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RESIDUAL CLOSENESS FOR HELM AND SUNFLOWER GRAPHS

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Abstract. Vulnerability is an important concept in network analysis related with the ability of the network to avoid intentional attacks or disruption when a failure is produced in some of its components. Often enough, the network is modeled as an undirected and unweighted graph in which vertices represent the processing elements and edges represent the communication channel between them. Different measures for graph vulnerability have been introduced so far to study different aspects of the graph behavior after removal of vertices or links such as connectivity, toughness, scattering number, binding number and integrity. In this paper, we consider residual closeness which is a new characteristic for graph vulnerability. Residual closeness is a more sensitive vulnerability measure than the other measures of vulnerability. We obtain exact values for closeness, vertex residual closeness (VRC) and normalized vertex residual closeness (NVRC) for some wheel related graphs namely helm and sunflower.

Keywords: network vulnerability, closeness, network design and communication, stability, communication network, Helm graph; Sunflower graph.

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1. Introduction

Networks are used for modeling different systems such as chemical systems, neural networks, social systems or the Internet and the World Wide Web. The stability of a communication network, composed of processing nodes and communication links, is of prime importance to network designers. The vulnerability of a communication network measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. Communication networks must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network. If we think of a graph as modeling a network, there have been several proposals for measures of the stability of a communication network including connectivity, toughness, scattering number, binding number and integrity [3, 4, 10, 14].

The concept of residual closeness is introduced as a measure of graph vulnerability by Chavdar Dangalchev [6]. The vulnerability of a network can be measured by the residual closeness of the graph describing the network. The aim of residual closeness is to measure the vulnerability even when the removal of the vertices do not disconnect the graph while other parameters except binding number measure vulnerability so the resulting graph is

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disconnected. Consider two different graph having the same graph vulnerability characteristics: their connectivity, toughness, integrity, binding number and scattering number is equal. In such a case, residual closeness recognizes the difference between these two graphs. As a measure for the graph vulnerability, the need and advantages of residual closeness is explained in [6], and examples are given to show that the residual closeness can reflect the vulnerability of graphs better than or independent of the other parameters in existing literature. Clearly, this parameter is of particular interest because it is considered to be a reasonable measure for the vulnerability of graphs and can be studied as a useful parameter.

In this paper, we consider simple finite undirected graphs without loops and multiple edges. Let $G = (V, E)$ be a graph with a vertex set $V = V(G)$ and an edge set $E = E(G)$. For vertices u and i of a graph G, the open neighborhood of u is $N(u) = \{v \in V(G)|(u, v) \in E(G)\}$ and $N_i(u) = \{v \in V(G\backslash i)|(u, v) \in E(G\backslash i)\}$. We define analogously for any $S \subseteq V(G)$ the open neighborhood $N(S) = \cup_{u\in S} N(u)$ and $S \subseteq V(G\backslash i)$ the open neighborhood $N_i(S) = \cup_{u\in S} N(u)$. The distance $d(u, v)$ between two vertices u and v in is the length of a shortest path (or geodesic) between them. If u and v are not connected, then $d(u, v) = \infty$, and for $u = v$, $d(u, v) = 0$. The eccentricity of a vertex v in G is the distance from v to a vertex farthest away from v in G, denoted by $e(v)$. The diameter of G, denoted by $diam(G)$, is the largest distance between two vertices in $V(G)$. The degree $deg_G(v)$ of a vertex $v \in V(G)$ is the number of edges incident to v. A vertex of degree one is called a pendant vertex, and its neighbor is called a support vertex [13, 7]. We shall use $\lfloor x \rfloor$ for the largest integer not larger than x.

Our aim in this paper is to consider the computing the closeness, vertex residual closeness (VRC) and the normalized vertex residual closeness (NVRC) of some wheel related networks. In section 2 and 3, definitions and known results for closeness, VRC and NVRC are given, respectively. In section 4 and 5, closeness, VRC and NVRC of helm and sunflower are, respectively, determined and exact values are given. Conclusions are addressed in Section 6.

2. Closeness, Residual Closeness and Normalized Residual Closeness

The closeness, VRC and NVRC of a graph are a new characteristic for graph vulnerability introduced in [6]. Their definitions are in the following:

- The closeness of a graph is defined as $C = \sum\limits_{i} C(i)$, where $C(i)$ is the closeness of a vertex i, and defined as $C = \sum\limits_{j\neq i} \frac{1}{2^{d(i,j)}}$. We can also use this definition for not connected graphs.

