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HESITANT FUZZY IDEAL EXTENSION IN PO-SEMIGROUPS

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Abstract. In this paper, the notions of extension of hesitant fuzzy ideals, hesitant fuzzy prime ideals, hesitant fuzzy semiprime ideals and hesitant 3-prime fuzzy ideals in po-semigroups are introduced with some of their properties are investigated. We discuss the relationship between between prime(semi prime) ideals and 3-prime ideals in po-semigroup by means of the extensions hesitant fuzzy ideals.

Keywords: Hesitant fuzzy ideal, hesitant fuzzy ideal extension, hesitant fuzzy (prime, semiprime)ideal, 3-prime hesitant fuzzy ideal

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1. Introduction

Given a set S, a fuzzy subset of S (or a fuzzy set in S) is described as an arbitrary mapping f : $S \to$ [0,1], where [0,1] is the usual interval of real numbers. The concept of fuzzy set was introduced by Zadeh[19]. Rosenfeld[13] gave the concept of fuzzy groups and fuzzy semigroups by Kuroki[8, 9]. Kehayopulu and Tsingelis[7] introduced fuzzy bi-ideals in po-semigroups (ordered semigroups). Xie et al.[16, 17, 18] introduced the idea of extensions of fuzzy ideals in semigroups. Torra[14, 15] initiated the hesitant fuzzy set theory. Hesitant fuzzy sets have attracted the attention of many researchers in a short period of time because hesitant situations are very common in different real world problems. Hesitant fuzzy set theory has been applied to different algebraic structures. Jun et al.[4, 5, 6] applied the notion of hesitant fuzzy sets to semigroups and hesitant fuzzy soft sets to subalgebras and BCK/BCI-algebras. They introduced the notion of hesitant fuzzy semigroups and hesitant fuzzy left (resp right ideals). Abbasi et al.[1, 2] applied the notion of hesitant fuzzy sets to po-semigroups.

Our main aim in this study is to use the idea of Torra and Xie to introduce the notion of extension of hesitant fuzzy ideals and 3-prime hesitant fuzzy ideals in po-semigroups and investigate some related properties. We study the relationships between hesitant fuzzy prime ideals and 3-prime hesitant fuzzy ideals in po-semigroups. We give an example

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to show that a 3-prime hesitant fuzzy ideal is not necessarily prime in a po-semigroup. Furthermore, we show that in a po-semigroup with identity element, the 3-prime hesitant fuzzy ideal coincides with the hesitant fuzzy prime ideal.

2. Preliminaries

Throughout the paper unless otherwise mentioned S denotes a po-semigroup. In this section we discuss some elementary definitions and results that we use in the sequel.

Definition 2.1. *Let S be a reference set, a hesitant fuzzy set on S is a function H that returns a subset of values in [0,1]:*

$$H : S \to \mathcal{P}([0,1])$$

where $\mathcal{P}([0,1])$ denotes the set of all subsets of [0,1].

Definition 2.2. *An ordered semigroup (or po-semigroup) is a system $(S, ., \leq)$ satisfying the following properties:*
(1) $(S, .)$ is a semigroup;
(2) $(S, \leq)$ is a poset;
(3) for all $x \in S, a \leq b$ implies $xa \leq xb$ and $ax \leq bx$.

Definition 2.3. *Let S be a po-semigroup. A hesitant fuzzy set H is called a hesitant fuzzy subsemigroup of S if it satisfies:*

$$(\forall\ x, y\ \in S)(H(x) \cap H(y) \subseteq H(xy)).$$

Definition 2.4. *Let S be a po-semigroup. A hesitant fuzzy set H on S is called a hesitant fuzzy left ideal of S if it satisfies :*
(1) $(\forall\ x, y \in S)\ x \leq y \Rightarrow H(x) \supseteq H(y)$
(2) $(\forall\ x, y \in S)\ H(y) \subseteq H(xy)$

Definition 2.5. *Let S be a po-semigroup. A hesitant fuzzy set H on S is called a hesitant fuzzy right ideal of S if it satisfies:*
(1) $(\forall\ x, y \in S)\ x \leq y \Rightarrow H(x) \supseteq H(y)$
(2) $(\forall\ x, y \in S)\ H(x) \subseteq H(xy)$

Definition 2.6. *A hesitant fuzzy subset H of a po-semigroup S is called a hesitant fuzzy ideal of S, if it is both a hesitant fuzzy left ideal and a hesitant fuzzy right ideal of S.*

