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A CHARACTERIZATION OF WAVE PACKET FRAMES FOR $L^2(\mathbb{R}^d)$

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ABSTRACT. In this paper we present necessary and sufficient conditions with explicit frame bounds for a finite sum of wave packet frames to be a frame for $L^2(\mathbb{R}^d)$. Further, we illustrate our results with some examples and applications.

Keywords: Frames, wave packet system.

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1. Introduction and Preliminaries

The wave packet system is a system of functions generated by combined action of translation, dilation and modulation operators on $L^2(\mathbb{R}^d)$. Cordoba and Fefferman in [6] introduced wave packet frames in the study of some classes of singular integral operators. Labate et al. [15] adopted the same expression to describe, more generally, any collection of functions which are obtained by applying the same operations to a finite family of functions in $L^2(\mathbb{R}^d)$. Lacey and Thiele [16, 17] gave applications of wave packet systems in boundedness of the Hilbert transforms. The wave packet systems have been studied by several authors, see [3, 5, 7, 10, 11, 12, 13, 14, 18, 19, 20].

In this paper we consider a system of the form

$$\{D_{A_j} T_{Bk} E_{C_m} \psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d},$$

where $\psi \in L^2(\mathbb{R}^d)$, $\{A_j\}_{j\in\mathbb{Z}} \subset GL_d(\mathbb{R})$, $B \in GL_d(\mathbb{R})$ and $\{C_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}^d$ and call it *wave packet system* in $L^2(\mathbb{R}^d)$. A frame for $L^2(\mathbb{R}^d)$ of the form $\{D_{A_j} T_{Bk} E_{C_m} \psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is called a *wave packet frame*. We present necessary and sufficient conditions for a finite sum of wave packet frames to be a frame for $L^2(\mathbb{R}^d)$ in terms of scalars and frame bounds associated with the given finite sum of wave packet frames in $L^2(\mathbb{R}^d)$. We conclude this paper with some examples and applications.

First we recall basic notations and definitions to make the paper self-contained. The characteristic function of a set E is denoted by χ_E. By $GL_d(\mathbb{R})$ we denote the set of all invertible $d \times d$ matrices over $\mathbb{R}$. Let $a, b \in \mathbb{R}^d$ and C be a real $d \times d$ matrix. We consider

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bounded linear operators $T_a,\ E_b\ D_C : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ given by

Translation by $a \leftrightarrow\ T_a f(t) = f(t-a)$,

Modulation by $b \leftrightarrow\ E_b f(t) = e^{2\pi i b\cdot t} f(t),\ b\cdot t$ denotes the inner product of b and t in $\mathbb{R}^d$,

Dilation by $C \leftrightarrow D_C f(t) = |detC|^{\frac{1}{2}} f(Ct)$.

A countable sequence $\{f_k\} \subset \mathcal{H}$ in a separable real (or complex) Hilbert space $\mathcal{H}$ is called a *frame* (or *Hilbert frame*) for $\mathcal{H}$, if there exist numbers $0 < a_o \leq b_o < \infty$ such that

$$a_o\|f\|^2 \leq \|\{\langle f, f_k\rangle\}\|^2_{\ell^2} \leq b_o\|f\|^2 \text{ for all } f \in \mathcal{H}.$$

The numbers a_o and b_o are called *lower* and *upper frame bounds* of the frame, respectively. Associated with a frame $\{f_k\}$ for $\mathcal{H}$, there are three bounded linear operators:

$$\textit{synthesis operator} \quad V : \ell^2 \to \mathcal{H}, \quad V(\{c_k\}) = \sum_{k=1}^{\infty} c_k f_k,\ \{c_k\} \in \ell^2,$$

$$\textit{analysis operator} \quad V^* : \mathcal{H} \to \ell^2, \quad V^*(f) = \{\langle f, f_k\rangle\}, \quad f \in \mathcal{H},$$

$$\textit{frame operator} \quad S = VV^* : \mathcal{H} \to \mathcal{H}, \quad S(f) = \sum_{k=1}^{\infty}\langle f, f_k\rangle f_k, \quad f \in \mathcal{H}.$$

