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ON THE MOMENTS FOR ERGODIC DISTRIBUTION OF AN INVENTORY MODEL OF TYPE (s, S) WITH REGULARLY VARYING DEMANDS HAVING INFINITE VARIANCE

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Abstract. In this study a stochastic process X(t) which represents a semi Markovian inventory model of type (s,S) has been considered in the presence of regularly varying tailed demand quantities. The main purpose of the current study is to investigate the asymptotic behavior of the moments of ergodic distribution of the process X(t) when the demands have any arbitrary distribution function from the regularly varying subclass of heavy tailed distributions with infinite variance. In order to obtain renewal function generated by the regularly varying random variables, we used a special asymptotic expansion provided by Geluk [14]. As a first step we investigate the current problem with the whole class of regularly varying distributions with tail parameter $1 < \alpha < 2$ rather than a single distribution. We obtained a general formula for the asymptotic expressions of n^{th} order moments $(n = 1, 2, 3, \dots)$ of ergodic distribution of the process X(t). Subsequently we consider this system with Pareto distributed demand random variables and apply obtained results in this special case.

Keywords: Semi Markovian Inventory Model, Renewal Reward Process, Regular Variation, Moments, Asymptotic Expansion.

AMS Subject Classification: 60K05, 60K15

1. Introduction

Heavy tailed distributions attracts growing attention in recent years because they have a wide application area in many disciplines such as, telecommunications, computer systems, risk, insurance and stock control. One of the common application areas of heavy tailed distributions is inventory models. Specifically there are plenty of studies which provide empirical examples for existence of regularly varying demands in inventory models (see [8], [13]). The main purpose of the current study is to investigate the impact of regularly varying demands with infinite variance on the stochastic process X(t) which represents a semi-Markovian inventory model of type (s,S). Now let us give some essential notations and the explanation of the model as follows:

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The Model:

Suppose a company want to create the optimal inventory policy. Assume that s is the stock control level, S is the maximum stock level and X(t) represents stock level in a depot at time t. Moreover z is the initial stock level in this company's depot at time $t = 0$, hence $X(0) = X_0 = z \in [s, S], 0 \leq s < S < \infty$. In addition suppose that $\{\eta_n\}, n \geq 1$ which describe the random amount of demands are coming to the system at random times $T_1, T_2, \ldots, T_n, \ldots$. Here $T_n = \sum_{i=1}^{n} \xi_i$, where $\{\xi_n\}, n \geq 1$ represents inter arrival times between two successive demands. Hence the stock level $X(t)$ decreases by $\eta_1, \eta_2, \ldots \eta_n, \ldots$ at random times $T_1, T_2, \ldots T_n, \ldots$ until $X(t)$ falls below s, at random time τ_1. In this instance the stock level changes as follows:

$$X(T_1) \equiv X_1 = z - \eta_1,\ X(T_2) \equiv X_2 = z - (\eta_1 + \eta_2), \ldots, X(T_n) \equiv X_n = z - \sum_{i=1}^{n} \eta_i.$$

where, η_n represents the amount of n^{th} demand, $n = 1, 2, 3 \ldots$. τ_1 is the first time, that the stock level falls below the control level s. After the stock level falls below s, it is immediately refilled up to the level ζ_1, and the first period is completed. Second period starts with a new initial stock level ζ_1 and continues in a similar manner to the first period. Note that $\{\eta_n, \zeta_n, \xi_n\}$, $n = 1, 2, \ldots$ is a sequence of i.i.d. random variables here. This model is referred in the literature as "Semi Markovian Inventory Model of Type (s,S)".

Investigation of semi Markovian inventory model of type (s,S) is a classical research area. So many characteristics of these models have been investigated in the literature (see [1], [2], [16], [3], [17], [18]). When analyzing an inventory model of type (s,S), the most common approach is assuming that the demand random variables are light tailed with finite variance.

Main departure point of this paper that distinguishes it from all previous literature is we consider mentioned stochastic process with heavy tailed demand random variables with infinite variance. More specifically we used regularly varying subclass of heavy tailed distributions with tail parameter $1 < \alpha < 2$.

Regular variation is one of the most important theories which come out in various contexts of applied probability theory. For more details about regularly varying functions and random variables we refer the reader to the textbooks ([5], [6], [10], [12], [20], [21]). We gave a short summary in preliminaries section. The main purpose of this paper is to investigate the asymptotic behavior of the model when the demand quantities are regularly varying with infinite variance.

2. Preliminaries

Let us give the essential notations and explain this model mathematically before analyzing the main problem. The well known content is taken from [12], [5].

Definition 2.1. *A distribution F on $\mathbb{R}$ is said to be (right) heavy tailed if*

$$\int_{-\infty}^{\infty} e^{\lambda x} F(dx) = \infty \text{ for all } \lambda > 0.$$

For a detailed information see the books by [4], [10], [6], [20].

Definition 2.2. *(Regularly Varying Functions) A positive, measurable function f is called regularly varying at ∞ with index $\alpha \in \mathbb{R}$, if for all $\lambda > 0$*

$$\lim_{x \longrightarrow \infty} \frac{f(x\lambda)}{f(x)} = \lambda^{\alpha}.$$

If $\alpha = 0$, then f is called slowly varying function. The family of regularly varying functions with index α is denoted by $RV(\alpha)$.