  Let $d_k(i, j)$ be the distance between vertices i and j in the graph, received from the original graph where all links of vertex k are deleted. Then the closeness after removing vertex k is defined as $C_k = \sum\limits_{i} \sum\limits_{j\neq i} \frac{1}{2^{d_k(i,j)}}$. This definition can also be used for disconnected graphs. For a connected graph G, the polynomial $H(G; x)$ is the Hosoya (or Wiener) polynomial defined as $H(G; \lambda) = \sum\limits_{k\geq 1} d(G, k)\lambda^k$ where $d(G, k)$ is the number of vertex pairs at distance k and introduced in [9]. Clearly, the closeness of a connected graph can be derived in terms of the Hosoya polynomial as $C = \sum\limits_{i} C(i) = \sum\limits_{i} \sum\limits_{j\neq i} 2^{-d(i,j)} = 2H(G; 1/2)$ [1, 2, 5, 6, 11]

- The VRC of the graph is defined as $R = \min_k \{C_k\}$.
- The NVRC of the graph is defined as dividing the residual closeness by the closeness C; $R' = R/C$.

3. Basic Results

Theorem 3.1. [1, 6] *The closeness of*

(a) *the complete graph K_n with n vertices is $C(K_n) = (n(n-1))/2$;*
(b) *the star graph S_n with n vertices is $C(S_n) = \frac{(n-1)(n+2)}{4}$;*
(c) *the path P_n with n vertices is $C(P_n) = 2n - 4 + \frac{1}{2^{n-2}}$;*
(d) *the cycle C_n with n vertices is* $C(C_n) = \begin{cases} 2n(1 - 1/2^{(n-1)/2}), & \text{if } n \text{ is odd,} \\ n(2 - 3/2^{n/2}), & \text{if } n \text{ is even.} \end{cases}$

Theorem 3.2. [1, 6, 11] *The VRC of*

(a) *the complete graph K_n with n vertices is $R(K_n) = ((n-1)(n-2))/2$;*
(b) *the star graph S_n with n vertices is $R(S_n) = 0$;*
(c) *the cycle C_n with n vertices is $R(C_n) = 2n - 6 + 1/2^{n-3}$.*

Theorem 3.3. [6] *For a graph G, $0 \leq R'(G) < 1$.*

Theorem 3.4. [6] *If H is a proper subgraph of graph G, then $R(H) < R(G)$.*

Theorem 3.5. [5] *If a vertex k does not belong to any unique geodesic (shortest path) of graph G, then $C(G \backslash k) = C(G) - 2C(k)$.*

Corollary 3.1. [2] *Let G be a graph. Then, for an endvertex u of G, $C_u(G) = C(G) - 2C(u)$.*

Corollary 3.2. [2] *If a vertex v has eccentricity two in G, then $C(v) = (|V(G)| + deg(v) - 1)/4$.*

Lemma 3.1. [12] *For any two graphs G_1 and G_2,*

$$H(G_1 \circ G_2) = (1 + |G_2|x)^2 H(G_1) + |G_1|(\binom{|G_2|}{2}) - |E(G_2)|)x^2 + |G_1|(|G_2| + |E(G_2)|)x.$$

Theorem 3.6. [2] *Let G be a graph and $\{u, v\} \in V(G)$. If u is an endvertex of the support vertex v in G, then $C_v(u) = 0$.*

4. Residual Closeness of Helm

Helm H_n is a graph of order $2n + 1$ obtained from a wheel W_n with cycle C_n having a pendant edge attached to each vertex of the cycle. Helm H_n consists of the vertex set $V(H_n) = \{v_i | 0 \leq i \leq n-1\} \cup \{a_i | 0 \leq i \leq n-1\} \cup \{c\}$ and edge set $E(H_n) = \{v_i v_{i+1} | 0 \leq i \leq n-1\} \cup \{v_i a_i | 0 \leq i \leq n-1\} \cup \{v_i c | 0 \leq i \leq n-1\}$, where $i + 1$ is taken modulo n [8].

Let c be the central vertex of H_n. The central vertex c has a vertex degree of n. The vertices of $H_n \backslash \{c\}$ are of two kinds: vertices of degree four and one, respectively. The vertices of degree one will be referred to as pendant vertices and vertices of degree four to as support vertices [9].

