geq 4. \end{cases}$$

Proof. Let $\{u_1, u_2, ..., u_p\}$ be the vertices of path P_p and $\{v_1, v_2, ..., v_p\}$ be the new vertices corresponding to $\{u_1, u_2, ..., u_p\}$ which are added to obtain $S'(P_p)$. As $N(v_1) = \{u_2\}$, $N(v_p) = \{u_{p-1}\}$, $N(u_2) = \{u_1, u_3, v_1, v_3\}$ and $N(u_{p-1}) = \{u_{p-2}, u_p, v_p, v_{p-2}\}$, we have the two cases:
Case 1: for $p = 2$, consider $S = \{u_1, u_2\}$ is a hub set for $S'(P_2)$, and clearly the set S is a minimum hub set. Hence $h(S'(P_2)) = 2$.
Case 2: for $p = 3$, consider $S = \{u_2, v_2\}$ is a hub set for $S'(P_3)$, and clearly the set S is a minimum hub set. Hence $h(S'(P_3)) = 2$.
Case 3: for $p \geq 4$, Consider $S = \{u_2, ..., u_{p-1}\}$ is a hub set for $S'(P_p)$ and $|S| = p - 2$. As v_1 is adjacent to u_2 and v_p is adjacent to u_{p-1} and $N(v_i) = \{u_{i-1}, u_{i+1}, 2 \leq i \leq p - 1\}$, so for any $v_i, v_j \in V(S'(P_p)) - S, 1 \leq i, j \leq p$, then there exist S-path between them. As $N(u_1) = \{u_2, v_2\}$ and $N(u_p) = \{u_{p-1}, v_{p-1}\}$ then there exist S-path between $(u_p, u_1), (u_p, v_i)$ and (u_1, v_i) where $1 \leq i \leq p$, now we claim that set $S = \{u_2, u_3, ..., u_{p-1}\}$ is a minimum hub set. If some $u_i, 2 \leq i \leq p - 1$ is removed from set S then there do not exist path between u_1 and u_{i+1}. Thus S is minimum hub set. Hence $h(S'(P_p)) = p-2$. □

Theorem 2.2. *For $p \geq 2$,*

$$HI(S'(P_p)) = \begin{cases} 3 & \text{if } p = 2, \\ p & \text{if } p \geq 3. \end{cases}$$

Proof. Let $\{u_1, u_2, ..., u_p\}$ be the vertices of path P_p and $\{v_1, v_2, ..., v_p\}$ be the new vertices corresponding to $\{u_1, u_2, ..., u_p\}$ which are added to obtain $S'(P_p)$.
Case 1: $p = 2$. From Theorem 2.1, we have $h(S'(P_2)) = 2$ and $H = \{u_1, u_2\}$ is a h-set of $S'(P_2)$. Then $m(S'(P_2) - H) = 1$. This implies that $HI(S'(P_2)) = h(S'(P_2)) + m(S'(P_2) - H) = 2 + 1 = 3$. Clearly there does not exist any hub set S_1 of $S'(P_2)$ such that $|S_1| + m(S'(P_2) - S_1) < h(S'(P_2)) + m(S'(P_2) - H)$. Hence, $HI(S'(P_2)) = 3$.
Case 2: $p = 3$. From Theorem 2.1, we have $h(S'(P_2)) = 2$ and $H = \{u_2, v_2\}$ is a h-set of $S'(P_3)$. Then $m(S'(P_3) - H) = 1$. This implies that $HI(S'(P_2)) = h(S'(P_2)) + m(S'(P_2) - H) = 2 + 1 = 3$. Moreover, for any hub set S of $S'(P_3)$ we have, $|S| + m(S'(P_3) - S) \geq |H| + m(S'(P_3) - H)$. Hence $HI(S'(P_3)) = 3$.
Case 3: $p \geq 4$. From Theorem 2.1, we have $h(S'(P_p)) = p - 2$. Let $H = \{v_2, v_3, ..., v_{p-1}\}$ be a h-set of graph $S'(P_p)$. Then $m(S'(P_p) - H) = 2$. Therefore,

