TWMS J. App. Eng. Math. V.8, N.1, 2018, pp. 44-50

INDEPENDENTLY SATURATED GRAPHS

Z.N. BERBERLER[1], M.E. BERBERLER[2], §

ABSTRACT. The independence saturation number $IS(G)$ of a graph $G = (V, E)$ is defined as $min\{IS(V) : v \in V\}$, where $IS(v)$ is the maximum cardinality of an independent set that contains v. In this paper, we consider and compute exact formulae for the independence saturation in specific graph families and composite graphs.

Keywords: Independence, Independence saturation, Graph theory

AMS Subject Classification: 05C69

1. Introduction

Graph theoretic techniques provide a convenient tool for the investigation of networks. It is well-known that an interconnection network can be modeled by a graph with vertices representing sites of the network and edges representing links between sites of the network. Therefore various problems in networks can be studied by graph theoretical methods. The independence based parameters reveal an underlying efficient and stable communication network. A subset of pairwise nonadjacent vertices in a graph G is called independent (or stable / internally stable). The cardinality of a maximum size independent set in G is called the independence (or stability) number (or coefficient of internal stability [1]) of G and is denoted by $\beta(G)$. The maximum stable set problem is one of the central problem in combinatorial optimization, and has been the subject of extensive study. The problem of determining a stable set of maximum cardinality is a basic algorithmic graph problem occurring in many models in computer science and operations research and finds important applications in various fields, including computer vision and pattern recognition. Finding a maximum independent set is a well-known widely-studied NP-hard problem. We refer to [3] for a review concerning algorithms, applications, and complexity issues of this problem. Among the independence-type parameters that have been studied, the independence saturation number is one of the fundamental ones introduced by Subramanian [4]. For a vertex v of a graph G, $IS(v)$ denotes the maximum cardinality of an independent set in G which contains v. The independence saturation number of G, denoted by $IS(G)$, is the value $\min\{IS(V) : v \in V\}$. Thus $IS(G)$ is the largest positive integer k such that every vertex of G lies in an independent set of cardinality k. Let $v \in V$ be such that $IS(v) = IS(G)$. Then any independent set of cardinality $IS(G)$ containing v is called an

[1] Faculty of Science, Department of Computer Science, Dokuz Eylul University, 35160, Izmir,Turkey.
e-mail: zeynep.berberler@deu.edu.tr; ORCID: http://orcid.org/0000-0002-3083-6314.

[2] Faculty of Science, Department of Computer Science, Dokuz Eylul University, 35160, Izmir,Turkey.
e-mail: murat.berberler@deu.edu.tr; ORCID: http://orcid.org/0000-0002-9227-2040.

§ Manuscript received: October 3, 2016; accepted: December 20, 2016.

IS-set. The problem of determining whether $IS(G) \geq k$ for any graph G is NP -complete. Independence saturation number of some classes of graphs are computed in [6].

In this paper, we consider finite undirected graphs without loops and multiple edges. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The order of G is the number of vertices in G. The open neighborhood of is v is $N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and for a set $S \subseteq V$, $N(S) = \bigcup_{v \in S} N(v)$. The degree of a vertex v is $deg_G(v) = |N(v)|$. A graph is regular if its vertices all have the same degree. A r-regular graph is the graph in which the degree of each vertex is r. A vertex of degree zero is an isolated vertex or an isolate. A leaf or an endvertex or a pendant vertex is a vertex of degree one and its neighbor is called a support vertex. The maximum degree of G is $\triangle(G) = \max\{deg_G(v) | v \in V(G)\}$ whereas the minimum degree of G is $\delta(G) = \min\{deg_G(v) | v \in V(G)\}$ [2].

We use $\lfloor x \rfloor$ to denote the largest integer not greater than x, and $\lceil x \rceil$ to denote the least integer not less than x.

The paper proceeds as follows. In section 2, existing literature on independence saturation number is reviewed. The independence saturation numbers for specific graph types and graph operations are computed and exact formulae are derived.