Theorem 4.1. *If H_n is a helm, then the closeness for the helm H_n with $2n + 1$ vertices is*

$$C(H_n) = \frac{n(9n + 49)}{16}.$$

Proof. We have three cases depending on the vertices of H_n:

Case 1. Let c be the central vertex of H_n. Then, c is adjacent to all support vertices and pendant vertices are at distance 2 from c. Thus, $deg(c) = n$ and $e(c) = 2$. By Corollary 3.2, the closeness of c is

$$C(c) = \frac{(2n+1) - 1 + n}{4} = \frac{3n}{4}.$$

Case 2. Let v_i be a major vertex. Then, v_i is adjacent to one of the pendant vertices, the central vertex c, and two support vertices. Since $d(v_i, c) = 1$, other remaining $n-3$ support vertices and $n-3$ pendant vertices attached to support vertices are at distance 2 and 3, respectively, from v_i. If the distance between v_i and two support vertices is 1, then two pendant vertices attached to two support vertices are at distance 2 from i. Thus,

$$C(v_i) = 4(\frac{1}{2^1}) + (n-3+2)(\frac{1}{2^2}) + (n-3)(\frac{1}{2^3}) = \frac{3n+11}{8}.$$

Case 3. Let a_i be a minor vertex. Since a_i is only adjacent to a major vertex v_i, by Case 2 of this theorem, the closeness of a minor vertex a_i is

$$C(a_i) = 1(\frac{1}{2^1}) + 3(\frac{1}{2^{1+1}}) + (n-3+2)(\frac{1}{2^{2+1}}) + (n-3)(\frac{1}{2^{3+1}}) = \frac{3n+15}{16}. \tag{1}$$

By Case 1, 2 and 3, the closeness of helm is $C(H_n) = C(c) + \sum_{i=0}^{n-1} C(v_i) + \sum_{i=0}^{n-1} C(a_i)$

$$C(H_n) = \frac{3n}{4} + n(\frac{3n+11}{8}) + n(\frac{3n+15}{16}) = \frac{n(9n+49)}{16}.$$

The proof is completed. □

Theorem 4.2. *If H_n is a helm with $2n+1$ vertices, then the VRC of the helm is*

$$R(H_n) = \begin{cases} \begin{cases} \frac{n}{2}(11 - \frac{9}{2^{\frac{n-1}{2}}}), & \text{if } n \text{ is odd;} \\ \frac{n}{2}(11 - \frac{27}{2^{\frac{n}{2}+1}}), & \text{if } n \text{ is even;} \end{cases} & \text{if } n > 5; \\ \frac{9n^2+31n-58}{16}, & \text{if } n \leq 5. \end{cases}$$

Proof. We have three cases depending on the vertices of H_n:

Case 1. Removing the central vertex c of H_n:

If c is removed from H_n, then the remaining graph is $G_1 = C_n \circ K_1$. G_1 is a thorn graph C_n^* of the graph C_n, with parameters $p_1 = p_2 = ... = p_n = 1$ [7]. The thorn graph C_n^* is obtained by attaching a degree-one vertex to the every vertex of C_n. By the definition of closeness, we have

$$C_c = C(G_1) = C(C_n \circ K_1) = 2H(C_n \circ K_1; \tfrac{1}{2}).$$

By Lemma 3.1, we obtain

$$C_c = 2((1 + \tfrac{1}{2})^2 2H(C_n; \tfrac{1}{2}) + n(\tfrac{1}{2})).$$

By the definition of closeness and Theorem 3.1(d), we have

$$C_c = \begin{cases} \frac{n}{2}(11 - \frac{9}{2^{\frac{n-1}{2}}}), & \text{if } n \text{ is odd;} \\ \frac{n}{2}(11 - \frac{27}{2^{\frac{n}{2}+1}}), & \text{if } n \text{ is even.} \end{cases}$$

Case 2. Removing a support vertex v_i of H_n:
If v_i is removed from H_n, then the remaining graph is $G_2 = H_n \backslash \{v_i\}$. We have six subcases depending on the vertices of G_2:

Subcase 1. Let c be central vertex of H_n:
Since the vertex c is adjacent to $n-1$ support vertices, the $n-1$ minor vertices are at distance 2 from c. There is not any path between the c and minor vertices a_i which are adjacent to removed v_i, that is $d(c, a_i) = \infty$. Hence, we have

$$C_{G_2}(c) = (n-1)(\frac{1}{2^1}) + (n-1)(\frac{1}{2^2}) + 1(\frac{1}{2^\infty}) = \frac{3n-3}{4}. \tag{2}$$

Subcase 2. Let x be support vertex in G_2 and $deg(x) = 3$. Then x is adjacent to three vertices, $n-2$ vertices and $n-3$ vertices are at distance 2 and 3, respectively, from x. There is not any path between x and minor vertices a_i which are adjacent to v_i in H_n. Thus, we get

$$C_{G_2}(x) = (3)(\frac{1}{2^1}) + (n-2)(\frac{1}{2^2}) + (n-3)(\frac{1}{2^3}) + (1)(\frac{1}{2^\infty})$$

$$C_{G_2}(x) = \frac{3n+5}{8}. \tag{3}$$

Subcase 3. Let y be pendant vertex which is adjacent to any vertex x shuch that $deg(x) = 3$ in G_2. By Subcase 2 of this theorem, we have