$$HI(S'(P_p)) \leq h(S'(P_p)) + m(S'(P_p) - H) = p. \tag{1}$$

For showing that the number $|H|+m(S'(P_p)-H)$ is minimum. The minimality of both $|H|$ and $m(S'(P_p) - H)$ is taken into consideration. The minimality of $|H|$ is guaranteed as H is h-set. It remains to show that if S is any hub set other then H, $|S|+m(S'(P_p)-S) \geq p$. If $m(S'(P_p) - S) = 1$, then $|S| \geq p > p - 2$, consequently $|S| + m(S'(P_p) - S) \geq p$. If $m(S'(P_p) - S) \geq 2$, then trivially $|S| + m(S'(P_p) - S) \geq p$. Hence for any hub set S,

$$|S| + m(S'(P_p) - S) > p. \tag{2}$$

From (1) and (2), $HI(S'(P_p)) = p$. □

Theorem 2.3. *For all $p \geq 3$,*

$$h(S'(C_p)) = \begin{cases} 2 & \text{if } p = 3, \\ p - 2 & \text{if } p \geq 4. \end{cases}$$

Proof. Let $\{u_1, u_2, ..., u_p\}$ be the vertices of cycle C_p and $\{v_1, v_2, ..., v_p\}$ be the new vertices corresponding to $\{u_1, u_2, ..., u_p\}$ which are added to obtain $S'(C_p)$.
Case 1: $p = 3$. Consider $S = \{u_1, u_2\}$, a hub set of $S'(C_3)$ as $N(u_1) = \{u_2, u_3, v_2, v_3\}$ and

$N(u_2) = \{u_1, u_3, v_1, v_3\}$, for any $x, y \in V(S'(C_3) - S)$, there is an S-path between them in $S'(C_3)$. To show that S is minimum hub set, if u_1 is removed from set S, then there is no S-path between v_1 and v_2. Thus S is minimum hub set. Hence $h(S'(C_3)) = 2$.

Case 2: $p \geq 4$. Consider $S = \{u_1, u_2, ..., u_{p-2}\}$, a hub set of $S'(C_p)$ and $|S| = p - 2$. As $N(u_1) = \{u_2, v_2, u_p, v_p\}$ and $N(u_{p-2}) = \{u_{p-3}, v_{p-3}, u_{p-1}, v_{p-1}\}$, for any $x, y \in V(S'(C_p) - S)$, there is an S-path between them in $S'(C_p)$.

We claim that set $S = \{u_1, u_2, ..., u_{p-2}\}$ is the minimum hub set. Since u_1 is adjacent to u_2, v_2, u_p, v_p , if u_1 is removed from set S then there does not exist an S-path between u_2 and u_p. Thus S is the minimum hub set. Hence $h(S'(C_p)) = p - 2$. □

Theorem 2.4. *For all* $p \geq 3$,

$$HI(S'(C_p)) = \begin{cases} 4 & \text{if } p = 3, \\ p+1 & \text{if } p \geq 4. \end{cases}$$

Proof. Let $\{u_1, u_2, ..., u_p\}$ be the vertices of cycle C_p and $\{v_1, v_2, ..., v_p\}$ be the new vertices corresponding to $\{u_1, u_2, ..., u_p\}$ which are added to obtain $S'(C_p)$.