2. Independence Saturation

2.1. Known results.

Theorem 2.1. [5] *If G is an r-regular graph on n vertices with $r > 0$, then $IS(G) \leq n/2$. Further equality holds if and only if G is bipartite.*

Theorem 2.2. [5] *For any graph G on n vertices, $IS(G) \leq n - \Delta$. Further for a tree T, $IS(G) = n - \Delta$, if and only if $V - N(v)$ is an independent set for every vertex v of degree Δ and $p_u \leq p_v$ for every $u \in N(v)$, where p_x is the number of pendant vertices adjacent to x.*

Theorem 2.3. [5] *Let G be any graph on p $(p \geq 3)$ vertices. Then*

(i) $3 \leq IS(G) + IS(\bar{G}) \leq p + 1 - (\Delta - \delta)$ *and* $2 \leq IS(G).IS(\bar{G}) \leq (p - \Delta)(\delta + 1)$.

(ii) The following are equivalent.

- *(a)* $IS(G) + IS(\bar{G}) = 3$.
- *(b)* $IS(G).IS(\bar{G}) = 2$.
- *(c) G or $\bar{G}$ has the property that it has a unique vertex of degree $p - 1$ and has at least one pendant vertex.*

(iii) $IS(G) + IS(\bar{G}) = p + 1$ *if and only if G is either K_p or $\overline{K_p}$.*

Theorem 2.4. [5] *The independence saturation of*

- *(a) the complete graph K_n is 1;*
- *(b) the cycle C_n is $\lfloor n/2 \rfloor$;*
- *(c) the complete bipartite graph $K_{m,n}$ is* $\min\{m, n\}$;
- *(d) the star $K_{1,n}$ is 1.*

2.2. **Specific families.** We begin this subsection by determining the independence saturation number of specific families of graphs.

Proposition 2.1. *(a)* $\beta(P_n) = \lceil n/2 \rceil$

(b) $\beta(C_n) = \lfloor n/2 \rfloor$

(c) For $n > 3$ $\beta(\bar{C_n}) = 2$

Definition 2.1. *Let G_1 and G_2 be two disjoint graphs. The union of G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph $G = G_1 \cup G_2$ with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2)$.*

Proposition 2.2. *Let $G_1, G_2, ..., G_n$ be disjoint graphs. If $G = \bigcup_{i=1}^{n} G_i$, then $\beta(G) = \sum_{i=1}^{n} \beta(G_i)$.*

Theorem 2.5. *The independence saturation of*

(a) the null graph $\overline{K_n}$ *is* n*;*

(b) the path P_n $(n \geq 2)$ *is* $\lfloor n/2 \rfloor$*;*

(c) the wheel W_n *is* 1*;*

(d) any complete multipartite graph of order p and least partite set of order r is r;

(e) the comet $C_{t,r}$ *is* $\lceil t/2 \rceil$.

Proof. The proofs to (a) and (d) are routine and we omit them.
To prove (a), there exist two cases according to the number of vertices of P_n:
Case 1. If n is odd;
If v is an endvertex of P_n, then $IS(v)$-set of P_n is also the independent set of G with maximum cardinality achieving the independence number of P_n, that is,

$$IS(v) = \beta(P_n) = \lceil n/2 \rceil \tag{1}$$

If v is not an endvertex of P_n, then $IS(v)$-set of P_n has the maximum cardinality of

$$IS(v) = 1 + \beta(P_i \cup P_{n-3-i})$$

where $0 \leq i \leq (n-3)/2$.
By Proposition (2.2), we have that

$$IS(v) = 1 + \beta(P_i) + \beta(P_{n-3-i})$$

Now, we have two subcases according to i:

Subcase 1. If i is even, then

$$IS(v) = 1 + i/2 + (n-3-i)/2 = (n-1)/2 \tag{2}$$

Subcase 2. If i is odd, then

$$IS(v) = 1 + \lceil i/2 \rceil + \lceil (n-3-i)/2 \rceil = (n+1)/2 \tag{3}$$

By Equation (1), (2), and (3), if n is odd, then

$$IS(P_n) = \min\{(n+1)/2, (n-1)/2\} = (n-1)/2$$

Case 2. If n is even;

If v is an endvertex of P_n, then $IS(v)$-set of P_n is also the independent set of G with maximum cardinality achieving the independence number of P_n, that is,

$$IS(v) = \beta(P_n) = n/2 \tag{4}$$

If v is not an endvertex of P_n, then $IS(v)$-set of P_n has the maximum cardinality of

$$IS(v) = 1 + \beta(P_i \cup P_{n-3-i})$$

where $0 \le i \le \lfloor (n-3)/2 \rfloor$

By Proposition (2.2), we have that

$$IS(v) = 1 + \beta(P_i) + \beta(P_{n-3-i})$$

Now, we have two subcases according to i:

Subcase 1. If i is even, then

$$IS(v) = 1 + i/2 + \lceil (n-3-i)/2 \rceil = n/2 \tag{5}$$

Subcase 2. If i is odd, then

$$IS(v) = 1 + \lceil i/2 \rceil + (n-3-i)/2 = n/2 \tag{6}$$

By Equation (4), (5), and (6), if n is even, then

$$IS(P_n) = n/2$$

By Case 1 and Case 2 we have that

$$IS(P_n) = \lfloor n/2 \rfloor$$

To prove (c), if G is a wheel of order $n+1$, since the center vertex c of G is adjacent to the vertices of the outer cycle, then $IS(c)$-set includes only vertex c with $IS(c) = 1$. For a vertex v of the outer cycle, by Theorem (2.4) (b)

$$IS(v) = \lfloor n/2 \rfloor$$

for every vertex v, and hence

$$IS(W_n) = \min\{1, \lfloor n/2 \rfloor\} = 1$$

To prove (e), define the comet $C_{t,r}$ to be the graph obtained by identifying one end of the path P_t with the center of the star $K_{1,r}$. Label the vertices of P_t sequentially as $v_1, v_2, ..., v_t$ being v_t the center of the star $K_{1,r}$ and label the endvertices of $K_{1,r}$ as $u_1, u_2, ..., u_r$. Then, there exist two cases depending on the number of vertices of P_t:

Case 1. t is odd:

For the vertex v_1 and for an endvertex u_i $(1 \le i \le r)$, both $IS(v_1)$ -set and $IS(u_i)$ -set include the independent set of maximum cardinality of P_{t-1} and all endvertices of $K_{1,r}$. So,

$$IS(v_1) = IS(u_i) = \beta(P_{t-1}) + r = (t + 2r - 1)/2 \tag{7}$$

For the vertex v_i, where $1 < i < t$ and i is odd, $IS(v_i)$-set includes the independent set of maximum cardinality of P_{i-2} and $P_{t-(i+1)-1}$, and all endvertices of $K_{1,r}$. Thus,

$$IS(v_i) = \beta(P_{i-2}) + 1 + \beta(P_{t-(i+1)-1}) + r = (t + 2r - 1)/2 \tag{8}$$

For the vertex v_i, where $1 < i < t$ and i is even, $IS(v_i)$-set includes the independent set of maximum cardinality of P_{i-2} and $P_{t-(i+1)}$, and all endvertices of $K_{1,r}$. Thus,

$$IS(v_i) = \beta(P_{i-2}) + 1 + \beta(P_{t-(i+1)}) + r = (t + 2r - 1)/2 \tag{9}$$

For the vertex v_t, $IS(v_t)$-set includes the independent set of maximum cardinality of P_{t-2}. Then,

$$IS(v_t) = 1 + \beta(P_{t-2}) = (t + 1)/2 \tag{10}$$

By Equation (7), (8), (9) and (10), we have that for t is odd,

$$IS(C_{t,r}) = \min\{(t + 2r - 1)/2, (t + 1)/2\} = (t + 1)/2$$

Case 2. t is even:

For the vertex v_1, $IS(v_1)$-set includes the independent set of maximum cardinality of P_t and all endvertices of $K_{1,r}$. So,

$$IS(v_1) = \beta(P_t) + r = (t + 2r)/2 \tag{11}$$

For the vertex v_i, where $1 < i < t$ and i is odd, $IS(v_i)$-set includes the independent set of maximum cardinality of P_{i-2} and $P_{t-(i+1)}$, and all endvertices of $K_{1,r}$. Thus,

$$IS(v_i) = \beta(P_{i-2}) + 1 + \beta(P_{t-(i+1)}) + r = (t + 2r)/2 \tag{12}$$

For the vertex v_i, where $1 < i < t$ and i is even, $IS(v_i)$-set includes the independent set of maximum cardinality of P_{i-2} and $P_{t-(i+1)-1}$, and all endvertices of $K_{1,r}$. Thus,

$$IS(v_i) = \beta(P_{i-2}) + 1 + \beta(P_{t-(i+1)-1}) + r = (t + 2r - 2)/2 \tag{13}$$

For the vertex v_t, $IS(v_t)$-set includes the independent set of maximum cardinality of P_{t-2}. Then,

$$IS(v_t) = 1 + \beta(P_{t-2}) = t/2 \tag{14}$$

For an endvertex $u_i \quad (1 \le i \le r)$ of $K_{1,r}$, $IS(u_i)$-set includes the independent set of maximum cardinality of P_{t-1} and all endvertices of $K_{1,r}$. Thus,

$$IS(u_i) = \beta(P_{t-1}) + r = (t + 2r)/2 \tag{15}$$

By Equation (11), (12), (13), (14) and (15), we have that for t is even,

$$IS(C_{t,r}) = \min\{(t + 2r)/2, (t + 2r - 2)/2, t/2\} = t/2$$

By Case 1 and Case 2 ,

$$IS(C_{t,r}) = \lceil t/2 \rceil$$

This completes the proof. □

2.3. **Graph operations.**

2.3.1. *Union.*

Theorem 2.6. *Let $G_1, G_2, ..., G_n$ be disjoint graphs. If $G = \bigcup_{i=1}^{n} G_i$, then*

$$IS(G) = \min_i \left\{ IS(G_i) + \sum_{j \neq i} \beta(G_j) \right\}$$

Proof. Denote the order of G_i by n_i. Hence $|V(G)| = \sum_{i=1}^{n} n_i$ Label the vertices of G as $v_1, v_2, ..., v_{n_1}, v_{n_1+1}, ..., v_{n_1+n_2}, v_{n_1+n_2+1}, ..., v_{\sum_i n_i}$ such that $V(G_i) = \left\{ v_{\left(\sum_{t=1}^{i-1} n_t\right)+1}, ..., v_{\sum_{t=1}^{i} n_t} \right\}$.

Then, we have that $IS(G) = \min \left\{ IS_G(v_1), IS_G(v_2), ..., IS_G\left(v_{\sum_i n_i}\right) \right\}$. It is clear that, for any vertex v_j of G where $\left(\sum_{t=1}^{i-1} n_t\right) + 1 \le j \le \sum_{t=1}^{i} n_t$ and $1 \le i \le n$, $IS_G(v_j) = IS_{G_i}(v_j) + \sum_{k \ne i} \beta(G_k)$. Thus,

$$\begin{aligned} IS(G) &= \min \Big\{ IS_{G_1}(v_1) + \sum_{k\ne 1} \beta(G_k), ..., IS_{G_1}(v_{n_1}) + \sum_{k\ne 1} \beta(G_k), ..., IS_{G_n}\left(v_{\left(\sum_{t=1}^{n-1} n_t\right)+1}\right) + \\ &\quad \sum_{k\ne n} \beta(G_k), ..., IS_{G_n}\left(v_{\sum_{t=1}^{n} n_t}\right) + \sum_{k\ne n} \beta(G_k) \Big\} \end{aligned}$$