$$C_{G_2}(y) = (1)(\frac{1}{2^1}) + (2)(\frac{1}{2^2}) + (n-2)(\frac{1}{2^3}) + (n-3)(\frac{1}{2^4}) + (1)(\frac{1}{2^\infty})$$

$$C_{G_2}(y) = \frac{3n+9}{16}. \tag{4}$$

Subcase 4. Let z be support vertex in G_2 and $deg(x) = 4$. Then z is adjacent to four vertices. $n-2$ vertices and $n-4$ pendant vertices are at distance 2 and 3, respectively, from z. There is not any path between z and pendant vertices a_i which are adjacent to v_i in H_n. Thus, we get

$$C_{G_2}(z) = (4)(\frac{1}{2^1}) + (n-2)(\frac{1}{2^2}) + (n-4)(\frac{1}{2^3}) + (1)(\frac{1}{2^\infty})$$

$$C_{G_2}(z) = \frac{3n+8}{8}. \tag{5}$$

Subcase 5. Let w be pendant vertex which is adjacent to any vertex z shuch that $deg(x) = 4$ in G_2. By Subcase 4 of this theorem, we have

$$C_{G_2}(w) = (1)(\frac{1}{2^1}) + (3)(\frac{1}{2^2}) + (n-2)(\frac{1}{2^3}) + (n-4)(\frac{1}{2^4}) + (1)(\frac{1}{2^\infty})$$

$$C_{G_2}(w) = \frac{3n+12}{16}. \tag{6}$$

Subcase 6. Let a_i be pendant vertex which is adjacent to v_i in H_n. Thus, by Theorem 3.6, we have

$$C(G_2)(a_i) = 0. \tag{7}$$

By summing up (2), (3), (4), (5), (6) and (7), we obtain

$$C(v_i) = C(G_2)$$
$$C(v_i) = C_{G_2}(c) + 2C_{G_2}(x) + 2C_{G_2}(y) + (n-3)C_{G_2}(z) + (n-3)C_{G_2}(w) + C_{G_2}(a_i)$$
$$C_{v_i} = \tfrac{9n^2+31n-58}{16}.$$

Case 3. Removing a pendant vertex a_i of H_n:
Let a_i be a pendant vertex. If a pendant vertex a_i is removed from H_n, then the remaining graph is $G_3 = H_n\backslash a_i$. Hence , by Corollary 3.1, the closeness of subgraph G_3 is

$$C_{a_i} = C(G_3) = C(H_n) - 2C_{H_n}(a_i).$$

By Theorem 4.1 and (1), we have

$$C_{a_i} = \frac{n(9n+49)}{16} - 2(\frac{3n+5}{16}) = \frac{9n^2+43n-30}{16}.$$

Consequently, let us show how to deduce $min\{C_c, c_{v_i}, C_{a_i}\}$:
It is easy to see that for $n \geq 3$, $C_{v_i} < C_{a_i}$.
If n is odd, then $C_c = \frac{n}{2}(11 - \frac{9}{2^{\frac{n-1}{2}}})$.
Assume that $C_c = \frac{n}{2}(11 - \frac{9}{2^{\frac{n-1}{2}}}) < \frac{9n^2+31n-58}{16} = C_{v_i}$.
Then, we obtain $\frac{-9n}{2^{\frac{n-1}{2}}} < \frac{9n^2-57n-58}{8}$.
Since n is integer-valued and positive, it is evident that this leads a contradiction for $n \geq 8$. If n is even, the proof is similar to the case when n is odd and is omitted. Moreover the values for $3 \leq n < 8$ are in the following Table 1.

As seen in Table1 above, $R(H_n) = min\{C_c, C_{v_i}, C_{a_i}\} = \begin{cases} C_c, & \text{if } n > 5; \\ C_{v_i}, & \text{if } n \leq 5. \end{cases}$

TABLE 1. The values for $3 \leq n < 8$

n	3	4	5	6	7
C_c	9.75	15.25	21.875	27.9375	34.5625
C_c	7.25	13.125	20.125	28.25	37.5

Thus the proof of Theorem 4.2 is completed.

