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PARTITIONING A GRAPH INTO MONOPOLY SETS

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Abstract. In a graph $G = (V, E)$, a set $M \subseteq V(G)$ is said to be a monopoly set of G if every vertex $v \in V - M$ has, at least, $\frac{d(v)}{2}$ neighbors in M. The monopoly size of G, denoted by $mo(G)$, is the minimum cardinality of a monopoly set. In this paper, we study the problem of partitioning $V(G)$ into monopoly sets. An M-partition of a graph G is the partition of $V(G)$ into k disjoint monopoly sets. The monatic number of G, denoted by $\mu(G)$, is the maximum number of sets in M-partition of G. It is shown that $2 \leq \mu(G) \leq 3$ for every graph G without isolated vertices. The properties of each monopoly partite set of G are presented. Moreover, the properties of all graphs G having $\mu(G) = 3$, are presented. It is shown that every graph G having $\mu(G) = 3$ is Eulerian and have $\chi(G) \leq 3$. Finally, it is shown that for every integer $k \notin \{1, 2, 4\}$, there exists a graph G of order $n = k$ having $\mu(G) = 3$.

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1. Introduction

The concept of monopoly in a graph was introduced by Khoshkhak K. et al. [10]. Some mathematical properties of monopoly in graphs have been studied in [12], other types of monopoly in graphs have been subsequently proposed by the authors ([13]-[16]). In particular, the monopoly in graphs is a dynamic monopoly (dynamos) that, when colored black at a certain time step, will cause the entire graph to be colored black in the next time step under an irreversible majority conversion process. Dynamos were first introduced by Peleg D. [17]. For more details in monopoly and dynamos in graphs, we refer the reader to [2, 3, 7, 11, 19]. In this paper, we focus our attention on the problem of partitioning of the vertex set of a graph G into disjoint monopoly sets. We denote by M-partition to the partition of $V(G)$ into k disjoint monopoly sets. The idea of M-partition of G closely related to unfriendly partition [5, 1], and an offensive k-alliances partition [18].

We begin by stating the terminology and notations used through this article. A graph $G = (V, E)$ is a simple graph, that is finite, having no loops no multiple and directed edges. As usual, we denote by $n = |V|$ and $m = |E|$ to the number of vertices and edges in a graph G, respectively. For a vertex $v \in V$, the open neighborhood of v in a graph G, denoted $N(v)$, is the set of all vertices that are adjacent to v and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The degree of vertex v in G is $d(v) = |N(v)|$, and the degree of a vertex v with respect to a subset $S \subset V(G)$ is $d_S(v) = |N(v) \cap S|$. We denote by

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$\Delta(G)$ and $\delta(G)$ to maximum and minimum degree among the vertices of G, respectively. $\lfloor x \rfloor$ ($\lceil x \rceil$) denotes the greatest (smallest) integer number less (greater) than or equal to x. An isolated vertex in G is a vertex with degree zero. As usual, $\overline{G}$ denotes the complement of G, for a subset $S \subseteq V$, $\overline{S} = V - S$ and kG denotes the k disjoint copies of G. A k-partite graph is a graph G whose vertex set $V(G)$ can be partitioned into k disjoint sets so that no two vertices within the same set are adjacent. A k-partite graph in which each partite set has the same number of vertices is said to be a balanced k-partite graph. The Friendship graph F_n, for $n \geq 2$, is the graph constructed by joining n copies of K_3 graph with a common vertex. A set $I \subseteq V$ is independent if no two vertices in I are adjacent. The independent sets of maximum cardinality are called maximum independent sets. The number of vertices in a maximum independent set is the independence number (or vertex independence number) of G and is denoted by $\alpha(G)$. For more terminologies and notations in graph theory, we refer the reader to the books [4, 8].

A set $D \subseteq V(G)$ is called a dominating set of a graph G if every vertex $v \in V(G) - D$ adjacent to some vertex in D. The minimum cardinality of such set is called the domination number of G and denoted by $\gamma(G)$. A dominating set of cardinality $\gamma(G)$ is called a γ-set for G. A thorough treatment of domination in graphs can be found in the book by Haynes at el. [9]. The domatic number $d(G)$ of a graph G is the maximum positive integer k such that $V(G)$ can be partitioned into k pairwise disjoint dominating sets. A partition V into pairwise disjoint dominating sets is called a domatic partition. The concept of a domatic number was introduced by E. J. Cockayne at el. [6]. A proper coloring of a graph G is a k-coloring in which each color class is an independent set. The minimum k for which a graph is k-colorable is called its chromatic number and denoted by $\chi(G)$ [4].