Definition 2.3. *(Regularly varying random variables) The non negative random variable X and its distribution are called regularly varying with index $\alpha \geq 0$ if the right tail distribution $\bar{F}(x) \in RV(-\alpha)$.*

Remark: Any regularly varying distribution can be represented in following way:

$$P(X > x) = x^{-\alpha} L(x), \ \text{where } \alpha > 0 \text{ and } L(x) \in RV(0).$$

In the rest of this study we will refer following propositions (Proposition 2.1 and Proposition 2.2) when integrating regularly varying functions.

Proposition 2.1. *((Karamata Theorem) Bingham et. al. [5]) Let L be slowly varying function in $[x_0, \infty)$ for some $x_0 \geq 0$. Then*

(1) *for $\alpha > -1$,*

$$\int_{x_0}^{x} t^{\alpha} L(t) dt \sim (\alpha + 1)^{-1} x^{\alpha+1} L(x).$$

(2) *for $\alpha < -1$,*

$$\int_{x}^{\infty} t^{\alpha} L(t) dt \sim -(\alpha + 1)^{-1} x^{\alpha+1} L(x).$$

Proposition 2.2. *(Seneta, E. [21]) Let L be a slowly varying function on $(0, \infty)$, and suppose that the integral*

$$\int_{0}^{\beta} f(t) L(tx) dt$$

is well defined for $0 < \beta < \infty$ and some given real function f. Then as $x \longrightarrow \infty$

$$\int_{0}^{\beta} f(t) L(tx) dt \sim L(x) \int_{0}^{\beta} f(t) dt.$$

Proposition 2.3 and Proposition 2.4 allows us to make some operations on regularly varying functions.

Proposition 2.3. *(Bingham et. al. [5])*

(1) *If L varies slowly, so does $(L(x))^{\alpha}$ for every $\alpha \in R$.*
(2) *If L_1, L_2 varies slowly, so do $L_1 L_2$, $L_1 + L_2$. Moreover if $L_2(x) \longrightarrow \infty$ as $x \longrightarrow \infty$, then $L_1(L_2(x))$ varies slowly.*
(3) *If L varies slowly and $\alpha > 0$ then $x^{\alpha} L(x) \longrightarrow \infty$, $x^{-\alpha} L(x) \longrightarrow 0$.*

Proposition 2.4. *(Bingham et. al. [5])*

(1) *If $f(x) \in RV(\alpha)$ then $(f(x))^p \in RV(\alpha p)$ for any $p \in \mathbb{R}$.*
(2) *If $f_i \in RV(\alpha_i)$, $i = 1, 2$, and $f_2(x) \longrightarrow \infty$ as $x \longrightarrow \infty$ then, $f_1(f_2(x)) \in RV(\alpha_1 \alpha_2)$.*
(3) *If $f_i \in RV(\alpha_i)$, $i = 1, 2$, then $f_1(x) + f_2(x) \in RV(\alpha)$, $\alpha = max(\alpha_1, \alpha_2)$.*

3. Mathematical Construction of The Process $X(t)$

Let $(\Omega, \Im, P)$ be probability space and $\{(\xi_n, \eta_n, \zeta_n)\}$, $n \geq 1$ be a vector of i.i.d. random variables defined on $(\Omega, \Im, P)$. Here $\{\xi_n\}$, $n \geq 1$ and $\{\eta_n\}$, $n \geq 1$ are positive valued random variables. The random variable ζ_n takes values in the interval $[s, S]$ and ξ_n, η_n and ζ_n are also independent from each other.

Let the distributions of ξ_n, η_n and ζ_n be denoted by $\Phi(t), F(x)$ and $\pi(z)$ respectively and these distributions defined as follows:

$$\Phi(t) = P\{\xi_1 \leq t\}, F(x) = P\{\eta_1 \leq x\}, \pi(z) = P\{\zeta_1 \leq z\},\ t \geq 0, x \geq 0,\ z \in [s, S].$$

We assume here that the random variables $\{\zeta_n\}$, $n \geq 1$ which represents the discrete interference of chance have uniform distribution on the interval $[s, S]$. $\{\eta_n\}$, $n \geq 1$ are regularly varying random variables with infinite variance.

Now we can construct the process with all these information above.

As a first step we need to define the renewal sequences $\{T_n\}$ and $\{S_n\}$ as:

$$T_0 = S_0 = 0,\ T_n = \sum_{i=1}^{n} \xi_i,\ S_n = \sum_{i=1}^{n} \eta_i,\ n \geq 1.$$

Now define a sequence of integer-valued random variables $\{N_n\}$, $n \geq 0$ as follows:

$$N_0 = 0,\ N_1 = N(z - s) = inf\{k \geq 1 :\ z - S_k \leq s\},\ \ z \in [s, S].$$

$$N_{n+1} = inf\{k \geq N_n + 1 :\ \zeta_n - (S_k - S_{N_n}) < s\},\ n \geq 1.$$

Let

$$\tau_0 = 0,\ \tau_n = T_{N_n} = \sum_{i=1}^{N_n} \xi_i,\ n \geq 1,$$

$$\nu(t) = max\{n \geq 0 :\ T_n \leq t\},\ t \geq 0.$$

Under these assumptions the desired stochastic process $X(t)$ constructed as follows:

$$X(t) = \zeta_n - \left(\eta_{N_n+1} + \ldots + \eta_{\nu(t)}\right) = \zeta_n - \left(S_{\nu(t)} - S_{N_n}\right),\ t \in [\tau_n, \tau_{n+1}),\ n \geq 0. \qquad (1)$$

The process $X(t)$ represents the variation of a stock level in the depot. A realization of this process is given as in Figure 1.