The frame operator S is a positive, self-adjoint and invertible operator on $\mathcal{H}$. This gives the *reconstruction formula* for all $f \in \mathcal{H}$,

$$f = SS^{-1}f = \sum_{k=1}^{\infty}\langle S^{-1}f, f_k\rangle f_k \quad \left(= \sum_{k=1}^{\infty}\langle f, S^{-1}f_k\rangle f_k \right).$$

Thus, a frame allows every element in a Hilbert space $\mathcal{H}$ to be written as a linear combination (need not be unique) of the frame elements. This reflects one of the important properties of frames in applied mathematics. For applications of frames in various directions in applied mathematics, see [1, 4, 8, 9].

The following inequality is known as Cauchy-Bunyakovsky-Schwarz inequality, which can be found in [2] (p. 13).

Lemma 1.1. *Let* $x = (\xi_1, \xi_2, ..., \xi_n),\ y = (\eta_1, \eta_2, ..., \eta_n) \in \mathbb{R}^n$. *Then,*

$$\left(\sum_{s=1}^{n} \xi_i\eta_i\right)^2 \leq \sum_{s=1}^{n}\xi_i^2 \ \sum_{s=1}^{n}\eta_i^2.$$

2. Finite Sum of Wave Packet Frames

Let $\{A_j\}_{j\in\mathbb{Z}} \subset GL_d(\mathbb{R})$, $B \in GL_d(\mathbb{R})$, $\{C_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}^d$ and let $\Lambda_p = \{1, 2, 3, ..., p\}$ be a finite subset of $\mathbb{N}$. For each $s \in \Lambda_p$, assume that $\{D_{A_j}T_{Bk}E_{C_m}\psi_s\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ ($\psi_s \in L^2(\mathbb{R}^d)$) is a wave packet frame for $L^2(\mathbb{R}^d)$. We consider a system of finite sum of wave packet frames:

$$\Psi_p \equiv \left\{\sum_{s=1}^{p}\alpha_s D_{A_j}T_{Bk}E_{C_m}\psi_s\right\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}, \text{ where } \alpha_1, \alpha_2, ..., \alpha_p \text{ are scalars.}$$

The finite sum Ψ_p is not a frame for $L^2(\mathbb{R}^d)$, in general.

It would be interesting to know the relation between frame bounds and scalars associated with the sum Ψ_p such that the system Ψ_p constitutes a frame for $L^2(\mathbb{R}^d)$. In this direction, the following theorem provides a sufficient condition for Ψ_p to be a frame for $L^2(\mathbb{R}^d)$.

Theorem 2.1. *Let* $\{D_{A_j}T_{Bk}E_{C_m}\psi_s\}_{\substack{j,m\in\mathbb{Z},k\in\mathbb{Z}^d\\ s\in\Lambda_p}}$ *be a finite family of wave packet frames for* $L^2(\mathbb{R}^d)$ *with frame bounds* a_s, b_s *and let* $\alpha_1, \alpha_2, ..., \alpha_p$ *be any scalars. If*

$$0 \ < \ \sum_{s=1}^{p}|\alpha_s|^2 a_s - \sum_{s,t=1,s\neq t}^{p}|\alpha_s\alpha_t|\sqrt{b_s b_t}, \tag{1}$$

then $\Psi_p \equiv \left\{\sum_{s=1}^{p}\alpha_s D_{A_j}T_{Bk}E_{C_m}\psi_s\right\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ *is a frame for* $L^2(\mathbb{R}^d)$ *with frame bounds* $\beta_o = \left(\sum_{s=1}^{p}|\alpha_s|^2 a_s - \sum_{s,t=1,s\neq t}^{p}|\alpha_s\alpha_t|\sqrt{b_s b_t}\right)$ *and* $\gamma_o = p\sum_{s=1}^{p}|\alpha_s|^2 b_s$.