□

Corollary 4.1. *If H_n is a helm with $2n+1$ vertices, then the NVRC of the helm is*

$$R'(H_n) = \begin{cases} \begin{cases} \frac{8(11-\frac{9}{2^{\frac{n-1}{2}}})}{9n+49}, & if\ n\ is\ odd; \\ \frac{8(11-\frac{27}{2^{\frac{n}{2}+1}})}{9n+49}, & if\ n\ is\ even; \end{cases} & if\ n > 5; \\ 1 - \frac{18n+58}{9n^2+49}, & if\ n \leq 5. \end{cases}$$

5. RESIDUAL CLOSENESS OF SUNFLOWER

Sunflower graph SF_n consists of a wheel with central vertex c and an n-cycle $v_0, v_1, v_2, ..., v_{n-1}$ and additional n vertices $w_0, w_1, w_2, ..., w_{n-1}$ where w_i is joined by edges to (v_i, v_{i+1}) for $i = 0, 1, 2, ...n-1$ where $i+1$ is taken modulo n. SF_n has order $2n+1$ and size $4n$ [8]. Let c be the central vertex of SF_n. The central vertex c has a vertex degree of n. The vertices of $SF\backslash\{c\}$ are of two kinds: vertices of degree five and two, respectively. The vertices of degree two will be referred to as minor vertices and vertices of degree five to as major vertices [9].

Theorem 5.1. *If SF_n $n > 4$ is a sunflower graph, then the closeness for the sunflower graph SF_n with $2n+1$ vertices is*

$$C(SF_n) = \frac{n(9n+67)}{16}.$$

Proof. We have three cases depending on the vertices of SF_n:

Case 1. For the central vertex c of SF_n, as being adjacent to all the support vertices, $|N(c)| = n$. Since a minor vertex w_i is adjacent to exactly two support vertices, $d(c, w_i) = 2$. Thus, $deg(c) = n$ and $e(c) = 2$. By Corollary 3.2, the closeness of c is $C(c) = \frac{(2n+1)-1+n}{4} = \frac{3n}{4}$.

Case 2. Let v_i be a major vertex of SF_n. Major vertex v_i is exactly adjacent to five vertices: two major vertices, two minor vertices and the central vertex c. Since $d(v_i, c) = 1$, other remaining $n-3$ major vertices are at distance 2 from v_i. Moreover two minor vertices are joined by edges to two major vertices which v_i is adjacent to. Thus, these minor vertices are also at distance 2 from v_i . Consequently, there remain $n-4$ minor vertices joined by edges to major vertices which are at distance 3 from v_i. Hence, we have $C(v_i) = 5(\frac{1}{2^1}) + (n-1)\frac{1}{2^2} + (n-4)(\frac{1}{2^3}) = \frac{3n+14}{8}$.

Case 3. Let w_i be a minor vertex of SF_n. Since w_i is joined by edges to two major vertices, $|N(w_i)| = 2$. Then, w_i is at distance 2 to five vertices: two major vertices, two minor vertices and the central vertex c. If $d(w_i, c) = 2$, then other remaining $n-4$ major vertices and 2 minor vertices are at distance 3 from w_i. Thus, remaining $n-5$ minor vertices are at distance 4 from w_i. Hence, we have

$$C(w_i) = 2(\frac{1}{2^1}) + (5)\frac{1}{2^2} + (n-2)(\frac{1}{2^3}) + (n-5)(\frac{1}{2^4}) = \frac{3n+27}{16}. \quad (8)$$

Thus, by Case 1, 2 and 3, the closeness of SF_n is

$$C(SF_n) = C(c) + \sum_{i=0}^{n-1} C(v_i) + \sum_{i=0}^{n-1} C(w_i)$$

$$C(SF_n) = \frac{3n}{4} + n(\frac{3n+14}{8}) + n(\frac{3n+27}{18}) = \frac{n(9n+67)}{16}.$$

The proof is completed.

□

Theorem 5.2. *If SF_n $n > 4$ is a sunflower graph with $2n+1$ vertices, then the VRC of a sunflower graph is*

$$R(SF_n) = \begin{cases} \frac{9n^2+55n-74}{14}, & \text{if } n = 5, \\ \begin{cases} n(7 - \frac{17}{2^{\frac{n}{2}+1}}), & \text{if } n \text{ is even;} \\ n(7 - \frac{12}{2^{\frac{n+1}{2}}}), & \text{if } n \text{ is odd;} \end{cases} & \text{if } n > 5. \end{cases}$$

Proof. We have three cases depending on the vertices of:

*Case 1.*Removing the central vertex c of SF_n:

If c is removed from SF_n, then the survival subgraph is $G_1 = SF_n \backslash \{c\}$. The closeness of G_1 is calculated in a similar manner to C_n . We have two cases depending on the vertices of G_1;

Subcase 1. Let v_i be a vertex which is $deg(v_i) = 4$ of G_1. The closeness of v_i is

$$C_{G_1}(v_i) = \begin{cases} (\sum_{j=1}^{\frac{n}{2}-1} 4(\frac{1}{2^j})) + 3(\frac{1}{2^{\frac{n}{2}}}), & \text{if } n \text{ is even} \\ (\sum_{j=1}^{\frac{n-1}{2}} 4(\frac{1}{2^j})) + \frac{1}{2^{\frac{n+1}{2}}}, & \text{if } n \text{ is odd} \end{cases}$$