Example 2.1. Let $S = \{a, b, c, d\}$ be the po-semigroup with the following multiplication table and the order below:

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$\leq := \{(a, a); (a, b); (b, b); (c, c); (d, d)\}$.
Clearly S is a po-semigroup.
Define a hesitant fuzzy subset H of S such that

$$H(a) = [0, 1];\ H(b) = [0.1, 0.9];\ H(c) = \emptyset;\ H(d) = [0.2, 0.8]$$

Clearly

$$H(xy) = \begin{cases} H(b) \text{ if } (x,y) := \begin{cases} (c,c) \\ (d,d) \\ (d,c) \end{cases} \\ H(a) \text{ otherwise} \end{cases}$$

for every x and y of S. This implies that $H(y) \subseteq H(xy)$ and $H(x) \subseteq H(xy)$. Moreover for all $x, y \in S,\ x \leq y$ we have $H(x) \supseteq H(y)$. Hence H is a hesitant fuzzy ideal on S.

Definition 2.7. *Let S be a po-semigroup. A hesitant fuzzy subsemigroup H of S is called a hesitant fuzzy bi-ideal of S if it satisfies:*
(1) $(\forall\ x, y \in S)\ x \leq y \Rightarrow H(x) \supseteq H(y)$
(2) $(\forall\ x, y, z \in S)\ (H(xyz) \supseteq H(x) \cap H(z))$

Definition 2.8. *A po-semigroup S is called left(right) regular if for every $a \in S$ there exists $x \in S$ such that $a \leq xa^2$ (resp., $a \leq a^2x$).*

Definition 2.9. *A po-semigroup S is called regular if for every $a \in S$, there exists $x \in S$ such that $a \leq axa$.*

Definition 2.10. *A po-semigroup S is called intra regular if for every $a \in S$ there exists $x, y \in S$ such that $a \leq xa^2y$.*

Lemma 2.1. [1] *Let S be left regular po-semigroup. Then for every hesitant fuzzy left ideal H of S, $H(a) = H(a^2)$ holds for all $a \in S$.*

Proof. Let H be any hesitant fuzzy ideal of S and a be any element of S. Then, since S is left regular, there exist an element x in S such that $a \leq xa^2$. Then we have
$H(a^2) \supseteq H(a) \supseteq H(x(a^2)) \supseteq H(a^2)$ and so $H(a) = H(a^2)$. □

3. Hesitant fuzzy ideals extensions in po-semigroups

Definition 3.1. *Let S be a po-semigroup, H a hesitant fuzzy subset of $S, x \in S$. The hesitant fuzzy subset $< x, H >$ of S defined by:*

$$< x, H > (y) = H(xy), \forall y \in S$$

is called the extension of H by x.

Example 3.1. Let $S = \{a, b, c, d\}$ be the po-semigroup with the following multiplication table and the order below:

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$\leq := \{(a, a); (a, b); (b, b); (c, c); (d, d)\}$.
Clearly S is a po-semigroup.
Let H be a hesitant fuzzy subset of S such that

$$H(x) = \begin{cases} [0,1] & \text{if } x = a \\ \{0.1, 0.2\} & \text{if } x = b \\ \{0.2\} & \text{if } x = c, d \end{cases}$$

For $x = a$, the hesitant fuzzy subset $< a, H >$ of S is defined by

$$< a, H > (y) \ = \ [0,1] \ \forall\ y \in S.$$

For $x = b$, the hesitant fuzzy subset $< b, H >$ of S is defined by

$$< b, H > (y) \;=\; [0, 1] \;\forall\; y \in S.$$

For $x = c$, the hesitant fuzzy subset $< c, H >$ of S is defined by

$$< c, H > (y) = \begin{cases} [0,1] & \text{if } y = a, b, d \\ \{0.1, 0.2\} & \text{if } y = c \end{cases}$$

For $x = d$, the hesitant fuzzy subset $< d, H >$ of S is defined by

$$< d, H > (y) = \begin{cases} [0,1] & \text{if } y = a, b \\ \{0.1, 0.2\} & \text{if } y = c, d \end{cases}$$

Lemma 3.1. *Let S be a commutative po-semigroup. If H is a hesitant fuzzy ideal of S and $x \in S$, then the extension of H by x is a hesitant fuzzy ideal of S.*