Proof. By using Lemma 1.1 and upper frame inequality for each of the wave packet frame $\{D_{A_j}T_{Bk}E_{C_m}\psi_s\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ $(s\in\Lambda_p)$, we compute

$$\begin{aligned}
&\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\left|\left\langle \sum_{s=1}^{p}\alpha_s D_{A_j}T_{Bk}E_{C_m}\psi_s, f\right\rangle\right|^2\\
&\leq p\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}|\alpha_1\langle D_{A_j}T_{Bk}E_{C_m}\psi_1, f\rangle|^2 + p\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}|\alpha_2\langle D_{A_j}T_{Bk}E_{C_m}\psi_2, f\rangle|^2 + ...\\
&\qquad + p\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}|\alpha_p\langle D_{A_j}T_{Bk}E_{C_m}\psi_p, f\rangle|^2\\
&\leq p\left(\sum_{s=1}^{p}|\alpha_s|^2\ b_s\right)\|f\|^2 \text{ for all } f\in L^2(\mathbb{R}^d).
\end{aligned}\tag{2}$$

Therefore, upper frame condition for Ψ_p

is satisfied. For lower frame condition, we compute

$$\begin{aligned}
&\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\left|\left\langle \sum_{s=1}^{p}\alpha_s D_{A_j}T_{Bk}E_{C_m}\psi_s, f\right\rangle\right|^2\\
&= \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\Big[\sum_{s=1}^{p}|\alpha_s|^2\ |\langle D_{A_j}T_{Bk}E_{C_m}\psi_s, f\rangle|^2\\
&\qquad + \sum_{s,t=1,s\neq t}^{p}\alpha_s\overline{\alpha_t}\langle D_{A_j}T_{Bk}E_{C_m}\psi_s, f\rangle\overline{\langle D_{A_j}T_{Bk}E_{C_m}\psi_t, f\rangle}\ \Big]\\
&\geq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}\Big[\sum_{s=1}^{p}|\alpha_s|^2\ |\langle D_{A_j}T_{Bk}E_{C_m}\psi_s, f\rangle|^2\\
&\qquad - \sum_{s,t=1,s\neq t}^{p}|\alpha_s\alpha_t|\ |\langle D_{A_j}T_{Bk}E_{C_m}\psi_s, f\rangle|\ |\overline{\langle D_{A_j}T_{Bk}E_{C_m}\psi_t, f\rangle}|\ \Big]\\
&= \sum_{s=1}^{p}|\alpha_s|^2\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}|\langle D_{A_j}T_{Bk}E_{C_m}\psi_s, f\rangle|^2\\
&\qquad - \sum_{s,t=1,s\neq t}^{p}|\alpha_s\ \alpha_t|\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}|\langle D_{A_j}T_{Bk}E_{C_m}\psi_s, f\rangle|\ |\langle D_{A_j}T_{Bk}E_{C_m}\psi_t, f\rangle|\\
&\geq \left(\sum_{s=1}^{p}|\alpha_s|^2 a_s\ -\sum_{s,t=1,s\neq t}^{p}|\alpha_s\alpha_t|\sqrt{b_s b_t}\right)\|f\|^2 \text{ for all } f\in L^2(\mathbb{R}^d). \qquad (3)
\end{aligned}$$

By (2) and (3), we conclude that Ψ_p is a frame for $L^2(\mathbb{R}^d)$. □

We now demonstrate by a concrete example that the condition given in Theorem 2.1 is sufficient but not necessary.

Example 2.1. *Let $\Lambda_2 = \{1, 2\} \subset \mathbb{N}$, $E_{C_m} = I_{L^2(\mathbb{R}^d)}$ (the identity operator on $L^2(\mathbb{R}^d)$) for all $m \in \mathbb{Z}$ and let A be any expansive $d \times d$ matrix (i.e., every eigenvalue ζ of A satisfies $|\zeta| > 1$). Choose $A_j = A^j$ for all $j \in \mathbb{Z}$. Then, there exist $\psi \in L^2(\mathbb{R}^d)$ such that $\hat{\psi} = \chi_E$, where E is a compact subset of $\mathbb{R}^d$, $\hat{\psi}$ is the Fourier transform of ψ and $\{D_{A_j}T_{Bk}E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} = \{D_{A^j}T_{Bk}\psi\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$ (see* [9], *p. 357). Thus, $\{D_{A_j}T_{Bk}E_{C_m}\psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is a wave packet frame for $L^2(\mathbb{R}^d)$ with frame bounds $a_o = b_o = 1$.*

Choose $\alpha_1 = \alpha_2 = 1$ and $\psi_1 = \psi_2 = \psi$. Then, $\{D_{A_j}T_{Bk}E_{C_m}\psi_s\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^d)$ with bounds $a_s = 1$, $b_s = 1$ ($s \in \Lambda_2$). Hence the finite sum Ψ_2 is a frame for $L^2(\mathbb{R}^d)$ with frame bounds $\beta_o = \gamma_o = 2$.