Subcase 2. Let w_i be a vertex which is $deg(w_i) = 2$of G_1. The closeness of w_i is

$$C_{G_1}(w_i) = \begin{cases} (2)\frac{1}{2^1} + (\sum_{j=2}^{\frac{n}{2}} 4(\frac{1}{2^j})) + 3(\frac{1}{2^{\frac{n}{2}+1}}), & \text{if } n \text{ is even} \\ (2)\frac{1}{2^1} + (\sum_{j=2}^{\frac{n-1}{2}} 4(\frac{1}{2^j})) + \frac{1}{2^{\frac{n+1}{2}}}, & \text{if } n \text{ is odd} \end{cases}$$

By summing up Subcase 1 and Subcase 2, the closeness of G_1 is

$$C_c = C(G_1)\sum_{i=0}^{n-1} C_{G_1}(v_i) + \sum_{i=0}^{n-1} C_{G_1}(w_i)$$

$$C_c = n(C_{G_1}(v_i)) + n(C_{G_1}(w_i))$$

$$C_c = \begin{cases} n(((\sum_{j=1}^{\frac{n}{2}-1} 4(\frac{1}{2^j})) + 3(\frac{1}{2^{\frac{n}{2}}})) + (2(\frac{1}{2^1}) + (\sum_{j=2}^{\frac{n}{2}} 4(\frac{1}{2^j})) + \frac{1}{2^{\frac{n}{2}+1}})), & \text{if } n \text{ is even} \\ n(((\sum_{j=1}^{\frac{n-1}{2}} 4(\frac{1}{2^j})) + \frac{1}{2^{\frac{n+1}{2}}}) + (2(\frac{1}{2^1}) + (\sum_{j=2}^{\frac{n-1}{2}} 4(\frac{1}{2^j})) + 3(\frac{1}{2^{\frac{n+1}{2}}}))), & \text{if } n \text{ is odd} \end{cases}$$

$$C_c = \begin{cases} 2(n(\sum_{j=1}^{\frac{n}{2}-1} (2)\frac{1}{2^j}) + \frac{1}{2^{\frac{n}{2}}}) + \frac{n}{2^{\frac{n}{2}}} + n(1 + 4(\sum_{j=2}^{\frac{n}{2}} \frac{1}{2^j}) + \frac{1}{2^{\frac{n}{2}+1}}), & \text{if } n \text{ is even} \\ 2(n(\sum_{j=1}^{\frac{n-1}{2}} (2)\frac{1}{2^j})) + \frac{n}{2^{\frac{n+1}{2}}} + n(1 + 4(\sum_{j=2}^{\frac{n-1}{2}} \frac{1}{2^j}) + \frac{3}{2^{\frac{n+1}{2}}}), & \text{if } n \text{ is odd} \end{cases}$$

By Theorem 3.1(d), we have

$$C_c = \begin{cases} 2C(C_n) + \frac{n}{2^{\frac{n}{2}}} + n(1 + 4(\sum_{j=2}^{\frac{n}{2}} \frac{1}{2^j}) + \frac{1}{2^{\frac{n}{2}+1}}), & \text{if } n \text{ is even} \\ 2C(C_n) + \frac{n}{2^{\frac{n+1}{2}}} + n(1 + 4(\sum_{j=2}^{\frac{n-1}{2}} \frac{1}{2^j}) + \frac{3}{2^{\frac{n+1}{2}}}), & \text{if } n \text{ is odd} \end{cases}$$

To calculate the above sums, we start from the equation $1 + X + X^2 + ... + X^{n-1} = \frac{X^n - 1}{X-1}$. We have,

$$C_c = \begin{cases} 2C(C_n) + \frac{n}{2^{\frac{n}{2}}} + n(1 + (4)\frac{1}{2^2}(1 + \frac{1}{2^1} + ... + \frac{1}{2^{\frac{n}{2}-2}}) + \frac{1}{2^{\frac{n}{2}+1}}), & \text{if } n \text{ is even} \\ 2C(C_n) + \frac{n}{2^{\frac{n+1}{2}}} + n(1 + (4)\frac{1}{2^2}(1 + \frac{1}{2^1} + ... + \frac{1}{2^{\frac{n-1}{2}-2}}) + \frac{3}{2^{\frac{n+1}{2}}}), & \text{if } n \text{ is odd} \end{cases}$$

$$C_c = \begin{cases} 2C(C_n) + \frac{n}{2^{\frac{n}{2}}} + n(1 + (4)\frac{1}{2}(1 - \frac{1}{2^{\frac{n}{2}-1}}) + \frac{1}{2^{\frac{n}{2}+1}}), & \text{if } n \text{ is even} \\ 2C(C_n) + \frac{n}{2^{\frac{n+1}{2}}} + n(1 + (4)\frac{1}{2}(1 - \frac{1}{2^{\frac{n-3}{2}}}) + \frac{3}{2^{\frac{n+1}{2}}}), & \text{if } n \text{ is odd} \end{cases}$$

$$C_c = \begin{cases} 2C(C_n) + 3n - \frac{5n}{2^{\frac{n}{2}+1}}, & \text{if } n \text{ is even} \\ 2C(C_n) + 3n - \frac{4n}{2^{\frac{n+1}{2}}}, & \text{if } n \text{ is odd} \end{cases}$$

