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LYAPUNOV-SCHMIDT REDUCTION IN THE STUDY OF PERIODIC TRAVELLING WAVE SOLUTIONS OF NONLINEAR DISPERSIVE LONG WAVE EQUATION

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ABSTRACT. This article studies the bifurcation of periodic travelling wave solutions of nonlinear dispersive long wave equation by using Lyapunov-Schmidt reduction. We determined the conditions for the existence of regular solutions for the reduced equation corresponding to the main problem, also we found the linear approximation of the solutions of the main problem.

Keywords: local bifurcation theory, local Lyapunov-Schmidt method, nonlinear dispersive long wave equation.

AMS Subject Classification: 34K18, 93C10.

1. Introduction

Many of the nonlinear problems that appear in Mathematics and Physics can be written in the operator equation form

$$F(x,\lambda) = b, \quad x \in O \subset X, \quad b \in Y, \quad \lambda \in R^n \tag{1}$$

where F is a smooth Fredholm map of index zero and X, Y are Banach spaces and O is open subset of X. For these problems, the method of reduction to finite dimensional equation,

$$\theta(\xi,\lambda) = \tilde{\beta}, \quad \xi \in \tilde{M}, \quad \tilde{\beta} \in \tilde{N}. \tag{2}$$

can be used, where $\tilde{M}$ and $\tilde{N}$ are smooth finite dimensional manifolds. A passage from (1) into (2) (variant local scheme of Lyapunov -Schmidt) with the conditions that equation (2) has all the topological and analytical properties of (1) (multiplicity, bifurcation diagram, etc) can be found in [8]. In the method of finite dimensional reduction (local method of Lyapunov-Schmidt) the solutions of equations in infinite dimensional spaces are in one-to-one corresponding with the solutions of equations in finite dimensional spaces [8]. For this reason the method become an important in the study of many problems arising in nonlinear sciences. In [2] Boiti introduced the following system of nonlinear partial differential equations

$$\begin{aligned} u_{ty} + v_{xx} + \frac{1}{2}(u^2)_{xy} &= 0, \\ v_t + (uv + u + u_{xy})_x &= 0. \end{aligned} \tag{3}$$

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which describe the nonlinear dispersive long wave in (2+1)-dimension. System (3) has interest in fluid dynamics. Also, good understanding of the solutions for system (3) is very helpful to coastal and civil engineers in applying the nonlinear water model to coastal harbor design.There are many studies of system (3) by different ways. In [6] Paquin and Winternitz studied the similarity solutions of system (3) by using symmetry algebra and the classical theoretical analysis. In [10] Tang and Lou studied the abundant localized coherent structures of a (2+1)-dimensional dispersive long-wave equation by using the variable separation approach. Chen and Yong in [3] obtained Several families of analytical solutions dispersive long-wave equation,their study based upon the extended projective Riccati equations method. In [5] Fan used the ansatz-based method to obtained some exact solutions of the dispersive long wave equation. Yomba In [11] obtained some new soliton-like solutions of the (2 + 1)-dimensional spaces long wave equation by using the improve extended tanh method. In [7] Rong and Tang studied the bifurcation of solitary and periodic waves for (2+1)-dimension nonlinear dispersive long wave equation by using the bifurcation theory of planar dynamical systems. In this paper we investigate the existence of bifurcation of periodic travelling wave solutions of the (2+1)-dimensional dispersive long-wave equation in some domain of parameters by using the Lyapunov-Schmidt method.

Theorem 1.1 (1). *Suppose X and Y are real Banach spaces and $F(x,\lambda)$ is a C^1 map defined in a neighborhood U of a point (x_0,λ_0) with range in Y such that $F(x_0,\lambda_0) = 0$ and $F_x(x_0,\lambda_0)$ is a linear Fredholm operator. Then all solutions (x,λ) of $F(u,\lambda) = 0$ near (x_0,λ_0) (with λ fixed) are in one-to-one correspondence with the solutions of a finite-dimensional system of N_1 real equations in a finite number N_0 of real variables. Furthermore, $N_0 = dim(kerL)$ and $N_1 = dim(cokerL)$, $(L = F_x(x_0,\lambda_0))$.*

Definition 1.1 (8). *The Nonlinear operator $F : U \subset X \to Y$ is called Fredholm if the first Fréchet derivative $dF(x)$ is a Fredholm for every $x \in U$. The index of the nonlinear Fredholm operator F is equal to the index of the linear operator $dF(x)$.*