A pigeonhole principle states that if n items are put into m containers, with $n > m$, then at least one container must contain more than one item. The pigeonhole principle has several generalizations and can be stated in various ways. In a more quantified version: for natural numbers k and m, if $n = km + 1$ objects are distributed among m sets, then the pigeonhole principle asserts that at least one of the sets will contain at least $k + 1$ objects. For arbitrary n and m this generalizes to $k + 1 = \lfloor \frac{n-1}{m} \rfloor + 1$ [20].

A set $M \subseteq V(G)$ is called a monopoly set of G if for every vertex $v \in V(G) - M$ has at least $\frac{d(v)}{2}$ neighbors in M. The monopoly size of G, denoted by $mo(G)$, is the minimum cardinality of a monopoly set in G. An M-partition of a graph G is the partition of $V(G)$ into k disjoint monopoly sets. The monatic number of G, denoted by $\mu(G)$, is the maximum number of sets in M-partition of G. The word "monatic" was created from monopoly and chromatic in the same way the word "domatic" which created from domination and chromatic. It is shown that $2 \leq \mu(G) \leq 3$ for every graph G without isolated vertices. The properties of each monopoly partite set of G are presented. Moreover, the properties of all graphs G having $\mu(G) = 3$, are presented. It is shown that every graph G having $\mu(G) = 3$ is Eulerian and have $\chi(G) \leq 3$. Finally, it is shown that for every integer $k \notin \{1, 2, 4\}$, there exists a graph G of order $n = k$ with $\mu(G) = 3$.

The following are some fundamental results which will be required for many of our arguments in this paper:

Theorem 1.1. [8] *A graph G is eulerian if and only if every vertex of G is of even degree.*

The following results appear in paper [6].

Proposition 1.1. **(a):** *For any graph G, $d(G) \leq \delta + 1$.*
(b): *$d(G) \geq 2$, if and only if G has no isolated vertices.*
(c): *For any tree T with $n \geq 2$ vertices, $d(T) = 2$.*

2. Partitioning Vertex Set of a Graph into Monopoly sets

Theorem 2.1. *Any non-trivial graph G without isolated vertices has an M-partition.*

Proof. Let $\{X, Y\}$ be a partition of $V(G)$ such that the edge-cut between X and Y has maximum cardinality. Then X and Y are dominating sets. Moreover, for every vertex $x \in X$, has at least $\frac{d(v)}{2}$ neighbors in Y, then we have that Y is a monopoly set in G. Analogously, we obtain that X is a monopoly set in G. Hence, $\{X, Y\}$ is a partition of $V(G)$ into two monopoly sets in G. This complete the proof. □

Since any monopoly set M of a graph G must be contain every isolated vertices in G, then we have the following result.

Proposition 2.1. *Let G be a graph of order n. Then $\mu(G) = 1$, if and only if G having an isolated vertex.*

Accordingly to Theorem 2.1 and Proposition 2.1, we obtain the following fundamental result.

Theorem 2.2. *For any graph G without isolated vertices,*

$$2 \leq \mu(G) \leq 3.$$

Proof. By Theorem 2.1 and Proposition 2.1, we have $\mu(G) \geq 2$. For the upper bound, since, the M-partition of G is a partition of $V(G)$ into k monopoly subset, it follows by the definition of a monopoly set, every vertex $v \in V(G)$ must be adjacent to, at least, $\frac{d(v)}{2}$ vertices in every subset other then its own. If a graph G has $\mu(G) = k$, then every vertex $v \in V(G)$ must be adjacent to, at least, $(k-1)\frac{d(v)}{2}$ vertices, $\frac{d(v)}{2}$ vertices in each partite set of an M-partition. Hence, we have $(k-1)\frac{d(v)}{2} \leq d(v)$, this implies that $(k-3)d(v) \leq 0$. But since $d(v) > 0$, for every $v \in V(G)$, it follows that $k - 3 \leq 0$. Therefore, $\mu(G) \leq 3$. □

Corollary 2.1. *For any graph G, $1 \leq \mu(G) \leq 3$.*

Theorem 2.3. *For any graph G without isolated vertices. If G has a vertex of odd degree, then $\mu(G) = 2$.*