To show that the inequality (1) *does not holds, we compute*

$$\begin{aligned}
\sum_{s=1}^{2}|\alpha_s|^2 a_s - \sum_{s,t=1,s\neq t}^{2}|\alpha_s\alpha_t|\sqrt{b_s b_t} &= |\alpha_1|^2 a_1 + |\alpha_2|^2 a_2 - |\alpha_1\alpha_2|\sqrt{b_1 b_2} - |\alpha_2\alpha_1|\sqrt{b_2 b_1}\\
&= 1 + 1 - 1 - 1\\
&= 0.
\end{aligned}$$

Hence the condition (1) *in Theorem 2.1 is not satisfied.*

Next, we give necessary conditions for the finite sum Ψ_p to be a frame for $L^2(\mathbb{R}^d)$ in terms of frame bounds of frames associated with the sum Ψ_p and frame bounds of Ψ_p.

Theorem 2.2. *Let* $\{D_{A_j}T_{Bk}E_{C_m}\psi_s\}_{\substack{j,m\in\mathbb{Z},k\in\mathbb{Z}^d\\ s\in\Lambda_p}}$ *be a finite family of wave packet frames for* $L^2(\mathbb{R}^d)$ *with bounds* a_s, b_s *and let* $\alpha_1, \alpha_2, ..., \alpha_p$ *be any scalars. If* $\Psi_p \equiv \left\{\sum\limits_{s=1}^{p} \alpha_s D_{A_j}T_{Bk}E_{C_m}\psi_s\right\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ *is a frame for* $L^2(\mathbb{R}^d)$ *with frame bounds* $a_o,\ b_o$, *then*

$$\sum_{s=1}^{p} |\alpha_s|^2 a_s - \sum_{s,t=1,s\neq t}^{p} |\alpha_s\alpha_t|\sqrt{b_s b_t} \ \leq\ b_o \tag{4}$$

and

$$\sum_{s=1}^{p} |\alpha_s|^2 b_s + \sum_{s,t=1,s\neq t}^{p} |\alpha_s\alpha_t|\sqrt{b_s b_t} \ \geq\ a_o. \tag{5}$$

Proof. We compute

$$\begin{aligned}
a_o\|f\|^2 &\leq \textstyle\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left|\left\langle \sum\limits_{s=1}^{p} \alpha_s D_{A_j}T_{Bk}E_{C_m}\psi_s, f\right\rangle\right|^2 \\
&= \textstyle\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} [\ \sum\limits_{s=1}^{p} |\alpha_s|^2\ |\langle D_{A_j}T_{Bk}E_{C_m}\psi_s, f\rangle|^2 \\
&\quad + \sum\limits_{s,t=1,s\neq t}^{p} \alpha_s\overline{\alpha_t}\langle D_{A_j}T_{Bk}E_{C_m}\psi_s, f\rangle\overline{\langle D_{A_j}T_{Bk}E_{C_m}\psi_t, f\rangle}\ ] \\
&\leq \textstyle\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} [\ \sum\limits_{s=1}^{p} |\alpha_s|^2\ |\langle D_{A_j}T_{Bk}E_{C_m}\psi_s, f\rangle|^2 \\
&\quad + \sum\limits_{s,t=1,s\neq t}^{p} |\alpha_s\ \alpha_t|\ |\langle D_{A_j}T_{Bk}E_{C_m}\psi_s, f\rangle|\ |\overline{\langle D_{A_j}T_{Bk}E_{C_m}\psi_t, f\rangle}|\ ]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{s=1}^{p} |\alpha_s|^2\ \textstyle\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle D_{A_j}T_{Bk}E_{C_m}\psi_s, f\rangle|^2 \\
&\quad + \sum_{s,t=1,s\neq t}^{p} |\alpha_s\ \alpha_t| \textstyle\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle D_{A_j}T_{Bk}E_{C_m}\psi_s, f\rangle|\ |\langle D_{A_j}T_{Bk}E_{C_m}\psi_t, f\rangle| \\
&\leq \left(\sum_{s=1}^{p} |\alpha_s|^2 b_s\ +\ \sum_{s,t=1,s\neq t}^{p} |\alpha_s\alpha_t|\sqrt{b_s b_t}\right) \|f\|^2 \text{ for all } f \in L^2(\mathbb{R}^d).
\end{aligned}$$