Proof. Let H be a hesitant fuzzy ideal of S. For any $x \in S, < x, H >$ is a hesitant fuzzy subset of S. Let $y, z \in S$ such that $y \leq z$. Then

$$xy \leq xz \Rightarrow H(xy) \supseteq H(xz) \Rightarrow < x, H > (y) \supseteq < x, H > (z).$$

Also,

$$< x, H > (yz) = H(xyz) \supseteq H(xy) =< x, H > (y).$$

Thus $< x, H >$ is a hesitant fuzzy right ideal of S. Hence S being commutative $< x, H >$ is a hesitant fuzzy ideal of S. □

Remark 3.1. *Commutativity of a po-semigroup S is not necessary to prove that$< x, H >$ is a hesitant fuzzy right ideal of S, when H is a hesitant fuzzy right ideal of S.*

Definition 3.2. *Let S be a po-semigroup and H be a hesitant fuzzy subset of S. Then we define*

$$\text{supp H} = \{\text{x} \in \text{S} : \text{H(x)} \neq \emptyset\}.$$

Theorem 3.1. *Let S be a po-semigroup and H be a hesitant fuzzy ideal of S and $x \in S$. Then the following hold:*

(1) $H \subseteq < x, H >$.
(2) $< x^n, H > \subseteq < x^{n+1}, H >, \forall n \in N$.
(3) If $H(x) \neq \emptyset$, then supp $< x, H >= S$.
(4) If $x \leq y$, then $< x, H > \supseteq < y, H >$.

Proof. (1) Let $y \in S$. Since H is a hesitant fuzzy ideal of S, we have

$$< x, H > (y) = H(xy) \supseteq H(y).$$

Hence $H \subseteq < x, H >$.

(2) Let $n \in N$ and $y \in S$. Since H is a hesitant fuzzy ideal of S, we have

$$\begin{aligned} < x^{n+1}, H > (y) &= H(x^{n+1}y) \\ &= H(xx^n)y \\ &\supseteq H(x^n y) \\ &= < x^n, H > (y) \end{aligned}$$

Hence $< x^n, H > \subseteq < x^{n+1}, H >, \forall n \in N$.

(3) Let $H(x) \neq \emptyset$. Since $< x, H >$ is a hesitant fuzzy subset of S, we have supp $< x, H > \subseteq S$. Let $y \in S$. Since H is a hesitant fuzzy ideal of S, we have

$$< x, H > (y) = H(xy) \supseteq H(x) \neq \emptyset.$$

This implies that $< x, H > (y) \neq \emptyset$ and so $y \in$ supp $< x, H >$. Hence supp $< x, H >= S$.

(4) Let $x \leq y, \forall\ x, y \in S$ and S is a po-semigroup, we have $xz \leq yz\ \forall\ z\ \in S$. Then $< x, H > (z) = H(xz) \supseteq H(yz) =< y, H > (z)$. Hence $< x, H >\supseteq< y, H >$. □

Definition 3.3. *A hesitant fuzzy ideal H in a semigroup S is called hesitant fuzzy semiprime if for all $x \in S$,*

$$H(x) \supseteq H(x^2).$$

Definition 3.4. *A hesitant fuzzy ideal H in a semigroup S is called hesitant fuzzy prime if for all $x, y \in S$,*

$$H(xy) = H(x) \cup H(y).$$

Theorem 3.2. *Let S be a commutativity po-semigroup and H be a hesitant fuzzy prime ideal of S. Then $< x, H >$ is a hesitant fuzzy prime ideal of S and $< x, H >=< x^2, H >$ $\forall x \in S$.*

Proof. Let H be a hesitant fuzzy prime ideal of S. Then by Lemma 3.1, $< x, H >$ is a hesitant fuzzy ideal of S. Let $y, z \in S$. Then

$$\begin{aligned} < x, H > (yz) = H(xyz) &= H(xy) \cup H(z) \\ &= H(x) \cup H(y) \cup H(z) \\ &= (H(x) \cup H(y)) \cup (H(x) \cup H(z)) \\ &= H(xy) \cup H(xz) \\ &= < x, H > (y) \cup < x, H > (z). \end{aligned}$$