Proof. Let $v \in V(G)$ be a vertex with odd degree. i.e., $d(v) = 2k + 1$, for any $k \geq 0$. Suppose, to the contrary, that $\mu(G) = 3$ and let $\{M_1, M_2, M_3\}$ be the M-partition of G. Assume, without loss of generality, that $v \in M_1$. Then by the definition of a monopoly set, $d_{M_2}(v) \geq \frac{d(v)}{2} \geq k+1$ and also, $d_{M_3}(v) \geq k+1$. Hence, $d(v) \geq d_{M_2}(v) + d_{M_3}(v) \geq 2k+2$, a contradiction. Therefore, by Theorem 2.2, $\mu(G) = 2$. □

Corollary 2.2. *For any graph G. If G has $\mu(G) = 3$, then every vertex of G is of even degree.*

Theorem 2.4. *Let G be a graph without isolated vertices and every vertex of G is of even degree. If G has a cycle, of order $k \equiv 1, 2 \pmod 3$, as an endblock. Then $\mu(G) = 2$.*

Proof. Let G be a graph with a cycle endblock C_k, for $k \equiv 1, 2 \pmod 3$, and let $\{v_1, v_2, ..., v_k\}$ be the vertex set of C_k, such that v_1 is the cut vertex of G on C_k. Clearly, $d(v_1) \geq 4$ and $d(v_i) = 2$, for every $2 \leq i \leq k$. Suppose, to the contrary, that $\mu(G) = 3$ and let $\{M_1, M_2, M_3\}$ be the M-partition of G. Assume, without loss of generality, that $v_1 \in M_1$

and $v_2 \in M_2$, then $v_3 \in M_3$. Furthermore, $M_1 = \{v_i : i \equiv 1 \pmod 3\}$, $M_2 = \{v_i : i \equiv 2 \pmod 3\}$ and $M_3 = \{v_i : i \equiv 0 \pmod 3\}$. Then we have the following two cases.

Case 1: If $k \equiv 1 \pmod 3$, then $v_k \in M_1$ and hence, $d_{M_3}(v_1) = 0$, a contradiction.

Case 2: If $k \equiv 2 \pmod 3$, then $v_k \in M_2$ and hence a gain, $d_{M_3}(v_1) = 0$, a contradiction.

Therefore, $\mu(G) \neq 3$. □

Theorem 2.5. *For any graph G, $\mu(G) \leq d(G)$. Furthermore, if $\Delta(G) \leq 2$, then $\mu(G) = d(G)$.*

Proof. Clearly, from the definition of the monopoly set that any monopoly set of a graph G is a dominating set. Then, $\mu(G) \leq d(G)$. Now, let$G$ be a graph with $\Delta(G) \leq 2$. and let $D_1, D_2, ..., D_k$ be the partition of G into k dominating set. Since, $d_{D_i}(v) \geq 1 \geq \frac{\Delta}{2} \geq \frac{d(v)}{2}$, for every vertex $v \notin D_i$, and for every $i = 1, 2, .., k$, it follows that D_i is a monopoly set of G, for every $i = 1, 2, .., k$. Hence, $d(G) \leq \mu(G)$, but we have $\mu(G) \leq d(G)$. Then $\mu(G) = d(G)$. □

The converse of Theorem 2.5, is not true. For example, the star graph $K_{1,n}$, for every $n \geq 3$, has $d(K_{1,n}) = \mu(K_{1,n}) = 2$, but $\Delta(K_{1,n}) \geq 3$. The following result immediate consequences of Proposition 1.1 and Theorem 2.5.

Corollary 2.3. *For any tree T with $n \geq 2$ vertices, $\mu(G) = d(G) = 2$.*

In the following result, the exact values of the monatic number $\mu(G)$ for some standard graphs G are determined.

Proposition 2.2. .

(1) $\mu(P_n) = 2$, *for every* $n \geq 2$.

(2) $\mu(C_n) = \begin{cases} 3, & \text{if } n \equiv 0 \pmod 3; \\ 2, & \text{otherwise.} \end{cases}$

(3) $\mu(\overline{K_n}) = 1$, *for every* $n \geq 2$.

(4) $\mu(K_n) = \begin{cases} 1, & \text{if } n = 1; \\ 3, & \text{if } n = 3; \\ 2, & \text{otherwise.} \end{cases}$

(5) $\mu(K_{r,s}) = 2$, *for* $1 \leq r \leq s$.

(6) $\mu(F_n) = 3$, *for every* $n \geq 2$.