Therefore, $\sum_{s=1}^{p} |\alpha_s|^2 b_s + \sum_{s,t=1,s\neq t}^{p} |\alpha_s\alpha_t|\sqrt{b_s b_t} \ \geq \ a_o$. The inequality (5) is proved. To prove the inequality (4), we compute

$$\begin{aligned}
b_o\|f\|^2 &\geq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left|\left\langle \sum_{s=1}^{p} \alpha_s D_{A_j} T_{Bk} E_{C_m}\psi_s, f \right\rangle\right|^2 \\
&= \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} [\sum_{s=1}^{p} |\alpha_s|^2 \ |\langle D_{A_j} T_{Bk} E_{C_m}\psi_s, f\rangle|^2 \\
&\qquad + \sum_{s,t=1,s\neq t}^{p} \alpha_s\overline{\alpha_t}\langle D_{A_j} T_{Bk} E_{C_m}\psi_s, f\rangle\overline{\langle D_{A_j} T_{Bk} E_{C_m}\psi_t, f\rangle} \ ] \\
&\geq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} [\sum_{s=1}^{p} |\alpha_s|^2 \ |\langle D_{A_j} T_{Bk} E_{C_m}\psi_s, f\rangle|^2 \\
&\qquad - \sum_{s,t=1,s\neq t}^{p} |\alpha_s \ \alpha_t| \ |\langle D_{A_j} T_{Bk} E_{C_m}\psi_s, f\rangle| \ |\overline{\langle D_{A_j} T_{Bk} E_{C_m}\psi_t, f\rangle}| \ ] \\
&= \sum_{s=1}^{p} |\alpha_s|^2 \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle D_{A_j} T_{Bk} E_{C_m}\psi_s, f\rangle|^2 \\
&\qquad - \sum_{s,t=1,s\neq t}^{p} |\alpha_s \ \alpha_t| \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle D_{A_j} T_{Bk} E_{C_m}\psi_s, f\rangle| \ |\langle D_{A_j} T_{Bk} E_{C_m}\psi_t, f\rangle| \\
&\geq \left(\sum_{s=1}^{p} |\alpha_s|^2 a_s \ - \sum_{s,t=1,s\neq t}^{p} |\alpha_s\alpha_t|\sqrt{b_s b_t} \right) \|f\|^2 \text{ for all } f \in L^2(\mathbb{R}^d).
\end{aligned}$$

This gives

$$\sum_{s=1}^{p} |\alpha_s|^2 a_s - \sum_{s,t=1,s\neq t}^{p} |\alpha_s\alpha_t|\sqrt{b_s b_t} \ \leq \ b_o.$$

The theorem is proved. □

Remark 2.1. *The conditions* (4) *and* (5) *in Theorem 2.2 gives a relative estimate of frame bounds of frames associated with the finite system* Ψ_p.

The following theorem characterize the finite sum Ψ_p of wave packet frames as a frame for $L^2(\mathbb{R}^d)$.