By Theorem 3.1(d),we have

$$C_c = \begin{cases} n(7 - \frac{17}{2^{\frac{n}{2}+1}}), & \text{if } n \text{ is even} \\ n(7 - \frac{12}{2^{\frac{n+1}{2}}}), & \text{if } n \text{ is odd} \end{cases}$$

Case 2. Let v_i be vertex which is $deg(v_i) = 5$ of SF_n except central vertex c. Removing vertex v_i of SF_n, then the survival subgraph is $G_2 = SF_n \backslash \{v_i\}$. There are six subcases depending on the vertices of G_2:

Subcase 1. Let c be a central vertex of SF_n. c is adjacent to $n-1$ vertices in G_2. n vertices are at distance 2 from c in G_2. Thus, $deg_{G_2}(c) = n-1$ and $e_{G_2}(c) = 2$. By Corollary 3.2, the closeness of c in G_2 is

$$C_{G_2}(c) = \frac{(2n+1-1)-1+(n-1)}{4} = \frac{3n-2}{4}. \tag{9}$$

Subcase 2. Let x be a vertex of SF_n except central vertex c which is adjacent to v_i and the degree of x is 5 in SF_n. There are two vertices like this. Since x is adjacent to 4 vertices in G_2, $|N_{G_2}(x)| = 4$, remaining $n-2$ major vertices and $n-3$ minor vertices are at distance 2 and 3, respectively from x in G_2. Thus, the closeness of x in G_2 is

$$C_{G_2}(x) = (4)\frac{1}{2^1} + (n-2)\frac{1}{2^2} + (n-3)\frac{1}{2^3} = \frac{3n+9}{8}. \tag{10}$$

Subcase 3. Let y be a vertex of SF_n which is not adjacent to v_i and the degree of y is 5 in SF_n. There are $n-3$ vertices like this. Since y is adjacent to 5 vertices in G_2, $|N_{G_2}(y)| = 5$, remaining $n-2$ major vertices and $n-4$ minor vertices are at distance 2 and 3, respectively from y in G_2. Thus, the closeness of y in G_2 is

$$C_{G_2}(y) = (5)\frac{1}{2^1} + (n-2)\frac{1}{2^2} + (n-4)\frac{1}{2^3} = \frac{3n+12}{8}. \tag{11}$$

Subcase 4. Let z be a minor vertex which is adjacent to v_i in SF_n. There are two vertices like this. z is adjacent to one vertex in G_2. Hence, z is at distance 2 to three vertices: a major vertex, a minor vertex and the central vertex c. It is easily to see that remaining $n-2$ major vertices and $n-3$ minor vertices are at distance 3 and 4, respectively, from z. Thus, the closeness of z in G_2 is

$$C_{G_2}(z) = (1)\frac{1}{2^1} + (3)\frac{1}{2^2} + (n-2)\frac{1}{2^3} + (n-3)\frac{1}{2^4} = \frac{3n+13}{16}. \tag{12}$$

Subcase 5. Let u be a minor vertex which is at distance 2 from v_i in SF_n. There are two vertices like this. u is adjacent to two vertices in G_2. Hence, z is at distance 2 to four vertices. It is easily to see that remaining $n-3$ major vertices and $n-4$ minor vertices are at distance 3 and 4, respectively, from u. Thus, the closeness of u in G_2 is

$$C_{G_2}(u) = (2)\frac{1}{2^1} + (4)\frac{1}{2^2} + (n-3)\frac{1}{2^3} + (n-4)\frac{1}{2^4} = \frac{3n+22}{16}. \tag{13}$$

Subcase 6. Let t be a minor vertex which is at distance 3 from v_i in SF_n. There are $n-4$ vertices like this. t is adjacent to two vertices in G_2. Hence, t is at distance 2 to five vertices. It is easily to see that remaining $n-3$ major vertices and $n-5$ minor vertices are at distance 3 and 4, respectively, from t. Thus, the closeness of t in G_2 is

$$C_{G_2}(t) = (2)\frac{1}{2^1} + (5)\frac{1}{2^2} + (n-3)\frac{1}{2^3} + (n-5)\frac{1}{2^4} = \frac{3n+25}{16}. \tag{14}$$