We have the two following cases:

Case 1: For $p = 3$. From Theorem 2.3, we have $h(S'(C_3)) = 2$ and $H = \{u_1, u_2\}$ is a h-set of $S'(C_3)$. Then $m(S'(C_3) - H) = 3$. Therefore

$$HI(S'(C_3)) \leq h(S'(C_3)) + m(S'(C_3) - H) = 2 + 3 = 5. \tag{3}$$

If S_1 is any hub set of $S'(C_3)$ other than H with $m(S'(C_3) - S_1) = 2$, then $|S_1| \geq 3$. This implies that

$$|S_1| + m(S'(C_3) - S_1) \geq 3 + 2 = 5. \tag{4}$$

Let $S_2 = \{u_1, u_2, u_3\}$, a hub set of $S'(C_3)$, then $m(S'(C_3) - S_2) = 1$. This implies that

$$|S_2| + m(S'(C_3) - S_2) = 3 + 1 = 4. \tag{5}$$

Hence from (3), (4) and (5), $HI(S'(C_p)) = 4$.

Case 2: $p \geq 4$. From Theorem 2.3, we have $h(S'(C_p)) = p - 2$ and $H = \{u_1, u_2, ..., u_{p-2}\}$ is a h-set of $S'(C_p)$. Then $m(S'(C_p) - H) = 6$. Therefore

$$HI(S'(C_p)) \leq h(S'(C_p)) + m(S'(C_p) - H) = p - 2 + 6 = p + 4 \tag{6}$$

If S_1 is any hub set of $S'(C_p)$ other than H with $m(S'(C_p) - S_1) = 4$ *or* 5, then $|S_1| > h(S'(C_p)) = p - 2$. This implies that

$$|S_1| + m(S'(C_p) - S_1) > h(S'(C_p)) + 4 = p - 2 + 4 = p + 2. \tag{7}$$

If S_2 is any hub of $S'(C_p)$ set other than H with $m(S'(C_p) - S_2) = 2$ *or* 3, then $|S_2| \geq p - 1$. This implies that

$$|S_2| + m(S'(C_p) - S_2) \geq p - 1 + 2 = p + 1. \tag{8}$$

Let $S_3 = \{u_1, u_2, ..., u_p\}$, a hub set of $S'(C_p)$ then $m(S'(C_p) - S_3) = 1$. This implies that

$$|S_3| + m(S'(C_p) - S_3) = p + 1. \tag{9}$$

Hence from (6), (7), (8) and (9), $HI(S'(C_p)) = p + 1$. □

Theorem 2.5. *For all* $p \geq 4$, $h(S'(K_{1,p-1})) = 2$.

Proof. Let $\{u, u_1, ..., u_{p-1}\}$ be the vertices of star $K_{1,p-1}$ and $\{v, v_1, ..., v_{p-1}\}$ be the new vertices corresponding to $\{u, u_1, ..., u_{p-1}\}$ which are added to obtain $S'(K_{1,p-1})$.

Consider $S = \{u, v\}$, a h-set of $S'(K_{1,p-1})$. We claim that S is a minimum hub set of $S'(K_{1,p-1})$. Since $N(u) = \{u_1, u_2, ..., u_{p-1}, v_1, v_2, ..., v_{p-1}\}$ and $N(v) = \{u_1, u_2, ..., u_{p-1}\}$, and removal of u from set S leads to nonexistence of S-path between any two vertices of $v_1, v_2, ..., v_{p-1}$, it follows that S is a minimum hub set. Hence $h(S'(K_{1,p-1})) = 2$. □

Theorem 2.6. *For all $p \geq 4$, $HI(S'(K_{1,p-1})) = 3$.*

Proof. Let $\{u, u_1, ..., u_{p-1}\}$ be the vertices of star $K_{1,p-1}$ and $\{v, v_1, ..., v_{p-1}\}$ be the new vertices corresponding to $\{u, u_1, ..., u_{p-1}\}$ which are added to obtain $S'(K_{1,p-1})$. From Theorem 2.5, we have $h(K_{1,p-1})) = 2$ and $H = \{u, v\}$ is a h-set of $S'(K_{1,p-1})$. Then $m(S'(K_{1,p-1}) - H) = 1$. Therefore,