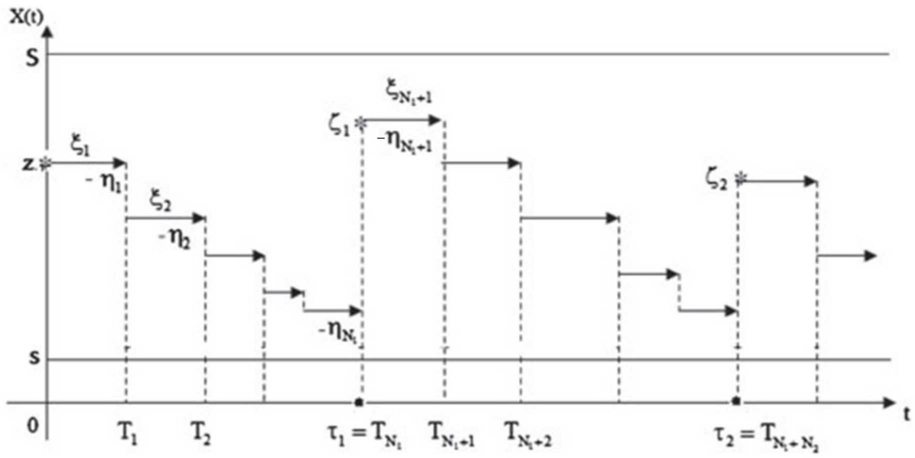

Figure 1. A realization of the process X(t)

4. Ergodicity of the process and exact formulas for the n^{th} order moments of the ergodic distribution of the process $X(t)$

Ergodicity of the process X(t) has proven by Khaniyev and Atalay [17] under some weak conditions. In addition to the mentioned conditions in the study by [17], we assumed here that the demand random variables $\{\eta_i\}, i \geq 1$ are regularly varying with index $1 < \alpha < 2$.

Proposition 4.1. *Let the initial sequence of random variables* $\{(\xi_n, \eta_n, \zeta_n)\}$, $n \geq 1$ *satisfy the following supplementary conditions:*

(1) $0 < E(\xi_1) < \infty$,
(2) $0 < E(\eta_1) < \infty$,
(3) $\{\eta_i\}$, $i \geq 1$ *are non-arithmetic random variables.*
(4) *The distribution functions of* $\{\eta_i\}$, $i \geq 1$ *are regularly varying with index* $1 < \alpha < 2$.
(5) *Markov chain* $\{\zeta_n\}$, $n \geq 1$ *has uniform distribution on the interval* $[s, S]$.

Then, the process $X(t)$ *is ergodic.*

Following proposition is the main result of Proposition 4.1.

Proposition 4.2. *Under the conditions of Proposition 4.1 the following relation is true with probability 1 for each measurable bounded function* $f(x)$, $(f : [s, S] \longrightarrow R)$:

$$\lim_{t\longrightarrow\infty} \frac{1}{t}\int_0^t f(X(u))\,du \equiv S_f = \frac{\int_s^S \int_s^S f(x)\left[U(z-s) - U(z-x)\right] d\pi(z)dx}{\int_s^S U(z-s)d\pi(z)}.$$

The ergodic distribution function of the process $X(t)$ is denoted by $Q_X(x)$ ande represented as:

$$Q_X(x) \equiv \lim_{t\longrightarrow\infty} P\{X(t) \leq x\} \;\; x \in [s, S].$$

Proposition 4.3 provides the exact expression for the ergodic distribution function $Q_X(x)$ and obtained by replacing f(x) with indicator function in Proposition 4.2

Proposition 4.3. *Let the conditions of Proposition 4.1 are satisfied. Then the ergodic distribution function* $Q_X(x)$ *of the process* $X(t)$ *is given as:*

$$Q_X(x) \equiv 1 - \frac{\int_x^S U(z-x)\,d\pi(z)}{\int_s^S U(z-s)\,d\pi(z)}\;,\; x \in [s, S]\,.$$

Corollary 4.1. *Assume that the conditions of the Proposition 4.1 are satisfied. Then the ergodic distribution function* $Q_X(x)$ *of the process* $X(t)$ *can be written as follows:*

$$Q_X(x) = 1 - \frac{E(U(\zeta - x))}{E(U(\zeta - s))},\; x \in [s, S]. \tag{2}$$

Here the random variable ζ *has a distribution* $\pi(z)$.