Theorem 2.3. *Let* $\{D_{A_j} T_{Bk} E_{C_m}\psi_s\}_{\substack{j,m\in\mathbb{Z},k\in\mathbb{Z}^d \\ s\in\Lambda_p}}$ *be a finite family of wave packet frames for* $L^2(\mathbb{R}^d)$. *Then,* $\Psi_p \equiv \left\{\sum_{s=1}^{p} \alpha_s D_{A_j} T_{Bk} E_{C_m}\psi_s\right\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ *is a frame for* $L^2(\mathbb{R}^d)$ *if and only if there exists* $\mu > 0$ *and some* $\nu \in \Lambda_p$ *such that*

$$\mu \sum_{j,m\in\mathbb{Z},k\in Z^d} |\langle D_{A_j} T_{Bk} E_{C_m}\psi_\nu, f\rangle|^2 \leq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left|\left\langle \sum_{s=1}^{p} \alpha_s D_{A_j} T_{Bk} E_{C_m}\psi_s, f \right\rangle\right|^2, \qquad (6)$$

for all $f \in L^2(\mathbb{R}^d)$ *and for any finite scalars* $\{\alpha_s\}_{s=1}^{p}$.

Proof. Assume first that Ψ_p is a frame for $L^2(\mathbb{R}^d)$ with frame bounds a_o, b_o. Then, for all $f \in L^2(\mathbb{R}^d)$, we have

$$\|f\|^2 \leq \frac{1}{a_o} \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left| \left\langle \sum_{s=1}^{p} \alpha_s D_{A_j} T_{Bk} E_{C_m} \psi_s, f \right\rangle \right|^2. \tag{7}$$

Choose $\mu = \frac{a_o}{b_\nu} > 0$, where b_ν is an upper frame bound of the frame $\{D_{A_j} T_{Bk} E_{C_m} \psi_\nu\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$. Then, by using (7), for all $f \in L^2(\mathbb{R}^d)$, we have

$$\mu \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle D_{A_j} T_{Bk} E_{C_m} \psi_\nu, f\rangle|^2 \leq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left| \left\langle \sum_{s=1}^{p} \alpha_s D_{A_j} T_{Bk} E_{C_m} \psi_s, f \right\rangle \right|^2.$$

For the reverse part, let a_s, b_s be frame bounds of the frame $\{D_{A_j} T_{Bk} E_{C_m} \psi_s\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ $(1 \leq s \leq p)$. Then, by using (6), we have

$$\begin{aligned} \mu a_\nu \|f\|^2 &\leq \mu \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle D_{A_j} T_{Bk} E_{C_m} \psi_\nu, f\rangle|^2 \\ &\leq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left| \left\langle \sum_{s=1}^{p} \alpha_s D_{A_j} T_{Bk} E_{C_m} \psi_s, f \right\rangle \right|^2, \ f \in L^2(\mathbb{R}^d). \end{aligned} \tag{8}$$

By using Lemma 1.1 (choose $\xi_s = 1$ and $\eta_s = |\alpha_s \langle D_{A_j} T_{Bk} E_{C_m} \psi_s, f\rangle|, (1 \leq s \leq p)$), we compute

$$\begin{aligned} &\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left| \left\langle \sum_{s=1}^{p} \alpha_s D_{A_j} T_{Bk} E_{C_m} \psi_s, f \right\rangle \right|^2 \\ &= \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left| \sum_{s=1}^{p} \alpha_s \langle D_{A_j} T_{Bk} E_{C_m} \psi_s, f\rangle \right|^2 \\ &\leq \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left[\sum_{s=1}^{p} |\alpha_s \langle D_{A_j} T_{Bk} E_{C_m} \psi_s, f\rangle| \right]^2 \\ &\leq p \sum_{s=1}^{p} \left(|\alpha_s|^2 \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle D_{A_j} T_{Bk} E_{C_m} \psi_s, f\rangle|^2 \right) \\ &\leq \left(p \max_{1\leq s\leq p} |\alpha_s|^2 \sum_{s=1}^{p} b_s \right) \|f\|^2, \ f \in L^2(\mathbb{R}^d). \end{aligned} \tag{9}$$

By (8) and (9), we conclude that Ψ_p is a frame for $L^2(\mathbb{R}^d)$. The theorem is proved. □