$$HI(S'(K_{1,p-1})) \leq h(S'(K_{1,p-1})) + 1. \tag{10}$$

To show that the number $|H| + m(S'(K_{1,p-1}) - H)$ is minimum, it is assumed that S is any hub set other than H and $m(S'(K_{1,p-1}) - S) \geq 1$, then $|S| + m(S'(K_{1,p-1}) - S) > h(S'(K_{1,p-1})) + 1$. Hence for any hub set S,

$$|S| + m(S'(K_{1,p-1}) - S) > h(S'(K_{1,p-1})) + 1. \tag{11}$$

From (10) and (11), we have $HI(S'(K_{1,p-1})) = 3$. □

Definition 2.1. [3] *The double star graph $S_{n,m}$ is the graph constructed from $K_{1,n-1}$ and $K_{1,m-1}$ by joining their centers v_0 and u_0. $V(S_{n,m}) = V(K_{1,n-1}) \cup V(K_{1,m-1})$ and $E(S_{n,m}) = \{v_0u_0, v_0v_i, u_0u_j \ : \ 1 \leq i \leq n-1, \ 1 \leq j \leq m-1\}$.*

Lemma 2.1. *For all $n, m \geq 2$, $h(S'(S_{n,m})) = 2$.*

Proof. Let $\{u, u_1, u_2, ..., u_{n-1}, v, v_1, v_2, ..., v_{m-1}\}$ be the vertex set of double star $S_{n,m}$ and $\{u', u'_1, u'_2, ..., u'_{n-1}, v', v'_1, v'_2, ..., v'_{m-1}\}$ be the new vertices corresponding to $\{u, u_1, u_2, ..., u_{n-1}, v, v_1, v_2, ..., v_{m-1}\}$ which are added to obtain $S'(S_{n,m})$. Consider $S = \{u, v\}$ is a hub set of $S'(S_{n,m})$ and $|S| = 2$. Let us claim that S is a minimum hub set of $S'(S_{n,m})$. Since $N(u) = \{v, v', u_1, u_2, ..., u_{n-1}, u'_1, u'_2, ..., u'_{n-1}\}$ and $N(v) = \{u, u', v_1, v_2, ..., v_{m-1}, v'_1, v'_2, ..., v'_{m-1}\}$ and removal of u or v from S, leads to non existence of S-path between any u_i with v_i. Thus S is a minimum hub set. Hence $HI(S'(S_{n,m})) = 2$. □

Theorem 2.7. *For all $n, m \geq 2$, $HI(S'(S_{n,m})) = 5$.*

Proof. Let $\{u, u_1, u_2, ..., u_{n-1}, v, v_1, v_2, ..., v_{m-1}\}$ be the vertex set of double star $S_{n,m}$ and $\{u', u'_1, u'_2, ..., u'_{n-1}, v', v'_1, v'_2, ..., v'_{m-1}\}$ be the new vertices corresponding to $\{u, u_1, u_2, ..., u_{n-1}, v, v_1, v_2, ..., v_{m-1}\}$ which are added to obtain $S'(S_{n,m})$. Consider $S = \{u, v\}$, a hub set of $S'(S_{n,m})$.

Case 1: $n = m = 2$. From Lemma 2.1, we have $h(S'(S_{2,2})) = 2$ and $S = \{u, v\}$ is a h-set of $S'(S_{2,2})$. Then $m(S'(S_{2,2}) - S) = 3$. Therefore

$$HI(S'(S_{2,2})) \leq h(S'(S_{2,2})) + m(S'(S_{2,2}) - S) = 5. \tag{12}$$

Consider S_1 is any hub set of $S'(S_{2,2})$ other than S with $m(S'(S_{2,2}) - S_1) = 2$, then $|S_1| \geq 4$. This implies that

$$|S_1| + m(S'(S_{2,2}) - S_1) \geq 2 + 4 = 6. \tag{13}$$

Let $S_2 = \{u, v, u', v'\}$ be a hub set of $S'(S_{2,2})$, then $m(S'(S_{2,2}) - S_2) = 1$. This implies that

$$|S_2| + m(S'(S_{2,2}) - S_2) = 4 + 1 = 5. \tag{14}$$

Hence from (12), (13) and (14), $HI(S'(S_{2,2})) = 5$.