To study the bifurcation of periodic travelling wave solutions of system (3) we first consider the travelling wave solutions in the form of

$$u(\eta) = u(x,y,t), \quad v(\eta) = v(x,y,t), \quad \eta = px + qy - ct, \quad pq \neq 0,$$

to reduce system (3) into following system,

$$\begin{aligned} -qcu^{''} + p^2v^{''} + pq(uu^{''} + (u^{'})^2) = 0,\\ -cv^{'} + p(uv + u + pqu^{''})^{'} = 0. \end{aligned} \tag{4}$$

where c denotes the wave speed and ($' = d/d\eta$). From the second equation of system (4) we have

$$v^{'} = \frac{p}{c}(uv + u + pqu^{''})^{'} \tag{5}$$

substitute (5) into first equation of (4) we get

$$-qcu^{''} + \frac{p^3}{c}(uv + u + pqu^{''})^{''} + pq(uu^{''} + (u^{'})^2) = 0. \tag{6}$$

by using the transformation $u = v$, equation (6) became

$$\alpha u^{''''} + \beta u^{''} + \mu(uu^{''} + (u^{'})^2) = 0. \tag{7}$$

where, $\alpha = p^4q, \quad \beta = p^3 - c^2q, \quad \mu = 2p^3 + pq.$

in our study we assume that u and v are periodic functions,

$$u(\eta) = u(\eta + T), \qquad v(\eta) = v(\eta + T), \qquad T = 2\pi.$$

In order to study the bifurcation of periodic travelling wave solutions of system (4) we shall study the bifurcation solutions of equation (7). In the next section we will apply the Lyapunov-Schmidt reduction to reduce equation (7) into an equivalent system of four nonlinear algebraic equations.

2. Reduction to the bifurcation equation corresponding to equation (7)

To apply Lyapunov-Schmidt method for equation (7) we first rewrite equation (7) in the form of operator equation

$$F(u,\lambda) = \alpha u^{''''} + \beta u^{''} + \mu(uu^{''} + (u^{'})^2), \tag{8}$$

where $F : \Pi_4([0,2\pi],R) \to \Pi_0([0,2\pi],R)$ is a nonlinear Fredholm operator of index zero, $\Pi_4([0,2\pi],R)$ is the space of all periodic continuous functions that have derivative of order at most four, $\Pi_0([0,2\pi],R)$ is the space of all periodic continuous functions, R is the real space, $u = u(\eta)$, $\eta \in [0,2\pi]$ and $\lambda = (\alpha,\beta)$. We note that the bifurcation solutions of equation (7) is equivalent to the bifurcation solutions of operator equation

$$F(u,\lambda) = 0. \tag{9}$$

The first step in this reduction is determines the linearized equation corresponding to the equation (9), which is given by the following equation

$$Lh = 0, \quad h \in \Pi_4([0,2\pi],R) \qquad ,$$

$$L = F_u(0,\lambda) = \alpha\frac{d^4}{d\eta^4} + \beta\frac{d^2}{d\eta^2}.$$

where $F_u(0,\lambda)$ is the Fréchet derivative of the operator F at the point $(0,\lambda)$.The periodic solutions of the linearized equation is given by

$$h_k(\eta) = r_k sin(k\eta) + t_k cos(k\eta), \qquad k = 1,2,3,...$$

and the characteristic equation corresponding to this solution is

$$\alpha k^4 - \beta k^2 = 0.$$

This equation gives in the $\alpha\beta$-plane characteristic lines l_k. The point of intersection of two lines is a bifurcation point [8]. In particular, the intersection of the lines l_1 and l_2 is the point (0,0).So the point $(\alpha,\beta) = (0,0)$ is a bifurcation point of equation (9). Localized parameters α, β as follows

$$\alpha = 0 + \delta_1, \qquad \beta = 0 + \delta_2, \qquad \delta_1,\ \delta_2 \textit{ are small parameters}$$

lead to bifurcation along the modes

$$e_1(\eta) = r_1 sin(\eta), \quad e_2(\eta) = t_1 cos(\eta), \quad e_3(\eta) = r_2 sin(2\eta), \quad e_4(\eta) = t_2 cos(2\eta)$$

where $\| e_i \|_H = 1$ and $r_i,\ t_i = \sqrt{2}$ for $i = 1,2$, ($H = L_2([0,2\pi],R)$ is a Hilbert space). Let $N = ker(L) = span\{e_1,e_2,e_3,e_4\}$,then the space $\Pi_4([0,2\pi],R)$ can be decomposed in direct sum of two subspaces, N and the orthogonal complement to N,

$$\Pi_4([0,2\pi],R) = N \oplus N^{\perp}, \qquad N^{\perp} = \{\tilde{v} \in \Pi_4([0,2\pi],R) : \tilde{v} \perp N\}.$$