Hence $< x, H >$ is a hesitant fuzzy prime ideal of S.
Let $x, y \in S$. We have $< x, H > (y) := H(xy), < x^2, H > (y) := H(x^2y)$. Since H is a hesitant fuzzy prime subset of S, we have $H(x) = H(x^2)$. Then
$< x, H > (y) = H(xy) = H(x) \cup H(y) = H(x^2) \cup H(y) = H(x^2y) =< x^2, H > (y)$.
Hence $< x, H >=< x^2, H > \forall\ x\ \in S$. □

Theorem 3.3. *Let S be a commutativity po-semigroup and H be a hesitant fuzzy semiprime ideal of S. Then $< x, H >$ is a hesitant fuzzy semiprime ideal of S.*

Proof. Let H be a hesitant fuzzy semiprime ideal of S. Then by Lemma 3.1, $< x, H >$ is a hesitant fuzzy ideal of S. Let $a \in S$. Then

$$\begin{aligned} < x, H > (a) = H(xa) &\supseteq H(xa)^2 \\ &= H(xaxa) \\ &= H(xa^2x) \\ &\supseteq H(xa^2) \\ &= < x, H > (a^2) \end{aligned}$$

Hence $< x, H >$ is a hesitant fuzzy semiprime ideal of S. □

Definition 3.5. [18] *Let S be a po-semigroup and $A \subseteq S$ and $x \in S$. Then we define*

$$< x, A >= \{y \in S \mid xy \in A\}.$$

Remark 3.2. *Let S be a po-semigroup and $\emptyset \neq A \subseteq S$. Let $x, y \in S$. Then*

$$< x, H_A > (y) = H_A(xy) = \begin{cases} [0, 1] & \text{if } xy \in A \\ \emptyset & \text{if } xy \notin A \end{cases}$$

On the other hand,

$$H_{<x,A>}(y) = \begin{cases} [0,1] & \text{if } y \in < x, A > \quad \text{i.e; if } xy \in A \\ \emptyset & \text{if } y \notin < x, A > \quad \text{i.e; if } xy \notin A \end{cases}$$

Hence $< x, H_A >= H_{<x,A>}$ *for every* $x \in S$.

Theorem 3.4. *Let* S *be a po-semigroup. If* H *is a hesitant fuzzy prime subset of* S *and* $x \in S$ *such that* $H(x) = \bigcap_{y \in S} H(y)$, *then* $< x, H >= H$.

Proof. Let H be hesitant fuzzy prime subset of S and $x \in S$ such that $H(x) = \bigcap_{y \in S}$ H(y).

$\Rightarrow H(x) \subseteq H(y) \ \forall \ y \in S.$

$\Rightarrow H(x) \cup H(y) = H(y) \ \forall \ y \in S.$

Since H is a hesitant fuzzy prime, we have

$$< x, H > (y) = H(xy) = H(x) \cup H(y) = H(y).$$

Hence $< x, H >= H$. □

Theorem 3.5. *Let* S *be a commutative po-semigroup and* H *be a hesitant fuzzy subset of* S *such that* $< x, H >= H$ *for every* $x \in S$. *Then* H *is a constant function.*

Proof. Let $x, y \in S$. Then by hypothesis we have

$$H(y) = < x, H > (y) = H(xy) = H(yx) = < y, H > (x) = H(x).$$

Hence H is a constant function. □

If H is a hesitant fuzzy ideal of S, we denote by H_ϱ the equivalent relation on S defined by:

$$H_\varrho = \{(x, y) :< x, H >=< y, H >\}$$

Theorem 3.6. *Let* S *be a commutative po-semigroup and* H *be a hesitant fuzzy ideal of* S. *Then*

(1) H_ϱ *is congruence on* S.

(2) If H *is semiprime then* H_ϱ *is a semilattice congruence on* S.

(3) If H *is prime and* $x \leq y$, *then* $(x, xy) \in H_\varrho$.

Proof. (1) To show that H_ϱ is a congruence on S, we need to show that H_ϱ is compatible with the operation on S. Let $(x, y) \in H_\varrho$, $a \in S$. Then $\forall z \in S$, we have

$< xa, H > (z) = H(xaz) =< x, H > (az) =< y, H > (az) = H(yaz) =< ya, H > (z)$

Thus $(xa, ya) \in H_\varrho$. Similarly $(ax, ay) \in H_\varrho$. Hence H_ϱ is a congruence on S.