Case 2: $n \geq 2, m > 2$ *or* $n > 2, m \geq 2$

From Lemma 2.1, $h(S'(S_{n,m})) = 2$, and $S = \{u, v\}$ is a h-set of $S'(S_{n,m})$. Then $m(S'(S_{n,m}) - S) = \ max \ \{n + 1, m + 1\}$. Therefore

$$HI(S'(S_{n,m})) \leq h(S'(S_{n,m})) + m(S'(S_{n,m}) - S) = 2 + \ max \ \{n + 1, m + 1\}. \tag{15}$$

Consider $S_1 = \{u, v, u', v'\}$, a hub set of $S'(S_{n,m})$, then $m(S'(S_{n,m}) - S_2) = 1$. This implies that

$$|S_1| + m(S'(S_{n,m}) - S_1) = 5. \quad (16)$$

We claim that S_1 is a minimum hub set. Since u is adjacent to $\{v, v', u_1, ..., u_n, u'_1, ..., u'_n\}$, and removal of u from S_1 leads to nonexistence of S_1-path between u_i and u'_i, it follows that S_1 is a minimum hub set. Hence from (15) and (16), $HI(S'(S_{n,m})) = 5$. □

Lemma 2.2. *For any wheel $W_{1,p-1}$, $h(S'(W_{1,p-1})) = 2$.*

Proof. Let $\{u, u_1, u_2, ..., u_{p-1}\}$ be the vertex set of wheel graph $W_{1,p-1}$ and $\{u', u'_1, u'_2, ..., u'_{p-1}\}$ be the new vertices corresponding to $\{u, u_1, u_2, ..., u_{p-1},\}$ which are added to obtain $S'(W_{1,p-1})$. Consider $S = \{u, u_1\}$, a hub set of $S'(W_{1,p-1})$. u and u_1 are adjacent to $u_2, ..., u_{p-1}$ and $u'_1, u'_2, ..., u'_{p-1}$, so for any $x, y \in V(S'(W_{1,p-1}) - S$, there exists S-path between them. Also S is minimum hub set because there do not exist S-path between the vertex u' and the vertices $u'_1, u'_2, ..., u'_{p-1}$ if the vertex u_1 is removed. Thus S is minimum hub set of $S'(W_{1,p-1})$, hence $h(S'(W_{1,p-1})) = 2$. □

Theorem 2.8. $HI(S'(W_{1,p-1})) = p + 1$.

Proof. Since $S'(W_{1,p-1})$ contains a wheel graph $W_{1,p-1}$ as its subgraph. If we choose the set S as all vertices of $W_{1,p-1}$ of $S'(W_{1,p-1})$, then there exist p components each contains only one vertex. So $HI(S'(W_{1,p-1})) = p + 1$. □

3. Duplication of Graph Elements

Lemma 3.1. *Let $(P_p)_v$ be a graph obtained by duplication of each edge by vertex of path P_p, then $h((P_p)_v) = p - 2, p \geq 3$.*

Proof. Let $(P_p)_v$ be a graph obtained by duplication of each edge v_iv_{i+1} of path P_p $(1 \leq i \leq p-1)$ by vertex u_i. The nature of the graph G, shows that from the vertices v_i at least one vertex must belong to any hub set S as u_i is adjacent to only v_i and v_{i+1}. Therefore if S is any hub set, then $|S| \geq p - 2$.
We claim that $S = \{v_2, ..., v_{p-1}\}$ is a minimum hub set of $(P_p)_v$. Since each v_i is adjacent to $v_{i-1}, v_{i+1}, u_i,$ *and* $u_{i-1}, (2 \leq i \leq p-2)$, if v_i is removed from set S, then u_i and u_{i-1} will not have S-path between them. Thus S is a minimum hub set. Hence $h((P_p)_v) = p-2$. □