In order to obtain asymptotic expansions for the moments of the ergodic distribution of the process $X(t)$, we need to know the exact formulas. Exact expressions are derived by using (2) by Khaniyev et. al. [18]. In the rest of this paper n^{th} order moments of the ergodic distribution of the process $X(t)$ will be denoted by $E(X^n)$. Let us define

$$\tilde{X}(t) = X(t) - s;\; E(\tilde{X}^n) = \lim_{t\longrightarrow\infty} E(\tilde{X}^n(t));\; \tilde{\zeta}_n = \zeta_n - s,\; n = 1, 2, 3, \ldots.$$

Following proposition by Khaniyev et. al. [18] states the exact expression for the moments of ergodic distribution of the process $\tilde{X}(t)$.

Proposition 4.4. *If n^{th} order (n=1,2,3,...) moments $(E(\tilde{X}^n))$ of the ergodic distribution of the process $\tilde{X}(t)$ exists and finite, then it can be represented as follows:*

$$E(\tilde{X}^n) = \frac{n}{E(U(\tilde{\zeta}))}\int_0^{2\beta} \upsilon^{n-1}E(U(\tilde{\zeta}-\upsilon))d\upsilon. \tag{3}$$

Here; $\tilde{\zeta} = \zeta - s$, $\beta \equiv \frac{S-s}{2}$, $\upsilon \in [0, 2\beta]$. Moreover $U(x)$ is the renewal function generated by the sequence $\{\eta_n\}$, $n = 1, 2, 3, \ldots$.

5. Asymptotic expansions for the n^{th} order moments of the ergodic distribution of the process $X(t)$

We assumed here, that the random variables $\{\eta_n\}$, $n \geq 1$ are heavy tailed with infinite variance. The main starting point of this current work is the study by Geluk [14] where he provided an asymptotic expansion for the renewal function generated by regularly varying distributions with infinite variance as follows:

Proposition 5.1. *(Geluk (1992) [14]) Let $F(.)$ be a c.d.f. on $(0,\infty)$ such that*

$$\bar{F}(.) \equiv 1 - F(.)$$

is regularly varying with exponent $-\alpha$, $1 < \alpha < 2$. Then

$$U(t) - \frac{t}{\mu} - \frac{1}{\mu^2}\int_0^t\int_s^\infty \bar{F}(v)\,dvds = O\left(t^4\left(\bar{F}(t)\right)^2\bar{F}\left(t^2\bar{F}(t)\right)\right) \; as\, t \longrightarrow \infty. \tag{4}$$

Here it is assumed that $\eta_1, \eta_2, \ldots$ is a sequence of i.i.d. real valued positive random variables with d.f. F and $U(t) = E(N(t))$ is the renewal function associated with $F(t)$.

Following Lemma is obtained by using Proposition 5.1.

Lemma 5.1. *Let $\{\eta_i\}$, $i \geq 1$ be a sequence of regularly varying random variables with exponent $-\alpha$, $1 < \alpha < 2$ i.e.:*

$$\bar{F}(t) = P\{\eta_1 > t\} = t^{-\alpha}L(t).$$

Then the renewal function generated by the random variables $\{\eta_i\}$, $i \geq 1$ obtained as follows:

$$U(t) = \frac{t}{\mu_1} + \frac{1}{\mu_1}G(t) + O\left(t^{(\alpha-2)^2}L_1(t)\right), \; t \longrightarrow \infty.$$

Where $\mu_k = E\left(\eta_1^k\right)$, $k = 1, 2, \ldots$. $L_1(t)$ is slowly varying and defined as:

$$L_1(t) = (L(t))^2 L\left(t^{2-\alpha}L(t)\right).$$

Note that $1 < \alpha < 2$ and $L(t)$ is slowly varying function associated with the random variable η_1. Moreover

$$G(t) = \frac{1}{\mu_1}\int_0^t\int_s^\infty \bar{F}(v)dvds.$$

Proof. Asymptotic expansion suggested by Geluk [14] generated by the regularly varying random variables with $1 < \alpha < 2$ is given as follows:

$$U(t) = \frac{t}{\mu_1} + \frac{1}{\mu_1}G(t) + O\left(t^4\left(\bar{F}(t)\right)^2\bar{F}\left(t^2\bar{F}(t)\right)\right); \; t \longrightarrow \infty.$$

Since $\bar{F}(t) \in RV(-\alpha)$, then $\bar{F}(t) = t^{-\alpha}L(t)$ where $1 < \alpha < 2$ and $L(t)$ is slowly varying at ∞. Moreover by Proposition 2.4 (2),

$$\bar{F}(t^{2-\alpha}L(t)) = (t^{2-\alpha})^{-\alpha}L(t^{2-\alpha}L(t)).$$

Hence

$$t^4(\bar{F}(t))^2\bar{F}\left(t^2\bar{F}(t)\right) = t^4t^{-2\alpha}(L(t))^2\bar{F}\left(t^2t^{-\alpha}L(t)\right) = t^{(\alpha-2)^2}(L(t))^2L\left(t^{2-\alpha}L(t)\right).$$

Let define $(L(t))^2\, L\left(t^{2-\alpha}L(t)\right) = L_1(t)$.

By Proposition 2.3, $(L(t))^2$ is slowly varying function. $t^{2-\alpha}L(t)$ is regularly varying with exponent $(2-\alpha)$ and $L(t)$ is slowly varying (regularly varying with exponent zero). Moreover by Proposition 2.3 (3), $t^{2-\alpha}L(t) \longrightarrow \infty$ as $t \longrightarrow \infty$. Hence by Proposition 2.4 (2), $L(t^{2-\alpha}L(t))$ is also regularly varying with exponent zero, which is a slowly varying function. So by Proposition 2.3 (1), $L_1(t) = (L(t))^2L\left(t^{2-\alpha}L(t)\right)$ is slowly varying function where $L(t)$ is slowly varying function associated with random variable η_1. This completes the proof.