There are Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $|V_1| = |V_2|$, $|E_1| = |E_2|$ and the sequence degrees $S_d(G_1) = S_d(G_2)$, where $S_d(G) = \{d_1, d_2, ..., d_n\}$ and d_i is the degree of vertex in G. But $\mu(G_1) \neq \mu(G_2)$. Figure 1, shows two graphs G_1 and G_2 with $n_1 = n_2 = 7$, $m_1 = m_2 = 9$ and $S_d(G_1) = S_d(G_2) = \{4, 4, 2, 2, 2, 2, 2\}$. But $\mu(G_1) = 3$ and $\mu(G_2) = 2$.

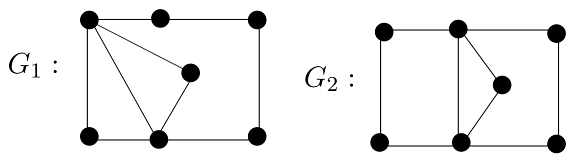

Figure 1

A bipartition (V_1, V_2) of a vertex set $V(G)$ of a graph G is called an unfriendly partition; if every vertex $u \in V_1$ has at least as many neighbors in V_2 as it does in V_1, and every vertex $v \in V_2$ has at least as many neighbors in V_1 as it does in V_2. This type of partition were defined and studied by Borodin et al. [5] and Aharoni et al. [1]. Clearly, for any graph G, if $\mu(G) = 2$, then the idea of M-partitions of a graph G is closely related to unfriendly partitions. Hence, in the following section, we shall focus our attention on the problem of partitioning a graph G into three monopoly sets.

3. Properties of the Monopoly Partite sets of Graphs G having $\mu(G) = 3$

In this section, we study the properties of every monopoly partite set of a graph G having $\mu(G) = 3$, number of edges which incident with every partite set.

Theorem 3.1. *For any graph G, if $\mu(G) = 3$, then every partite set in M-partition of G is an independent set.*

Proof. Let G be a graph with $\mu(G) = 3$ and let $\{M_1, M_2, M_3\}$ be the M-partition of G. On the contrary, suppose, without loss of generality, that M_1 is not an independent. Then there exists, at least, a vertex $v \in M_1$ such that $|N(v) \cap M_1| \geq 1$. Since M_2 is a monopoly set in G and $v \notin M_2$, it follows by definition of a monopoly set that

$$d_{M_2}(v) = |N(v) \cap M_2| \geq \frac{d(v)}{2}. \tag{1}$$

Similarly,

$$d_{M_3}(v) = |N(v) \cap M_3| \geq \frac{d(v)}{2}. \tag{2}$$

Hence, by the definition of the degree of a vertex in a graph G and by equations 1 and 2, we obtain $d(v) = d_{M_1}(v) + d_{M_2}(v) + d_{M_3}(v) \geq d(v) + 1$, a contradiction. Therefore, M_1 must be an independent set. For M_2 and M_3 the proof is similar to the proof of M_1. □

In the following two results, we investigate the sum of the degrees of vertices in every monopoly partite set of a graph G with $\mu(G) = 3$ and the edges which connected between any two monopoly partite sets in M-partition of G.

Proposition 3.1. *Let $\{M_1, M_2, M_3\}$ be an M-partition of a graph G. Then*

$$d_{M_i}(v) = d_{M_j}(v) = \frac{d(v)}{2}$$

for every $v \in M_k$, where i, j and $k \in \{1, 2, 3\}$ and $k \neq i \neq j$.

Theorem 3.2. *Let $\{M_1, M_2, M_3\}$ be the M-partition of G. Then*

$$\sum_{v \in M_i} d(v) = \frac{2m}{3}, \ for\ every\ 1 \leq i \leq 3.$$

Proof. Let G be a graph with $\mu(G) = 3$ and let $\{M_1, M_2, M_3\}$ be the M-partition of a graph G. By Theorem 3.1, every partite set M_i and for $1 \leq i \leq 3$, in M-partition of G is an independent. Then $d(v) = |N(v) \cap (V - M_i)| = d_{\overline{M_i}}(v)$, for every $v \in M_i$, $1 \leq i \leq 3$. Also, by Observation 3.1, we have $d_{M_i}(v) = d_{\overline{M_i}}(v)$, for every $v \in \overline{M_i}$.