Example 2.2. *Let $\Lambda_n = \{1, 2, ..., n\} \subset \mathbb{N}$, $E_{C_m} = I_{L^2(\mathbb{R}^d)}$ (the identity operator on $L^2(\mathbb{R}^d)$) for all $m \in \mathbb{Z}$ and let A be any expansive $d \times d$ matrix (i.e., every eigenvalue ζ of A satisfies $|\zeta| > 1$). Choose $A_j = A^j$ for all $j \in \mathbb{Z}$. Then, there exist $\psi \in L^2(\mathbb{R}^d)$ such that $\hat{\psi} = \chi_E$, where E is a compact subset of $\mathbb{R}^d$, $\hat{\psi}$ is the Fourier transform of ψ and $\{D_{A_j} T_{Bk} E_{C_m} \psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} = \{D_{A^j} T_{Bk} \psi\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$ (see [9], p. 357). Thus, $\{D_{A_j} T_{Bk} E_{C_m} \psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is a wave packet frame for $L^2(\mathbb{R}^d)$*

(i) *Let* $\psi_s = \psi$*, for all* $s \in \Lambda_n$*.*
Then

$$\sum_{s=1}^{n} D_{A_j} T_{Bk} E_{C_m} \psi_s = \sum_{s=1}^{n} D_{A^j} T_{Bk} \psi_s$$
$$= n D_{A^j} T_{Bk} \psi.$$

Now $\{D_{A^j} T_{Bk} \psi\}_{j\in\mathbb{Z},k\in\mathbb{Z}^d}$ *is an orthonormal basis for* $L^2(\mathbb{R}^d)$*.*
Therefore $\{\sum\limits_{s=1}^{n} D_{A_j} T_{Bk} E_{C_m} \psi_s\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ *is a frame for* $L^2(\mathbb{R}^d)$*.*

(ii) *If* $\psi_s = \psi$*, for all* $s = 1, 3, ..., n-1$ *and* $\psi_s = -\psi$ *for* $s = 2, 4, ..., n$*. Then, for even positive integer* n*, we have* $\sum\limits_{s=1}^{n} D_{A_j} T_{Bk} E_{C_m} \psi_s = 0$*. Hence* $\{\sum\limits_{s=1}^{n} D_{A_j} T_{Bk} E_{C_m} \psi_s\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ *is not a frame for* $L^2(\mathbb{R}^d)$*. However, for each* $s \in \Lambda_n$*,* $\{D_{A_j} T_{Bk} E_{C_m} \psi_n\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ *is a frame for* $L^2(\mathbb{R}^d)$*.*

2.1. **Application:** The following example gives an application of Theorem 2.3.

Example 2.3. *Let* $\{D_{A_j} T_{Bk} E_{C_m} \psi\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ *be a wave packet frame for* $L^2(\mathbb{R}^d)$*. Choose* $\psi_s = \psi$*,* $s \in \Lambda_p$ *and* $\alpha_1, ..., \alpha_p$ *be any scalars such that* $\sum\limits_{s=1}^{p} \alpha_s \neq 0$*.*
We compute

$$\begin{aligned}
\sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left|\left\langle \sum_{s=1}^{p} \alpha_s D_{A_j} T_{Bk} E_{C_m} \psi_s, f \right\rangle\right|^2 &= \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left|\left\langle \sum_{s=1}^{p} \alpha_s D_{A_j} T_{Bk} E_{C_m} \psi, f \right\rangle\right|^2 \\
&= \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left|\left\langle \left(\sum_{s=1}^{p} \alpha_s\right) D_{A_j} T_{Bk} E_{C_m} \psi, f \right\rangle\right|^2 \\
&= \left|\sum_{s=1}^{p} \alpha_s\right|^2 \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle D_{A_j} T_{Bk} E_{C_m} \psi, f\rangle|^2.
\end{aligned}$$

Choose $\mu = \left|\sum\limits_{s=1}^{p} \alpha_s\right|^2 > 0$*. Then, for any* $\nu \in \Lambda_p$ *we have*

$$\mu \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} |\langle D_{A_j} T_{Bk} E_{C_m} \psi_\nu, f\rangle|^2 = \sum_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d} \left|\left\langle \sum_{s=1}^{p} \alpha_s D_{A_j} T_{Bk} E_{C_m} \psi_s, f \right\rangle\right|^2.$$

Therefore, by Theorem 2.3, the finite sum Ψ_p *is a frame for* $L^2(\mathbb{R}^d)$*.*

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