□

Lemma 5.2. *For any bounded function $g : \mathbb{R} \longrightarrow \mathbb{R}$ the following asymptotic relation holds when $\beta \equiv \frac{S-s}{2} \longrightarrow \infty$:*

$$\int_0^{2\beta-\upsilon} x^{(\alpha-2)^2}L_1(x)g(x)dx = O\left(\beta^{(\alpha-2)^2+1}L_1(\beta)\right),\ \upsilon \in [0,2].$$

Here $L_1(\beta) = (L(\beta))^2L\left(\beta^{2-\alpha}L(\beta)\right)$ is slowly varying function and $L(\beta)$ is slowly varying function associated with the random variable η_1.

Proof. Since $g(x)$ is given as a bounded function, there exists a constant $K > 0$ such that:

$$\begin{aligned}\left|\int_0^{2\beta-\upsilon} x^{(\alpha-2)^2}L_1(x)g(x)dx\right| &\leq K\int_0^{2\beta-\upsilon}\left|x^{(\alpha-2)^2}L_1(x)\right|dx\\ &= K\int_0^{2\beta-\upsilon} x^{(\alpha-2)^2}L_1(x)dx\\ &\sim K\frac{(2\beta-\upsilon)^{(\alpha-2)^2+1}}{(\alpha-2)^2+1}L_1(2\beta-\upsilon),\ \upsilon\in[0,2].\end{aligned}$$

Note that we used Karamata Theorem in order to obtain following asymptotic relation:

$$K\int_0^{2\beta-\upsilon} x^{(\alpha-2)^2}L_1(x)dx \sim K\frac{(2\beta-\upsilon)^{(\alpha-2)^2+1}}{(\alpha-2)^2+1}L_1(2\beta-\upsilon).$$

Therefore

$$\int_0^{2\beta-\upsilon} x^{(\alpha-2)^2}L_1(x)g(x)dx = O\left(\beta^{(\alpha-2)^2+1}L_1(\beta)\right).$$

□

Lemma 5.3. *Under the conditions of Proposition 4.1 and Proposition 5.1 the following asymptotic expansion holds as $\beta \equiv \frac{S-s}{2} \longrightarrow \infty$:*

$$E(U(\tilde{\zeta}-\upsilon)) = \frac{1}{2\beta}\left[\frac{1}{\mu_1}\frac{(2\beta-\upsilon)^2}{2} + \frac{1}{\mu_1}G_0\left(2\beta-\upsilon\right) + O\left(\beta^{(\alpha-2)^2+1}L_1(\beta)\right)\right]. \tag{5}$$

Here

$L_1(\beta) = (L(\beta))^2L\left(\beta^{2-\alpha}L(\beta)\right)$ *is slowly varying,* $\upsilon \in [0,2]$, $1<\alpha<2$, *and*

$$G_0(x) = \int_0^x G(t)dt = \int_0^x\left[\frac{1}{\mu_1}\int_0^t\int_s^\infty \bar{F}(\upsilon)d\upsilon ds\right]dt,\ x \longrightarrow \infty. \tag{6}$$

Proof. We assumed here, the random variable ζ_n has uniform distribution on the interval [s, S]. Hence the random variable $\tilde{\zeta}_n = \zeta_n - s$ has the same distribution on the interval $[0, 2\beta]$, $\beta \equiv \frac{S-s}{2}$.

$$\tilde{\pi}(x) \equiv P\left\{\tilde{\zeta}_1 \leq x\right\} = P\left\{\zeta_1 - s \leq x\right\} = \pi\left(s+x\right).$$

Hence

$$E(U(\tilde{\zeta}-\upsilon)) = \int_{\upsilon}^{2\beta} U(x-\upsilon)d\tilde{\pi}(x) = \frac{1}{2\beta}\int_0^{2\beta-\upsilon} U(t)dt. \tag{7}$$

It is clear that:

$$\int_0^{2\beta-\upsilon} \frac{t}{\mu_1}dt = \frac{1}{\mu_1}\frac{(2\beta-\beta\upsilon)^2}{2},\ \upsilon \in [0,2]. \tag{8}$$

Moreover by using the definition of $G_0(x)$ and Karamata Theorem:

$$G_0(x) = \frac{1}{\mu_1}\int_0^x\int_0^t\int_s^\infty \upsilon^{-\alpha}L(\upsilon)d\upsilon ds dt \sim -\frac{1}{\mu_1}\frac{1}{(1-\alpha)}\frac{1}{(2-\alpha)}\frac{1}{(3-\alpha)}x^{3-\alpha}L(x). \tag{9}$$

Here $L(x)$ is slowly varying function associated with the random variable η_1. Result is obtained by using Lemma 5.2, (8) and asymptotic relation (9). □