Theorem 3.1. *Let $(P_p)_v$ be a graph obtained by duplication of each edge by vertex of path P_p, then $HI((P_p)_v) = p, p \geq 3$.*

Proof. Let $(P_p)_v$ be a graph obtained by duplication of each edge v_iv_{i+1} of path P_p $(1 \leq i \leq p-1)$ by vertex u_i. Then from Lemma 3.1, $h((P_p)_v) = p-2$ and $H = \{v_2, v_3, ..., v_{p-1}\}$ is a h-set of $(P_p)_v$, then $m((P_p)_v - H) = 2$ which implies,

$$HI((P_p)_v) \leq h((P_p)_v) + m((P_p)_v - H) = p - 2 + 2 = p. \quad (17)$$

We will show that the number $|H| + m((P_p)_v - H)$ is the minimum. If S_1 is a hub set of $(P_p)_v$ other than H and $m((P_p)_v - S_1) = 1$, then $|S_1| \geq p$ which implies that

$$|S_1| + m((P_p)_v - S_1) \geq p + 1. \quad (18)$$

Consider $m((P_p)_v - S_1) \geq 2$, then $|S_1| + m((P_p)_v - S_1) \geq p$. Hence for any hub set S_1,

$$|S_1| + m((P_p)_v - S_1) \geq p. \quad (19)$$

From (17), (18) and (19), we have $HI((P_p)_v) = p, p \geq 3$. □

Lemma 3.2. *Let $(C_p)_v$ be a graph obtained by duplication of each edge by vertex of cycle C_p. Then $h((C_p)_v) = p - 1, p \geq 3$.*

Proof. Let $(C_p)_v$ be a graph obtained by duplication of each edge $v_i v_{i+1}$ of cycle C_p $(1 \leq i \leq p-1)$ by vertex u_i and duplication edge $v_p v_1$ by vertex u_p. There are two types of vertices in $(C_p)_v$,

(1) $d(u_i) = 2 \; for \; 1 \leq i \leq p$,
(2) $d(v_i) = 4, \; for \; 1 \leq i \leq p$.

The nature of the graph $(C_p)_v$, shows that out of the vertices v_i and v_{i+1} at least one vertex must belong to any hub set S as u_i is adjacent to only v_i, v_{i+1} and u_p is adjacent to v_1, v_p. Therefore if S is any hub set then $|S| \geq p-1$. We claim that $S = \{v_1, v_2, ..., v_{p-1}\}$ is the minimum hub set of $(C_p)_v$. Since each $v_i, 2 \leq i \leq p-1$ is adjacent to v_{i-1}, v_{i+1}, u_{i-1}, and u_i, if v_i is removed from set S then there is no S-path between u_{i-1} and u_i. Thus S is the minimum hub set. Hence $h((C_p)_v) = p-1$. □

Theorem 3.2. *Let $(C_p)_v$ be a graph obtained by duplication of each edge by vertex of cycle C_p. Then $HI((C_p)_v) = p+1, p \geq 3$.*

Proof. Let $(C_p)_v$ be a graph obtained by duplication of each edge $v_i v_{i+1}$ of cycle C_p $(1 \leq i \leq p-1)$ by vertex u_i and edge $v_p v_1$ by vertex u_p. Then from Lemma 3.2, $h((C_p)_v) = p-1$, and $H = \{v_1, v_2, ..., v_{p-1}\}$ is a h-set of $(C_p)_v$. Then $m((C_p)_v - H) = 3$ which implies,

$$HI((C_p)_v) \leq h((C_p)_v) + m((C_p)_v - H) = p - 1 + 3 = p + 2. \tag{20}$$

Consider $S = \{v_1, v_2, ..., v_p\}$, a hub set of $(C_p)_v$, then $m((C_p)_v - S) = 1$ which implies that,