$$IS(G) = \min \left\{ IS(G_1) + \sum_{k\ne 1} \beta(G_k), ..., IS(G_n) + \sum_{k\ne n} \beta(G_k) \right\}$$

$$IS(G) = \min_i \left\{ IS(G_i) + \sum_{k\ne i} \beta(G_k) \right\}$$

□

2.3.2. *Join.*

Definition 2.2. *Let G_1 and G_2 be two disjoint graphs. The join of G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$ is the graph $G = G_1 + G_2$ with vertex set $V(G) = V(G_1) \cup V(G_2)$ and edge set $E(G) = E(G_1) \cup E(G_2) \cup \{(u, v) : u \in V(G_1), v \in V(G_2)\}$.*

Theorem 2.7. *Let G_1 and G_2 be two disjoint graphs,then*

$$IS(G_1 + G_2) = \min \{IS(G_1), IS(G_2)\}$$

Proof. Denote the order of G_1 and G_2 by n and m, respectively. Label the vertices of $V(G_1 + G_2)$ as $v_1, v_2, ..., v_n$, $u_1, u_2, ..., u_m$ such that $V(G_1) = \{v_1, v_2, ..., v_n\}$ and $V(G_2) = \{u_1, u_2, ..., u_m\}$. Then

$$\begin{aligned} IS(G_1 + G_2) &= \min \{IS_{G_1+G_2}(v_1), ..., IS_{G_1+G_2}(v_n), ..., IS_{G_1+G_2}(u_1), ..., IS_{G_1+G_2}(u_m)\} \\ &= \min \{IS_{G_1}(v_1), IS_{G_1}(v_2), ..., IS_{G_1}(v_n), IS_{G_2}(u_1), IS_{G_2}(u_2), ..., IS_{G_2}(u_m)\} \\ &= \min \left\{ \min_i \{IS_{G_1}(v_i) : 1 \le i \le n\}, \min_j \{IS_{G_2}(u_j) : 1 \le j \le m\} \right\} \\ &= \min \{IS(G_1), IS(G_2)\}. \end{aligned}$$

□

References

[1] Korshunov, A. D.,(1974), Coefficient of internal stability of graphs, Cybernetics 10 (1) pp. 19-33.
[2] West, D.B., (2001), Introduction to Graph Theory, Prentice Hall, NJ.
[3] Bomze, I., Budinich, M., Pardalos, P., Pelillo, M., (1999), The maximum clique problem, in D. Du and P. Pardalos (eds), Handbook of Combinatorial Optimization, Supplement Volume A, Kluwer Academic Press.
[4] Subramanian, M., (2004), Studies in Graph Theory-Independence saturation in Graphs, Ph.D thesis, Manonmaniam Sundaranar University.
[5] Arumugam, S., Subramanian, M., (2007), Independence saturation and extended domination chain in graphs, AKCE J. Graphs. Combin. 4(2) pp. 5969.
[6] Muthulakshmi, T., Subramanian, M., (2014), Independence saturation number of some classes of graphs, Far East Journal of Mathematical Sciences 86(1) pp. 11-21.

Zeynep Nihan BERBERLER received the MS and PhD degrees in mathematics-computer science from Ege University, Izmir, Turkey, in 2009 and 2014, respectively. She is currently an assistant professor in Computer Science Department, Dokuz Eylul University, Izmir, Turkey. Her research interests include graph theory, optimization, and operations research.

Murat Ersen BERBERLER received the MS and PhD degrees in mathematics-computer science from Ege University, Izmir, Turkey, in 2006 and 2009, respectively. He is currently an assistant professor in Computer Science Department, Dokuz Eylul University, Izmir, Turkey. His research interests include artificial intelligence, computational complexity, graph theory, soft computing and operations research.