now, for every $1 \leq i \leq 3$,

$$\begin{aligned} 2m &= \sum_{v\in V(G)} d(v) = \sum_{v\in M_i} d(v) + \sum_{v\in \overline{M_i}} d(v); \\ &= \sum_{v\in M_i} d_{M_i}(v) + \sum_{v\in M_i} d_{\overline{M_i}}(v) + \sum_{v\in \overline{M_i}} d_{M_i}(v) + \sum_{v\in \overline{M_i}} d_{\overline{M_i}}(v); \\ &= 0 + \sum_{v\in M_i} d_{\overline{M_i}}(v) + 2\sum_{v\in \overline{M_i}} d_{M_i}(v); \\ &= 3\sum_{v\in M_i} d_{\overline{M_i}}(v) = 3\sum_{v\in M_i} d(v). \end{aligned}$$

Therefore, $\sum_{v\in M_i} d(v) = \frac{2m}{3}$, for every $i = 1, 2, 3$. □

For any graph G with $\mu(G) = 3$, Theorem 3.2 shows that the number of edges between any partite set and both the others partite sets in M-partition of G is equal to $\frac{2m}{3}$. In the following result, $m(M_i, M_j)$ denotes the number of edges between M_i and M_j, $i, j \in \{1, 2, 3\}$.

Corollary 3.1. *Let $\{M_1, M_2, M_3\}$ be an M-partition of a graph G. Then*

$$m(M_i, M_j) = \frac{m}{3}, \ \textit{for every } i, j \in \{1, 2, 3\} \textit{ and } i \neq j.$$

Theorem 3.3. *Let $\{M_1, M_2, M_3\}$ be an M-partition of a graph G such that $|M_1| \leq |M_2| \leq |M_3|$. Then*

(1) $mo(G) \leq |M_1| \leq \lfloor \frac{n}{3} \rfloor$;
(2) $|M_1| \leq |M_2| \leq \frac{n-mo(G)}{2}$;
(3) $\lceil \frac{n}{3} \rceil \leq |M_3| \leq |M_1||M_2|$.

Proof. Let G be a graph of order n and let $\{M_1, M_2, M_3\}$ be an M-partition of a graph G such that $|M_1| \leq |M_2| \leq |M_3|$. Then

(1) Clearly that $|M_1| \geq mo(G)$. For the upper bound of $|M_1|$, assume, to the contrary, that $|M_1| \geq \lfloor \frac{n}{3} \rfloor + 1$. Since, $|M_1| \leq |M_2| \leq |M_3|$, then by the pigeonhole principle, $|M_3| \geq \lceil \frac{n}{3} \rceil$. We have the following Cases.
   Case 1: If $n \equiv 0 \pmod 3$, then $|M_1| \geq \frac{n}{3} + 1$. Hence, by the hypothesis, $n = |M_1| + |M_2| + |M_3| \geq n + 3$, a contradiction.
   Case 2: If $n \equiv 1 \pmod 3$, then $|M_1| \geq \frac{n-1}{3} + 1$ and $|M_3| \geq \frac{n+2}{3}$. Hence, we obtain, $n \geq 2(\frac{n-1}{3} + 1) + \frac{n+2}{3} = n + 2$, a contradiction.
   Case 3: if $n \equiv 2 \pmod 3$, then $|M_1| \geq \frac{n-2}{3} + 1$ and $|M_3| \geq \frac{n+1}{3}$. Hence, we obtain, $n \geq 2(\frac{n-2}{3} + 1) + \frac{n+1}{3} = n + 1$, a contradiction.
   Therefore, $|M_1| \leq \lfloor \frac{n}{2} \rfloor$.
(2) Form the hypothesis, we have $|M_1| \leq |M_2|$ and the cardinality of M_2 is maximum if and only if $|M_2| = |M_3|$. Since, $|M_2| \leq n - (|M_1| + |M_3|)$, it follows that and by the maximality of $|M_2|$,

$$|M_2| \leq \frac{n - |M_1|}{2} \leq \frac{n - mo(G)}{2}.$$

(3) By the hypothesis and the pigeonhole principle, we get $|M_3| \geq \lceil \frac{n}{3} \rceil$. Since $d_{M_1}(v) \geq 1$, for every $v \in M_3$, it follows that $\sum_{v\in M_3} d_{M_1}(v) \geq |M_3|$ and by Observation 3.1,