Corollary 5.1. *Under the conditions of Lemma 5.3 the following asymptotic expansion holds as* $\beta \longrightarrow \infty$ *:*

$$E(U(\tilde{\zeta})) = \frac{1}{2\beta}\left[\frac{1}{\mu_1}\frac{(2\beta)^2}{2} + \frac{G_0(2\beta)}{\mu_1} + O\left(\beta^{(\alpha-2)^2+1}L_1(\beta)\right)\right]. \tag{10}$$

Lemma 5.4. *For any bounded function* $h : \mathbb{R} \longrightarrow \mathbb{R}$ *the following asymptotic relation holds when* $\beta \longrightarrow \infty$ *:*

$$\int_0^{2\beta} \upsilon^{n-1}\beta^{(\alpha-2)^2+1}h\left(\upsilon\right)L_1(\beta)d\upsilon = O\left(\beta^{n+(\alpha-2)^2+1}L_1(\beta)\right),\ \upsilon \in [0,2].$$

Here

$$L_1(\beta) = (L(\beta))^2 L\left(\beta^{2-\alpha}L(\beta)\right) \tag{11}$$

is slowly varying function and $L(\beta)$ *is slowly varying function associated with the random variable* η_1.

Proof. Since $h(x)$ is given as a bounded function, there exists a constant $K > 0$ such that

$$\begin{aligned}\left|\int_0^{2\beta} \upsilon^{n-1}\beta^{(\alpha-2)^2+1}L_1(\beta)h(\upsilon)d\upsilon\right| &\leq K\beta^{(\alpha-2)^2+1}L_1(\beta)\int_0^{2\beta}\upsilon^{n-1}d\upsilon \\ &= \frac{K2^n}{n}\beta^{n+(\alpha-2)^2+1}L_1(\beta).\end{aligned} \tag{12}$$

$n \geq 1$, $1 < \alpha < 2$, $L_1(\beta)$ is defined as (11).

Result is straightforward from (12).

□

Lemma 5.5. *Under the conditions of Lemma 5.3 following asymptotic relation holds as* $\beta \longrightarrow \infty$*:*

$$\frac{1}{\mu_1}\int_0^{2\beta} \upsilon^{n-1}G_0\left(2\beta-\upsilon\right)d\upsilon \sim \frac{1}{\mu_1^2}L(2\beta)(2\beta)^{n+2-\alpha}B(n, 4-\alpha).$$

Here $B(x, y)$ *is Beta function and defined as:*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt;\ Re(x) > 0,\ Re(y) > 0.$$

Moreover

$$\upsilon \in [0,2],\ 1 < \alpha < 2,\ n \geq 1.$$

Proof. From Lemma 5.3;

$$G_0(x) \equiv -\frac{1}{\mu_1}\frac{1}{(1-\alpha)}\frac{1}{(2-\alpha)}\frac{1}{(3-\alpha)}x^{3-\alpha}L(x).$$

By using Proposition 2.2 and changing variables following asymptotic relation is obtained:

$$\begin{aligned}\frac{1}{\mu_1}\int_0^{2\beta}\upsilon^{n-1}G_0\left(2\beta-\upsilon\right)d\upsilon &\sim \frac{1}{\mu_1^2}\frac{1}{(\alpha-1)(2-\alpha)(3-\alpha)}\int_0^{2\beta}\upsilon^{n-1}(2\beta-\upsilon)^{3-\alpha}L(2\beta-\upsilon)d\upsilon \\ &= \frac{1}{\mu_1^2}\frac{(2\beta)^{4-\alpha}}{(\alpha-1)(2-\alpha)(3-\alpha)}\int_0^1 (2\beta t)^{n-1}(1-t)^{3-\alpha}L(2\beta-2\beta t)dt \\ &\sim \frac{1}{\mu_1^2}\frac{(2\beta)^{n-\alpha+3}}{(\alpha-1)(2-\alpha)(3-\alpha)}L(2\beta)\int_0^1 (1-u)^{n-1}u^{3-\alpha}du \\ &= \frac{1}{\mu_1^2}\frac{(2\beta)^{n-\alpha+3}}{(\alpha-1)(2-\alpha)(3-\alpha)}L(2\beta)B(n,4-\alpha). \qquad (13)\end{aligned}$$

□

Following corollary is obtained by using Lemma (5.4) and Lemma (5.5).

Corollary 5.2. *Let the conditions of Lemma (5.4) and Lemma (5.5) are satisfied. Moreover; define $J_n(\beta)$ as:*

$$J_n(\beta) = \int_0^{2\beta}\upsilon^{n-1}E(U(\tilde{\zeta}-\upsilon))d\upsilon.$$

Then the asymptotic expansion for $J_n(\beta)$ is obtained as $\beta \equiv \frac{S-s}{2} \longrightarrow \infty$ as follows:

$$\begin{aligned}J_n(\beta) = \frac{1}{2\beta}&\left\{\frac{1}{\mu_1}\frac{1}{n(n+1)(n+2)}(2\beta)^{n+2} + \left[\frac{1}{\mu_1^2}\frac{L(2\beta)B(n,4-\alpha)}{(\alpha-1)(2-\alpha)(3-\alpha)}\right](2\beta)^{n+3-\alpha}\right. \\ &\left. O\left(\beta^{n+(\alpha-2)^2+1}L_1(\beta)\right)\right\}. \qquad (14)\end{aligned}$$

Where $1 < \alpha < 2$, $n \geq 1$, $L(\beta)$ is slowly varying function associated with the random variable η_n and $L_1(\beta)$ is defined as (11).