$$HI((C_p)_v) \leq |S| + m((C_p)_v - S) = p + 1. \tag{21}$$

If S_1 is any hub set other than S and $m((C_p)_v - S_1) = 2$, then $|S_1| \geq p$, so

$$|S_1| + m((C_p)_v - S_1) \geq p + 2. \tag{22}$$

If S_2 is any hub set other than S and S_1, and $m((C_p)_v - S_2) = 1$, then $|S_2| > p$, thus

$$|S_2| + m((C_p)_v - S_2) > p + 1. \tag{23}$$

From (20), (21), (22) and (23), we have $HI((C_p)_v) = p+1, p \geq 3$. □

Lemma 3.3. *Let G_e be a graph obtained by duplication of each vertex by an edge of path P_p or cycle C_p. Then $h(G_e) = p$.*

Proof. Let G_e be a graph obtained by duplication of vertices $\{v_1, v_2, ..., v_p\}$ of path P_p or cycle C_p by an edge $u_{2i-1}u_{2i}, (1 \leq i \leq p)$. Now set $S = \{v_1, v_2, ..., v_p\}$ is a minimum hub set, since each v_i is adjacent to u_{2i} and u_{2i-1}, and removal of v_i from set S leads to nonexistence of S-path between u_{2i} and u_{2i-1}. Hence $h(G_e) = p$. □

Theorem 3.3. *Let G_e be a graph obtained by duplication of each vertex by an edge of path P_p or cycle C_p. Then $HI(G_e) = p+2$.*

Proof. Let G_e be a graph obtained by duplication of vertices v_i of path P_p or cycle C_p by an edge $u_{2i-1}u_{2i}(1 \leq i \leq p)$. Then from Lemma 3.3, we have $h(G_e) = p$, and $H = \{v_1, v_2, ..., v_p\}$ is a h-set of graph G_e. Then $m(G_e - H) = 2$. Therefore

$$HI(G_e) \leq h(G_e) + m(G_e - H) = p + 2. \tag{24}$$

For showing that the number $|H| + m(G_e - H)$ is minimum. The minimality of both $|H|$ and $m(G_e - H)$ is taken into consideration. The minimality of $|H|$ is guaranteed as H is h-set. Now if S is any hub set other than H and $m(G_e - S) = 1$, then $|S| \geq 2p > p+2$, consequently $|S| + m(G_e - S) > p+2$. Consider $m(G_e - S) \geq 2$, then $|S| + m(G_e - S) \geq p+2$. Hence for any hub set S,

$$|S| + m(G_e - S) \geq p + 2. \tag{25}$$

From (24) and (25), we have $HI(G_e) = p + 2$. □

Lemma 3.4. *Let $(K_{1,p-1})_e$ be a graph obtained by duplication of each vertex of star $K_{1,p-1}$ by an edge. Then $h((K_{1,p-1})_e) = p$.*

Proof. Let $(K_{1,p-1})_e$ be a graph obtained by duplication of vertices $\{v_0, v_1, ..., v_{p-1}\}$ of star $K_{1,p-1}$ by an edge $u_{2i}u_{2i+1}, (0 \leq i \leq p-1)$. We claim that set $S = \{v_0, v_1, ..., v_{p-1}\}$ is minimum hub set. Since each v_i is adjacent to u_{2i} and u_{2i+1}, also v_0 is adjacent to $u_0, u_1, v_1, ..., v_{p-1}$, if v_0 is removed from set S then there do not exist is S-path between u_0, u_1 with v_i and if some $v_i(1 \leq i \leq p-1)$ is removed from set S then there do not exist S-path between u_0, u_1 with u_{2i} and $u_{2i+1}(1 \leq i \leq p-1)$. So S is minimum hub set, hence $h((K_{1,p-1})_e) = p$. □

Theorem 3.4. *Let $(K_{1,p-1})_e$ be a graph obtained by duplication of each vertex of star $K_{1,p-1}$ by an edge. Then $HI((K_{1,p-1})_e) = p + 2$.*