Similarly, the space $\Pi_0([0,2\pi],R)$ can be decomposed in direct sum of two subspaces, N and the orthogonal complement to N

$$\Pi_0([0,2\pi],R) = N \oplus \tilde{N}^{\perp}, \qquad \tilde{N}^{\perp} = \{g \in \Pi_0([0,2\pi],R) : g \perp N\}.$$

Accordingly, there exist projections $p : \Pi_4([0, 2\pi], R) \to N$ and $I - p : \Pi_4([0, 2\pi], R) \to N^{\perp}$ such that $pu = w$, $(I - p)u = \tilde{v}$ and hence every vector $u \in \Pi_4([0, 2\pi], R)$ can be written in the form of,

$$u = w + \tilde{v}, \quad w = \sum_{i=1}^{4} x_i \, e_i \in N, \quad \tilde{v} \in N^{\perp}, \quad x_i = \Big\langle \, u \, , \, e_i \, \Big\rangle_H .$$

($\langle \cdot, \cdot \rangle$ is the inner product in Hilbert space $H = L_2([0, 2\pi], R)$). Similarly, there exists projections $\varrho : \Pi_0([0, 2\pi], R) \to N$ and $I - \varrho : \Pi_0([0, 2\pi], R) \to \tilde{N}^{\perp}$ such that

$$F(u, \lambda) = \varrho F(u, \lambda) + (I - \varrho) F(u, \lambda),$$

$$\varrho F(u, \lambda) = \sum_{i=1}^{4} v_i(u, \lambda) \, e_i \in N, \quad (I - \varrho) F(u, \lambda) \in \tilde{N}^{\perp},$$

$$v_i(u, \lambda) = \Big\langle \, F(u, \lambda) \, , \, e_i \, \Big\rangle_H .$$

Hence equation (9) can be written as

$$\varrho F(u, \lambda) = 0,$$
$$(I - \varrho) F(u, \lambda) = 0.$$

or

$$\varrho F(w + \tilde{v}, \lambda) = 0,$$
$$(I - \varrho) F(w + \tilde{v}, \lambda) = 0.$$

By the implicit function theorem, there exists a smooth map $\Theta : N \to N^{\perp}$ such that $\tilde{v} = \Theta(w, \lambda)$ and

$$(I - \varrho) F(w + \Theta(w, \lambda), \lambda) = 0$$

so to find the solutions of equation (9) in the neighborhood of the point $u = 0$ it is sufficient to find the solutions of the equation,

$$\varrho F(w + \Theta(w, \lambda), \lambda) = 0. \tag{10}$$

Equation (10) is the bifurcation equation corresponding to equation (7).Since,

$$\varrho F(u, \lambda) = \sum_{i=1}^{4} v_i(u, \lambda) \, e_i = 0, \quad v_i(u, \lambda) = \Big\langle \, F(u, \lambda) \, , \, e_i \, \Big\rangle_H$$

then the bifurcation equation can be written in the form of

$$\Xi(\xi, \lambda) = \sum_{i=1}^{4} v_i(u, \lambda) \, e_i = 0, \quad \xi = (x_1, x_2, x_3, x_4).$$

Equation (8) can be written as

$$F(w + \tilde{v}, \lambda) = L(w + \tilde{v}) + \mu B(w + \tilde{v}) = Lw + \mu(ww^{''} + (w^{'})^2) + ...$$

where $B(w + \tilde{v}) = (w + \tilde{v})(w + \tilde{v})^{''} + ((w + \tilde{v})^{'})^2$ and the dots denote the terms consists the element $\tilde{v}$ and its derivatives. Hence

$$\Xi(\xi, \lambda) = \sum_{i=1}^{4} \Big\langle \, Lw + \mu(ww^{''} + (w^{'})^2) \, , \, e_i \, \Big\rangle e_i + ... = 0. \tag{11}$$