By summing up (9), (10), (11), (12), (13) and (14), we have

$$C_{v_i} = C(G_2)$$

$$C_{v_i} = C_{G_2}(c) + 2C_{G_2}(x) + (n-3)C_{G_2}(y) + 2C_{G_2}(z) + 2C_{G_2}(u) + (n-4)C_{G_2}(t)$$

$$C_{v_i} = \frac{9n^2+55n-74}{16}.$$

Case 3. Removing a minor vertex w_i of SFn:
If a minor vertex w_i is removed from SF_n, then the remaining graph is $G_3 = SF_n \backslash \{w_i\}$. w_i

do not lie in between any geodesics. Thus, by Theorem 3.5, the closeness of subgraph G_3 is

$$C_{w_i} = C(G_3) = C(SF_n) - 2C_{SF_n}(w_i).$$

By Theorem 5.1 and (8), we have

$$C_{w_i} = \frac{n(9n+67)}{16} - 2(\frac{3n+27}{16}) = \frac{9n^2+61n-54}{16}.$$

Consequently, let us show how to deduce $min\{C_c, C_{v_i}, C_{w_i}\}$:
It is easy to see that $C_{v_i} < C_{w}i$.
If n is odd, then $C_c = n(7 - \frac{12}{2^{\frac{n+1}{2}}})$.
Assume that $C_c = n(7 - \frac{12}{2^{\frac{n+1}{2}}}) < \frac{9n^2-55n-74}{8} = C_{v_i}$.
Then, we obtain
$\frac{-12n}{2^{\frac{n+1}{2}}} < \frac{9n^2-57n-74}{8}$.

Since n is integer-valued and positive, it is evident that this leads a contradiction for $n \geq 8$. If n is even, the proof is similar to the case when n is odd and is omitted. Moreover the values for $4 < n < 8$ are in the following Table 2.

Table 2. The values for $4 < n < 8$

n	5	6	7
C_c	27.75	35.625	43.75
C_{v_i}	26.625	36.25	47

As seen in Table 2 above, $R(SF_n) = min\{C_c, C_{v_i}, C_{w_i}\} = \begin{cases} C_c & \text{if } n > 5, \\ C_{v_i} & \text{if } n = 5. \end{cases}$

Thus the proof of Theorem 5.2 is completed. □

Corollary 5.1. *If SF_n $(n > 4)$ is a sunflower graph with $2n+1$ vertices, then $NVRC$ of a sunflower graph is*

$$R'(SF_n) = \begin{cases} 1 - \frac{12n+74}{n(9n+67)}, & \text{if } n = 5, \\ \begin{cases} \frac{16(7-\frac{17}{2^{\frac{n}{2}+1}})}{9n+67}, & \text{if } n \text{ is even;} \\ \frac{16(7-\frac{12}{2^{\frac{n+1}{2}}})}{9n+67}, & \text{if } n \text{ is odd;} \end{cases} & \text{if } n > 5. \end{cases}$$

Moreover the values of $C(SF_n)$, $R(SF_n)$ and $R'(SF_n)$ for $n \leq 4$ are in the following Table 3.

Table 3. The values of $C(SF_n)$, $R(SF_n)$ and $R'(SF_n)$ for $n \leq 4$

n	3	4
$C(SF_n)$	$\frac{33}{2}$	$\frac{51}{2}$
$R(SF_n)$	$\frac{43}{4}$	$\frac{137}{8}$
$R'(SF_n)$	$\frac{43}{66}$	$\frac{137}{204}$

6. Conclusion

In this paper, we calculate the residual closeness of some wheel related networks. Residual closeness is a new characteristic for graph vulnerability introduced in Ref. [2] and more sensitive than the other known vulnerability measures. Calculation of closeness and residual closeness for simple graphs is important because the closeness and residual closeness of more complex graphs can be calculated (e.g. using formula 3 of [2]) by using closeness and residual closeness of its (simple) parts. It is important the residual closeness to be introduced to CS community. Very good practical results can be achieved if the residual closeness is calculated for some real networks (e.g. the Power grid). This parameter is of particular interest because it is considered to be a reasonable measure for the vulnerability of graphs. The residual closeness is not so closely related to connectivity, degrees or closeness as it seems [2]. The vertex supplying the smallest residual closeness may not be the one maintaining the same connectivity of the graph for example the helm H_n and a sunflower graph SF_n. The vertex supplying the smallest residual closeness may not be the one with the highest degree for example the helm H_5. The vertex with maximal closeness not always has the minimal residual closeness for example the helm H_5 and a sunflower graph SF_5.

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