Theorem 5.1. *Let the conditions of Proposition (4.1) and Proposition (5.1) are satisfied. Then the following two term asymptotic expansion is obtained for the n^{th} order moments, $n \geq 1$ of the ergodic distribution of the process $\tilde{X}(t) = X(t) - s$ as $\beta \equiv \frac{S-s}{2} \longrightarrow \infty$:*

$$\begin{aligned}E(\tilde{X}^n) &= \frac{2^{n+1}}{(n+1)(n+2)}\beta^n + \frac{1}{\mu_1}\left\{\frac{\left[(n^3+3n^2+2n)B(n,4-\alpha)-2\right](2^{n+2-\alpha})c}{(n+1)(n+2)}L\left(2\beta\right)\right\}\beta^{n+1-\alpha} \\ &+ O\left(\beta^{n+(\alpha-2)^2-1}L_1(\beta)\right). \qquad (15)\end{aligned}$$

Here $B(x,y)$ is the Beta function and,

$$c = \frac{1}{(\alpha-1)(2-\alpha)(3-\alpha)},\ 1 < \alpha < 2,\ \mu_1 = E(\eta_1),\ n \geq 1.$$

Moreover $L_1(x)$ is a slowly varying function defined as (11).

Proof. Define $J(0) = E\left(U\left(\tilde{\zeta}\right)\right)$, then

$$\begin{aligned}
E(\tilde{X}^n) &= \frac{nJ_n(\beta)}{J(0)} \\
&= \frac{\frac{1}{2\beta}\left\{\frac{1}{\mu_1}\frac{(2\beta)^{n+2}}{(n+1)(n+2)} + \frac{1}{\mu_1^2}\frac{nL(2\beta)B(n,4-\alpha)}{(\alpha-1)(2-\alpha)(3-\alpha)}(2\beta)^{n+3-\alpha} + O\left(\beta^{n+(\alpha-2)^2+1}L_1(\beta)\right)\right\}}{\frac{1}{2\beta}\left\{\frac{1}{\mu_1}\frac{(2\beta)^2}{2} + \frac{1}{\mu_1^2}\frac{L(2\beta)}{(\alpha-1)(2-\alpha)(3-\alpha)}(2\beta)^{3-\alpha} + O\left(\beta^{(\alpha-2)^2+1}L_1(\beta)\right)\right\}} \\
&= \left\{\frac{2^{n+1}}{(n+1)(n+2)}\beta^n + \left(\frac{n2^{n+2-\alpha}L(2\beta)cB(n,4-\alpha)}{\mu_1}\right)\beta^{n+1-\alpha} + O\left(\beta^{n+(\alpha-2)^2-1}L_1(\beta)\right)\right\} \\
&\cdot \left\{1 - \left(\frac{L(2\beta)2^{2-\alpha}c}{\mu_1}\right)\beta^{1-\alpha} + O\left(\beta^{(\alpha-2)^2-1}L_1(\beta)\right)\right\} \\
&= \frac{2^{n+1}}{(n+1)(n+2)}\beta^n + \frac{1}{\mu_1}\left\{\frac{\left[(n^3+3n^2+2n)B(n,4-\alpha)-2\right](2^{n+2-\alpha})c}{(n+1)(n+2)}L\left(2\beta\right)\right\}\beta^{n+1-\alpha} \\
&+ O\left(\beta^{n+(\alpha-2)^2-1}L_1(\beta)\right), \ \beta \longrightarrow \infty. \qquad (16)
\end{aligned}$$

□

Asymptotic Expansion (16) is a general formula, and can be used conveniently in order to obtain asymptotic expansion for the moments of the ergodic distribution of the considered process as long as demand random variables belongs to the regularly varying subclass of heavy tailed distributions with infinite variance. Now let us use Asymptotic Expansion (16) on an example by assuming that the demand random variables have regularly varying Pareto distribution with $1 < \alpha < 2$ as follows:

Example 5.1. *Let the conditions of Theorem 5.1 be satisfied. Moreover let $\{\eta_i\}$, $i \geq 1$ be a sequence of i.i.d. and regularly varying Pareto distributed random variables with parameters $b > 0$ and $1 < \alpha < 2$, i.e.:*

$$F(x) = P\{\eta_1 \leq x\} = 1 - \left(\frac{b}{x}\right)^\alpha.$$

Then the asymptotic expansion for the n^{th} order moments of the ergodic distribution of the process $\tilde{X}(t)$ can be obtained as follows:

$$\begin{aligned}
E(\tilde{X}^n) &= \frac{2^{n+1}}{(n+1)(n+2)}\beta^n + \frac{1}{\mu_1}\left\{\frac{\left[(n^3+3n^2+2n)B(n,4-\alpha)-2\right](2^{n+2-\alpha})b^\alpha c}{(n+1)(n+2)}\right\}\beta^{n+1-\alpha} \\
&+ O\left(\beta^{n+(\alpha-2)^2-1}\right) \qquad (17)
\end{aligned}$$

where

$$c = \frac{1}{(\alpha-1)(2-\alpha)(3-\alpha)}, \ 1 < \alpha < 2, \ \mu_1 = E(\eta_1), \ n \geq 1, \ \beta \equiv \frac{(S-s)}{2} \longrightarrow \infty,$$

and $B(x,y)$ is Beta function.