To find equation (11) we expand the summation then substitute $w = \sum_{i=1}^{4} x_i e_i$ in (11). After some calculations we have a system of four nonlinear algebraic equations given below

$$\begin{aligned}
x_1x_4 - x_2x_3 + q_1x_1 &= 0,\\
x_1x_3 + x_2x_4 + q_1x_2 &= 0,\\
x_1x_2 + q_3x_3 &= 0,\\
x_1^2 - x_2^2 + q_3x_4 &= 0,
\end{aligned} \tag{12}$$

where, $x_i,\ q_1,\ q_3$ are real and $Le_i = \kappa_i(\lambda)e_i$, i=1,2,3,4.
From theorem (1.1) the solutions of equation (9) are in one-to-one correspondence with solutions of a system (12).Thus the point

$$\bar{u} = \sum_{i=1}^{4} x_ie_i + \Theta(\sum_{i=1}^{4} x_ie_i, \lambda),$$

is a solution of equation (9) if and only if the point (x_1, x_2, x_3, x_4) is a solution of system (12) [8].

3. Analysis of bifurcation of system (12)

In this section we will investigate the existence of the regular solutions of system (12). From the third and the fourth equation of system (12) we have

$$x_3 = \frac{-x_1x_2}{q3}, \quad \textit{and} \quad x_4 = \frac{1}{q_3}(x_2^2 - x_1^2)$$

Simple calculations shows that system (12) has nine solutions given below

$$(0,0,0,0), \quad (0, \pm\sqrt{-q_1q_3}, 0, -q_1), \quad (\pm\sqrt{q_1q_3}, 0, 0, -q_1),$$

$$\Big(\pm\sqrt{\frac{q_1q_3}{3}}, \pm\sqrt{-\frac{q_1q_3}{3}}, \mp\frac{\sqrt{-q_1^2q_3^2}}{3q_3}, -\frac{2q_1}{3}\Big)$$

where $a = \frac{2}{q_3}$.
The determinant of the jacobian matrix of system (12) is given by

$$\begin{aligned}
\mid J \mid = q_3^2x_4^2 + 2x_4q_3^2q_1 - 3x_4x_1^2q_3 + 3x_4x_2^2q_3 + q_1^2q_3^2 - 3q_1x_1^2q_3\\
+3q_1x_2^2q_3 + x_3^2q_3^2 - 6x_3x_1x_2q_3 + 2x_2^4 + 4x_2^2x_1^2 + 2x_1^4
\end{aligned}$$

All degenerate solutions of system (12) are degenerate on the lines $q_1 = 0$ and $q_3 = 0$, it follows that the non-degenerate solutions will exist when $q_1 \neq 0$ and $q_3 \neq 0$. We note that the component $\frac{\sqrt{-q_1^2q_3^2}}{3q_3}$ never to be real for all values of q_1 and q_3, so the remaining nonzero solutions of system (12) are

$$(0, \pm\sqrt{-q_1q_3}, 0, -q_1), \quad (\pm\sqrt{q_1q_3}, 0, 0, -q_1)$$

The solutions

$$(0, \pm\sqrt{-q_1q_3}, 0, -q_1)$$

exist when $q_1q_3 < 0$ and the solutions

$$(\pm\sqrt{q_1q_3}, 0, 0, -q_1)$$

exist when $q_1q_3 > 0$. Accordingly, the corresponding linear approximation of the solutions of equation (7) are

$$w = \pm\sqrt{-2q_1q_3}\ cos(\eta) - \sqrt{2}\ q_1\ cos(2\eta), \qquad w = \pm\sqrt{2q_1q_3}\ sin(\eta) - \sqrt{2}\ q_1\ cos(2\eta).$$

From the obtained results we deduce the following theorem,

Theorem 3.1. *The bifurcation equation corresponding to the equation (7) is a nonlinear system of four nonlinear algebraic equations given in (12). The solutions of equation (7) are in one-to-one correspondence with solutions of a system (12).*

Suppose now

$$\begin{aligned} U_1 &= x_1x_4 - x_2x_3 + q_1x_1, \\ U_2 &= x_1x_3 + x_2x_4 + q_1x_2, \\ U_3 &= x_1x_2 + q_3x_3, \\ U_4 &= x_1^2 - x_2^2 + q_3x_4, \end{aligned} \tag{13}$$

and let $z_1 = x_1 + ix_2$, $\bar{z}_1 = x_1 - ix_2$, $z_2 = x_3 + ix_4$, $\bar{z}_2 = x_3 - ix_4$, then in the complex variables system (13) can be written in the form of $G_1 = U_1 + iU_2$, $G_2 = U_3 + iU_4$. It follows that

$$\begin{aligned} G_1(z_1, z_2) &= z_1\bar{z}_2 i + q_1z_1, \\ G_2(z_1, z_2) &= \frac{i}{4}(z_1^2 + 3\bar{z}_1^2) + q_3z_2. \end{aligned}$$

and system (12) become

$$\begin{aligned} z_1\bar{z}_2 i + q_1z_1 &= 0, \\ \frac{i}{4}(z_1^2 + 3\bar{z}_1^2) + q_3z_2 &= 0. \end{aligned} \tag{14}$$