$\sum_{v \in M_3} d_{M_1}(v) = \sum_{v \in M_2} d_{M_1}(v)$. Hence,

$$|M_3| \leq \sum_{v \in M_3} d_{M_1}(v) = \sum_{v \in M_2} d_{M_1}(v) \leq \sum_{v \in M_2} |M_1| \leq |M_2||M_1|.$$

□

Corollary 3.2. *Let $\{M_1, M_2, M_3\}$ be an M-partition of a graph G, such that $|M_1| \leq |M_2| \leq |M_3|$. If $|M_1| = 1$, then $|M_2| = |M_3| = \frac{n-1}{2}$. Furthermore, $G = K_3$ or $G \cong F_n$.*

4. Properties of Graphs G having $\mu(G) = 3$

In this section, we investigate the properties of the graphs G having $\mu(G) = 3$ and the relationships between the monatic number of G and some other parameters of G.

Theorem 4.1. *For any graph G, if $\mu(G) = 3$, then G is eulerian.*

Proof. The result is an immediate consequences of Theorem 1.1 and Corollary 2.2. □

Theorem 3.1, shows that for every graph G with $\mu(G) = 3$, every partite set in M-partition of G is independent set. Then we have the following result.

Corollary 4.1. *Every graph G having $\mu(G) = 3$ is a 3-partite graph.*

The converse of the Corollary 4.1, in general, is not true. For example, the complete 3-partite graph $K_{1,2,3}$ has a vertex of odd degree, then by Theorem 2.3, $\mu(K_{1,2,3}) = 2$. In the following result, we characterize each complete 3-partite graph G with $\mu(G) = 3$.

Theorem 4.2. *Let $G = K_{n_1,n_2,n_3}$ a complete 3-partite graph. Then $\mu(G) = 3$, if and only if $n_1 = n_2 = n_3$.*

Proof. Let $G = K_{n_1,n_2,n_3}$ a complete 3-partite graph with partite sets (V_1, V_2, V_3) such that $|V_1| \leq |V_2| \leq |V_3|$. Certainly, If $n_1 = n_2 = n_3$, then every partite set is a monopoly set of G. Thus, $\mu(G) = 3$.
Conversely, let $G = K_{n_1,n_2,n_3}$ a complete 3-partite with $\mu(G) = 3$, and let $\{M_1, M_2, M_3\}$ be the M-partition of G such that $|M_1| \leq |M_2| \leq |M_3|$. We claim that $|M_i| = |V_i|$ for every $i = 1, 2, 3$. Otherwise, there is at least a monopoly partite set $|M_i|$ form M-partition of G, for $i = 1, 2, 3$, such that $M_i \cup V_j$ and $M_i \cap V - V_j$ are not empty sets, for some $j = 1, 2, 3$. Hence, M_i is not independent set, a contradiction. Then the claim is true. Now, assume, without loss the generality, that $n_1 < n_2$. Then, there exists at least a vertex $v \in M_3$ such that $d_{M_2}(v) = |M_2| > |M_1| = d_{M_1}(v)$. Hence, either v of odd degree, a contradiction to Corollary 2.2, or a set M_1 is not a monopoly set of G, once again a contradiction to assumption. This complete a proof. □

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Theorem 4.3. *For any graph G of order n, if $\mu(G) = 3$, then*

$$n \leq m \leq \frac{n^2}{3}.$$

Proof. Let G be a graph with $\mu(G) = 3$ and let $\{M_1, M_2, M_3\}$ be the M-partition of G. Then by Corollary 2.2, every vertex in G is of even degree that means $\delta \geq 2$. Then the minimum number of edges in G, if G is a cycle graph hence $m \geq n$. For the upper bound, we denote $m(M_1, M_2)$ to the number of edges between M_1 and M_2. Since $m(M_1, M_2) \leq |M_1||M_2|$, it follows that the maximum value of $m(M_1, M_2)$ is $|M_1||M_2|$. Using calculus we can deduce that $m(M_1, M_2)$ is maximal when $|M_1| = |M_2|$ and Theorem 3.3, M_1 is maximal when $|M_1| = \frac{n}{2}$. Then by Corollary 3.1, $\frac{m}{3} = m(M_1, M_2) \leq \frac{n^2}{9}$. Therefore,, $m = \dfrac{n^2}{3}$. □

These bounds in Theorem 4.3 are sharp. The cycle C_n, for $n \equiv 0 \pmod 3$, gives the lower bound and the complete 3-partite $K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}$ gives the upper bound.