6. Summary and Conclusion

In this work the effect of heavy tailed distributions with infinite variance, were examined on a semi Markovian inventory model of type (s,S). Under the consideration of regularly varying demand quantities with infinite variance, asymptotic expansion for the n^{th} order moments of ergodic distribution of the considered process is obtained. Differently from current literature results of this study are obtained by using different asymptotic expansion for the renewal function U(x), based on the main results of the study by Geluk [14]. By using similar approach a semi Markovian inventory model of type (s,S) can be considered when demand random variables belongs to the different subclasses of heavy tailed distributions. Moreover, other stochastic processes, that incorporate renewal theory such as random walk process can be examined with heavy tailed components in the future.

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References

[1] Aliyev, R. T., (2016), On a stochastic process with a heavy tailed distributed component describing inventory model of type (s,S), Communications in Statistics-Theory and Methods, DOI: 10.1080/03610926.2014.1002932.

[2] Aliyev, R. T., Khaniyev, T., (2014), On the semi-Markovian random walk with Gaussian distribution of summands, Communication in Statistics -Theory and Methods, 43 (1), pp. 90-104.

[3] Aliyev, R. T., Khaniyev, T. A., Kesemen, T. (2010), Asymptotic expansions for the moments of a semi-Markovian random walk with Gamma distributed interference of chance, Communications in Statistics-Theory and Methods, 39 (1), pp. 130-143.

[4] Asmussen, S., (2000), Ruin Probabilities, World Scientific Publishing, Singapore.

[5] Bingham, N. H., Goldie, C. M., Teugels, J. L., (1987), Regular Variation, Cambridge University Press, Cambridge.

[6] Borokov, A. A., Borokov, K. A., (2008), Asymptotic Analysis of Random Walks, Heavy Tailed Distributions, Cambridge University Press, New York.

[7] Brown, M., Solomon, H. A., (1975), Second order approximation for the variance of a renewal reward process and their applications, Stochastic Processes and their Applications, 3, 301-314.

[8] Chevalier, Judith, Austan, G., (2003), Measuring prices and price competition online: Amazon.com and barnesandnoble.com., Quantitative Marketing and Economics, 1 (2), pp. 203-222.

[9] Chen, F., Zheng, Y. S., (1997), Sensitivity analysis of an inventory model of type (s,S) inventory model, Operation Research and Letters, 21, pp. 19-23.

[10] Embrechts, P., Kluppelberg, C., Mikosh, T., (1997), Modelling Extremal Events, Springer Verlag.

[11] Feller, W., (1971), Introduction to Probability Theory and Its Applications II, John Wiley, New York.

[12] Foss, S., Korshunov, D., Zachary, S., (2011), An Introduction to Heavy Tailed and Subexponential Distributions, Springer Series in Operaions Research and Financial Engineering, New York.

[13] Gaffeo, Edoardo, Antonello, E. S., Laura, V., (2008), Demand distribution dynamics in creative industries: The market for books in Italy, Information Economics and Policy, 20 (3), pp. 257-268.

[14] Geluk, J. L., (1997), A renewal theorem in the finite-mean case, Proceedings of the American Mathematical Society, 125 (11), pp. 3407-3413.

[15] Gikhman, I. I., Skorohod, A. V., (1975), Theory of Stochastic Processes II, Springer: Berlin.

[16] Khaniyev, T., Aksop, C., (2011), Asymptotic results for an inventory model of type (s,S) with generalized beta interference of chance, TWMS J.App.Eng.Math., 2, pp. 223-236.

[17] Khaniyev, T., Atalay, K. D., (2010), On the weak convergence of the ergodic distribution for an inventory model of type (s,S), Hacettepe Journal of Mathematics and Statistics, 39 (4), pp. 599-611.

[18] Khaniyev, T., Kokangul, A., Aliyev, R. (2013), An asymptotic approach for a semi Markovian inventory model of type (s,S), Applied Stochastic Models in Business and Industry, 29 (5), pp. 439-453.

[19] Levy, J., Taqqu, M. S, (1987), On renewal processes having stable inter-renewal intervals and stable rewards, Ann.Sci.Math., 11, pp. 95-110.

[20] Resnick, S. I., (2006), Heavy-Tail Phenomena: Probabilistic and Statistical Madeling, Springer Series in Operations Research and Financial Engineering, New York.
[21] Seneta, E., (1976), Regularly Varying Functions, Springer-Verlag, New York.
[22] Sgibnev, M. S., (2009), On a renewal function when second moment is infinite, Statistics and Probability Letters, 79, pp. 1242-1245.
[23] Smith, W. L., (1959), On the cumulants of renewal process, Biometrika 46 (1), pp. 1-29.
[24] Teugels, J. L., (1968), Renewal theorems when the second moment is infinite, The Annals of Mathematical Statistics 39 (4), pp. 1210-1219.

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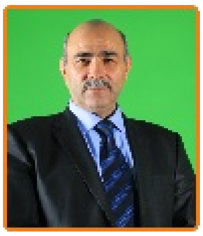

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