In polar coordinate system let $z_1 = r_1e^{i\theta_1}$, $z_2 = r_2e^{i\theta_2}$, then

$$\begin{aligned} Re(G_1(z_1, z_2)) &= r_1r_2sin(\theta_2) + q_1r_1, \\ Im(G_1(z_1, z_2)) &= r_1r_2cos(\theta_2), \\ Re(G_2(z_1, z_2)) &= \frac{sin(2\theta_1)}{2}r_1^2 + q_3r_2cos(\theta_2), \\ Im(G_2(z_1, z_2)) &= cos(2\theta_1)r_1^2 + q_3r_2sin(\theta_2). \end{aligned}$$

Hence, the real part of equations (14) is given by the following system

$$\begin{aligned} r_1r_2 + \lambda_1r_1 &= 0, \\ r_1^2 + \lambda_2r_2 &= 0. \end{aligned} \tag{15}$$

where, $\lambda_1 = \frac{q_1}{sin(\theta_2)}$ and $\lambda_2 = \frac{2q_3cos(\theta_2)}{sin(2\theta_1)}$, ($sin(\theta_2) \neq 0$ and $sin(2\theta_1) \neq 0$). The solutions (equilibrium points) of system (15) are

$$(r_1, r_2) = (0, 0), \quad (r_1, r_2) = (\pm\sqrt{\lambda_1\lambda_2}, -\lambda_1).$$

These solutions are degenerated on the line $\lambda_1 = 0$ or $\lambda_2 = 0$. It is clear that the nonzero real solutions exist when $\lambda_1\lambda_2 > 0$. The eigenvalues of the jacobian matrix of system (15) at the point (0,0) are $k_1 = \lambda_1$, $k_2 = \lambda_2$. If $\lambda_1 > 0$, $\lambda_2 > 0$ then the point (0,0) is unstable (source) and if $\lambda_1 < 0$, $\lambda_2 < 0$ then the point (0,0) is stable (sink).The eigenvalues of the jacobian matrix of system (15) at the points $(\pm\sqrt{\lambda_1\lambda_2}, -\lambda_1)$ are $k_1 = \frac{1}{2}(\lambda_2 + \sqrt{\lambda_2^2 + 8\lambda_1\lambda_2})$, $k_2 = \frac{1}{2}(\lambda_2 - \sqrt{\lambda_2^2 + 8\lambda_1\lambda_2})$. It is clear that $\lambda_2^2 + 8\lambda_1\lambda_2 > 0$ for all values of λ_1 and λ_2, so in the first and third quarter of $\lambda_1\lambda_2$-plane the eigenvalue k_1 is positive and the eigenvalue k_2 is negative. Accordingly, the points $(\pm\sqrt{\lambda_1\lambda_2}, -\lambda_1)$ are unstable (saddle points). The phase portrait of system (15) in the first and third quarter is given in the following figures

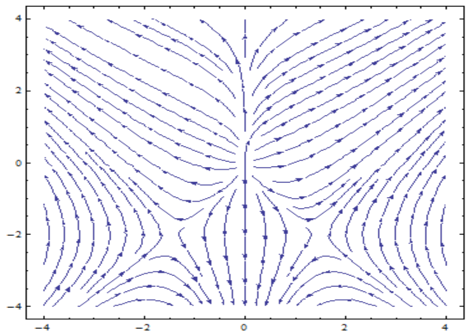

FIGURE 1. The phase portrait of system (15) in the first quarter.

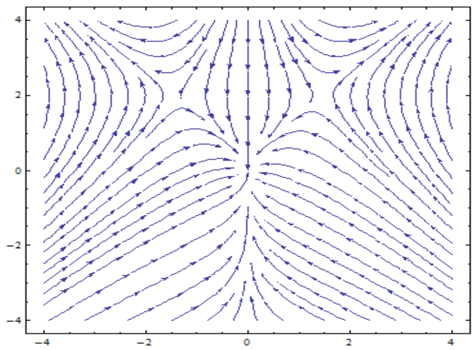

FIGURE 2. The phase portrait of system (15) in the third quarter.

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