Proposition 4.1. *For any graph G, if $\mu(G) = 3$, then $\chi(G) \leq 3$.*

Proof. The result is the consequence of Theorem 3.1. □

The bound in Proposition 4.1, is sharp, the cycle graphs C_{3n}, for every n is odd, and the complete 3-partite graphs K_{n_1,n_2,n_3} attending it. The example of graphs G with $\mu(G) = 3$ and $\chi(G) = 2$ is the graphs $G = C_{3n}$, for every n is even. The converse of the Proposition 4.1, in general, is not true. For example, $\chi(C_5) = 2$ but $\mu(C_5) = 2$.

Corollary 4.2. *For any non-bipartite graph G without isolated vertices. If $\mu(G) = 3$, then $\chi(G) = 3$.*

Theorem 4.4. *Let G be a graph with a clique number $\omega(G)$. If $\mu(G) = 3$, then $\omega(G) \leq 3$.*

Proof. Let G be a graph with $\mu(G) = 3$ and let $\{M_1, M_2, M_3\}$ be the M-partition of G. Suppose, on the contrary, that $\omega(G) \geq 4$. Then there exists a clique $C \subseteq V(G)$ with vertex set $V(C) = \{v_1, v_2, ..., v_k\}$, $k \geq 4$. Hence, by the pigeonhole principle, there is at least on set from M-partition of G contains at least $\lfloor \frac{k-1}{3} \rfloor + 1$ vertices from $V(C)$. Since, $k \geq 4$ then $\lfloor \frac{k-1}{3} \rfloor + 1 \geq 2$. Hence, there is at least one set form M-partition of G is not independent, a contradiction to Theorem 3.1. Therefore, $\omega(G) \leq 3$. □

The converse of Theorem 4.4, in general, is not true. For example, the Path graph P_n with $\omega(P_n) = 2$, but $\mu(P_n) = 2$.

Theorem 4.5. *For any graph G of order n, if $\mu(G) = 3$, then $\alpha(G) \geq \lceil \frac{n}{3} \rceil$.*

Proof. Let G be a graph of order n and $\mu(G) = 3$ and let $\{M_1, M_2, M_3\}$ be the M-partition of G. Then by the pigeonhole principle, there is at least one set from M-partitions of G contains at least $\lfloor \frac{n-1}{3} \rfloor + 1$ vertices from $V(G)$. Since, by Theorem 3.1, every set in M-partitions of G is an independent set, it follows that $\alpha(G) \geq \lfloor \frac{n-1}{3} \rfloor + 1 = \lceil \frac{n}{3} \rceil$. □

Corollary 4.3. *For any graph G of order n, if $\mu(G) = 3$, then the independence monopoly size, $imo(G)$, of G is defined. Furthermore, $imo(G) \leq \lceil \frac{n}{3} \rceil$.*

The bound in Corollary 4.3, is sharp. The cycle graphs C_{3n}, for every n, is attending it. The converse of Corollary 4.3, in general, is not true. For example, the star graph $K_{1,n}$ has $imo(K_{1,n}) = 1$ but $\mu(K_{1,n}) = 2$. For more details in the independence monopoly size of a graph, we refer the reader to [15].

Theorem 4.6. *Let G be a graph of order n and maximum degree $\Delta(G) = n - 1$. Then $\mu(G) = 3$, if and only if $G = K_3$ or $G \cong F_n$.*

Proof. Certainly, if $G = K_3$ or $G = F_n$, then $\Delta(G) = n - 1$ and $\mu(G) = 3$.
Conversely, Let G be a graph of order n, maximum degree $\Delta(G) = n-1$ and $\mu(G) = 3$ and let $\{M_1, M_2, M_3\}$ be the M-partition of G. Now, let a vertex $v \in V(G)$ with $d(v) = n - 1$ and assume, with loss of generality, that $v \in M_1$. Then by Theorem 3.1, $M_1 = \{v\}$ and by Observation 3.1, $|N(v) \cap M_2| = |N(v) \cap M_3| = \frac{n-1}{2}$.
On the other hand, once again by the Observation 3.1, $|N(u) \cap M_1| = |N(u) \cap M_3| = 1$, for every $u \in M_2$. Hence, $d(u) = 2$ for every $u \in M_2$. Similarly, $d(w) = 2$, for every $w \in M_2$.