Proof. Let $(K_{1,p-1})_e$ be a graph obtained by duplication of vertices $\{v_0, v_1, ..., v_{p-1}\}$ of star $K_{1,p-1}$ by an edge $u_{2i}u_{2i+1}, (0 \leq i \leq p-1)$. Then from Lemma 3.4, $h((K_{1,p-1})_e) = p$, and $H = \{v_0, v_1, ..., v_{p-1}\}$ is a h-set of graph $(K_{1,p-1})_e$. Then $m((K_{1,p-1})_e - H) = 2$. Therefore

$$HI((K_{1,p-1})_e) \leq h((K_{1,p-1})_e) + m((K_{1,p-1})_e - H) = p + 2. \tag{26}$$

We will show that the number $|H| + m((K_{1,p-1})_e - H)$ is minimum. The minimality of $|H|$ is guaranteed as H is h-set. It remains to show that if S is any hub set other than H and $m((K_{1,p-1})_e - S) = 1$, then $|S| \geq 2p$, hence $|S| + m((K_{1,p-1})_e - S) \geq 2p + 1$. If $m((K_{1,p-1})_e - S) \geq 2$, then $|S| + m((K_{1,p-1})_e - S) \geq p + 2$. Hence for any hub set S,

$$|S| + m((K_{1,p-1})_e - S) \geq p + 2. \tag{27}$$

From (26) and (27), $HI((K_{1,p-1})_e) = p + 2$. □

Lemma 3.5. *Let $(K_{1,p-1})_v$ be a graph obtained by duplication of each edge of star $K_{1,p-1}$ by vertex. Then $h((K_{1,p-1})_v) = 1$.*

Proof. Let $(K_{1,p-1})_v$ be a graph obtained by duplication of each edge vv_i of star $K_{1,p-1}$ by vertex $u_i, (1 \leq i \leq p-1)$. There are two types of vertices in $(K_{1,p-1})_v$,

(1) $d(u_i) = 2\ for\ 1 \leq i \leq p-1$,
(2) $d(v) = 2p - 2$,
(3) $d(v_i) = 2\ for\ 1 \leq i \leq p-1$.

The nature of the graph $(K_{1,p})_v$, shows that $\{v\}$ is a hub set of G. So $S = \{v\}$, is the minimum hub set. Therefore $h((K_{1,p-1})_v) = 1$. □

Theorem 3.5. *Let $(K_{1,p-1})_v$ be a graph obtained by duplication of each edge of star $K_{1,p-1}$ by vertex. Then $HI((K_{1,p-1})_v) = 3$.*

Proof. Let $(K_{1,p-1})_v$ be a graph obtained by duplication of each edge vv_i of star $K_{1,p-1}$ by vertex $u_i, (1 \leq i \leq p-1)$. Then from Lemma 3.5, $h((K_{1,p-1})_v) = 1$, and $H = \{v\}$ is a h-set of $(K_{1,p-1})_v$. Therefore $m((K_{1,p-1})_v - H) = 2$, which implies,

$$HI((K_{1,p-1})_v) \leq h((K_{1,p-1})_v) + m((K_{1,p-1})_v - H) = 1 + 2 = 3. \tag{28}$$

The minimality of $|H|$ is guaranteed as H is h-set. It remains to show that if S is any hub set other than H, then $|S| + m((K_{1,p-1})_v - S) \geq 3$. If $m((K_{1,p-1})_v - S) = 1$ then $|S| \geq p \geq 3$, which implies that $|S| + m((K_{1,p-1})_v - S) > h((K_{1,p-1})_v) + 2$. If $m((K_{1,p-1})_v - S) \geq 2$, then

$$|S| + m((K_{1,p-1})_v - S) \geq h(G) + 2. \tag{29}$$

From (28) and (29), $HI((K_{1,p-1})_v) = 3$. □

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