(2) Let S be a commutative semigroup and H be a hesitant semiprime fuzzy ideal of S. Then for any $y \in S$, we have

$< x, H > (y) = H(xy) \supseteq H(xy)^2 = H(xyxy) = H(yx^2y) \supseteq H(x^2y) =< x^2, H > (y)$

and so $< x^2, H > \subseteq < x, H >$. It follows from Theorem 3.1, $< x^2, H >=< x, H >$. Hence $(x, x^2) \in H_\varrho$.

(3) Let H be prime and $x \leq y$. Therefore, $H(x) \supseteq H(y)$. Then $\forall z \in S$, we have

$$\begin{aligned} < x, H > (z) = H(xz) &= H(y) \cup H(z) \\ &= H(x) \cup H(y) \cup H(z) \\ &= H(xy) \cup H(z) \\ &= H(xyz) =< xy, H > (z). \end{aligned}$$

Hence $< x, H >=< xy, H >$, i.,e $(x, xy) \in H_\varrho$. □

Theorem 3.7. *Let S be a po-semigroup and H be a hesitant fuzzy prime ideal of S. Then $H_\varrho = H_\sigma$,*

$$\text{where } H_\sigma = \{(x, y) : H(x) = H(y) \text{ or } H(xz) = H(yz)\ \forall z \in S\}$$

Proof. Let H be a hesitant fuzzy prime ideal of S and $(x, y) \in H_\varrho$. Then $< x, H >=< y, H >$. For any $z \in S$, we have

$$< x, H > (z) =< y, H > (z)$$

This implies that $H(xz) = H(yz)\ \forall z \in S$. Therefore, $(x, y) \in H_\sigma$.

Conversly, let $(x, y) \in H_\sigma$. Then $H(x) = H(y)$ *or* $H(xz) = H(yz)\ \forall z \in S$
If $H(xz) = H(yz)\ \forall z \in S$, implies that $< x, H >=< y, H >$. Hence $(x, y) \in H_\varrho$.
If $H(x) = H(y)$. Then for each $z \in S$, we have
$H(xz) = H(x) \cup H(z) = H(y) \cup H(z) = H(yz)$
This implies that $< x, H >=< y, H >$. Hence $(x, y) \in H_\varrho$. □

Theorem 3.8. *Let S be a commutative regular po-semigroup and H be a hesitant fuzzy right ideal of S. Then for any $x \in S, < x, H >$ is hesitant fuzzy semiprime in S.*

Proof. Let H be a hesitant fuzzy ideal of S and $a \in S$. Then, since S is regular, there exists an element y in S such that $a \leq aya$. Then for any $x \in S, xa \leq xaya$, we have $H(xa) \supseteq H(xaya)$. Therefore, $< x, H > (a) = H(xa) \supseteq H(xaya) = H(xa^2y) \supseteq H(xa^2) =< x, H > (a^2)$. Hence$< x, H >$ is hesitant fuzzy semiprime in S. □

The left-right dual of Theorem 3.8 reads as follows:

Theorem 3.9. *Let S be a commutative regular po-semigroup and H be a hesitant fuzzy left ideal of S. Then for any $x \in S, < x, H >$ is hesitant fuzzy semiprime in S.*

Theorem 3.10. *Let S be a commutative left regular po-semigroup and H be a hesitant fuzzy ideal of S. Then for any $x \in S, < x, H >$ is hesitant fuzzy semiprime in S.*

Proof. Let H be a hesitant fuzzy ideal of S and $a \in S$. Then, since S is left regular, there exists an element y in S such that $a \leq ya^2$. Then for any $x \in S, xa \geq xya^2$, we have$H(xa) \supseteq H(xya^2)$. Therefore,$< x, H > (a) = H(xa) \supseteq H(xya^2) = H(xa^2y) \supseteq H(xa^2) =< x, H > (a^2)$. Hence $< x, H >$ is hesitant fuzzy semiprime in S. □

Remark 3.3. *Commutativity of a po-semigroup S is not necessary to prove that$< x, H >$ is a hesitant fuzzy semiprime in S, when S is a right regular po-semigroup.*

Theorem 3.11. *Let S be an intra-regular commutative po-semigroup and H be a hesitant fuzzy ideal of S. Then for any $x \in S, < x, H >$ is hesitant fuzzy semiprime in S.*