Hence, a graph G has only a vertex v with $d(v) = n-1$ and each other vertex with degree two. Therefore, If $n = 3$, then $G = K_3$ and if $n \geq 4$, then $G = F_{\frac{n-1}{2}}$. □

Theorem 4.7. *Let G be a graph having $\mu(G) = 3$. Then $mo(G) \leq \frac{n}{3}$.*

Proof. Let G be a graph with $\mu(G) = 3$ and let $\{M_1, M_2, M_3\}$ be the M-partition of G. Since $|M_i| \geq mo(G)$, for every $i \in \{1, 2, 3\}$, it follows that $n = |M_1| + |M_2| + |M_3| \geq 3mo(G)$. Therefore, $mo(G) \leq \frac{n}{3}$. □

This bound is sharp, The cycle graphs C_n, for every $n \equiv 0 \pmod 3$, and a complete 3-partite $K_{\frac{n}{3},\frac{n}{3},\frac{n}{3}}$, attending it.

Corollary 4.4. *For any graph G, $\mu(G) \leq \frac{n}{mo(G)}$.*

It is clear that every graph G of order $n \leq 4$, $G \neq K_3$ has $\mu(G) \leq 2$. In the following result, we study the existences graph G of order $n = k$ having $\mu(G) = 3$ for every positive integer number $k \notin \{1, 2, 4\}$.

Theorem 4.8. *For every positive integer $k \notin \{1, 2, 4\}$, there exists a graph G of order $n = k$ having $\mu(G) = 3$.*

Proof. For $k = 3$ and 5, the result is true, since $G_1 = K_3$ and $G_2 = F_2$ have the required property. Now, we may assume that $k \geq 6$. Then we consider the following cases.

Case 1: If $k \equiv 0 \pmod 3$, then the cycle graph $G_3 = C_k$ is holding the property, since $\mu(C_k) = 3$.

Case 2: If $k \equiv 1 \pmod 3$, let $v_1, v_2, ..., v_k$ be the vertex set of the cycle C_k. Then the graph G_4 which formed from C_k by firstly, removed the edge e_{k-1} which join the vertices v_{k-1} with v_k, then insert three new edges e'_1, e'_2 and e'_3, such that e'_1 join v_1 with v_{k-1}, e'_2 join v_1 with v_{k-2} and e'_3 join v_{k-2} with v_k. Figure 2, shows the graph G_4.

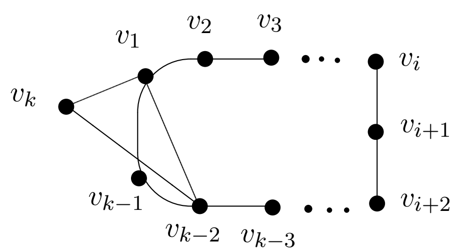

Figure 2: The graph G_4.

Then the partition $\{M_1, M_2, M_3\}$ where $M_1 = \{v_i : i \equiv 1 \pmod 3\} - \{v_k\}$, $M_2 = \{v_i : i \equiv 2 \pmod 3\}$ and $M_3 = \{v_i : i \equiv 0 \pmod 3\} \cup \{v_k\}$ is M-partition of G_4. Indeed, every partite set M_i for $i = 1, 2, 3$ is an independent monopoly set in G_4. Therefore, $\mu(G_4) = 3$.

Case 3: If $k \equiv 2 \pmod 3$, Then the graph G_5 which formed from the cycle C_k by removed the edge e which join the vertices v_{k-2} with v_{k-1} and then insert two new edges e'_1 join v_1 with v_{k-1} and e'_2 join v_1 with v_{k-2}. Figure 3, shows the graph G_5.

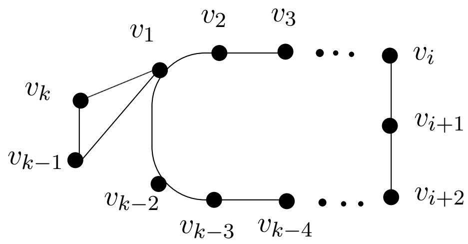

Figure 3: The graph G_5.

Then the partition $\{M_1, M_2, M_3\}$ where $M_1 = \{v_i : i \equiv 1 \pmod 3\} - \{v_{k-1}\}$, $M_2 = \{v_i : i \equiv 2 \pmod 3\}$ and $M_3 = \{v_i : i \equiv 0 \pmod 3\} \cup \{v_{k-1}\}$ is M-partition of G_5. Indeed, every partite set M_i for $i = 1, 2, 3$ is an independent monopoly set in G_5. Therefore, $\mu(G_5) = 3$.

□

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