Proof. Let H be a hesitant fuzzy ideal of S and $a \in S$: Then, since S is intra regular, there exist elements y and z in S such that $a \leq ya^2z$. Then for any $x \in S, xa \leq xya^2z$, we have $H(xa) \supseteq H(xya^2z)$. Therefore $< x, H > (a) = H(xa) = H(xya^2z) = H(yxa^2z) \supseteq H(xa^2) =< x, H > (a^2)$. Hence $< x, H >$ is hesitant fuzzy semiprime in S. □

Theorem 3.12. *If S is a po-semigroup and H a hesitant fuzzy semiprime ideal of S. Then $H = \bigcap_{x \in S} < x, H >$.*

Proof. Let H be a hesitant fuzzy semiprime ideal of S. By Theorem 3.1, $H \subseteq< x, H >$ $\forall x \in S$. This implies that $H \subseteq \bigcap_{x \in S} < x, H >$. Let G be a hesitant fuzzy subset of S such that $G \subseteq< x, H >\ \forall x \in S$. Since H is semiprime, for any $y \in S$ we have

$$G(y) \subseteq< y, H > (y) = H(y^2) \subseteq H(y).$$

Therefore, $G \subseteq H$. Hence $H = \bigcap_{x \in S} < x, H >$. □

Theorem 3.12 is illustrated by the following example:

Example 3.2. Let $S = \{0, 1, 2\}$ be a semigroup with the following cayley table .

.	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

$\leq := \{(0, 0); (1, 1); (2, 2)\}$
Let H be a hesitant fuzzy subset of S such that

$$H(x) = \begin{cases} [0,1] & \text{if } x = 0 \\ \{0.1\} & \text{if } x = 1 \\ \{0.1, 0.2\} & \text{if } x = 2 \end{cases}$$

For any $x, y \in S$. Let one of x, y be 0 then we have
$H(xy) = H(0) \supseteq H(x)$ *and* $H(xy) = H(0) \supseteq H(y)$.
In case x, y be different from 0. Then $H(xy)$ has the following cases:

$$\begin{aligned} H(11) &= H(1) \supseteq H(1) \\ H(12) &= H(2) \supseteq H(1) \\ H(21) &= H(2) \supseteq H(1) \\ H(22) &= H(2) \end{aligned}$$

Thus $H(xy) \supseteq H(x)$ and $H(xy) \supseteq H(y)$ $\forall x, y \in S$
Hence H is a hesitant fuzzy ideal of S.
Further $H(x^2) \subseteq H(x) \forall x \in S$. Hence H is a hesitant fuzzy semiprime ideal of S. For $x = 0$, we have

$$< 0, H > (y) = [0, 1] \ \forall \ y \in S.$$

For $x = 1$, we have

$$< 1, H > (y) = \begin{cases} [0,1] & \text{if } y = 0 \\ \{0.1\} & \text{if } y = 1 \\ \{0.1, 0.2\} & \text{if } y = 2 \end{cases}$$

For $x = 2$, we have

$$< 2, H > (y) = \begin{cases} [0,1] & \text{if } y = 0 \\ \{0.1, 0.2\} & \text{if } y = 1, 2 \end{cases}$$

Clearly

$$\begin{aligned} \bigcap_{x \in S} < x, H > &= \begin{cases} [0,1] & \text{if } y = 0 \\ \{0.1\} & \text{if } y = 1 \\ \{0.1, 0.2\} & \text{if } y = 2 \end{cases} \\ &= H. \end{aligned}$$

4. Hesitant 3-prime fuzzy ideals in po-semigroups

Definition 4.1. *A hesitant fuzzy subset H of a po-semigroup S is called 3-prime if for any $x, y, z \in S$,*

$$\begin{aligned} H(xyz) &= H(xy) \cup H(xz) \\ &= H(yx) \cup H(yz) \\ &= H(zx) \cup H(zy) \end{aligned}$$

Theorem 4.1. *Let S be a po-semigroup and H a hesitant fuzzy ideal of S. If H is prime, then H is 3-prime.*

Proof. Let H be any hesitant fuzzy prime ideal of S. Then for any $x, y, z \in S$, we have

$$\begin{aligned} H(xyz) = H((xy)z) &= H(xy) \cup H(z) \\ &\subseteq H(xy) \cup H(xz) \\ &= H(x) \cup H(y) \cup H(z) \\ &= H(xyz) \end{aligned}$$

Therefore, $H(xyz) = H(xy) \cup H(xz)$.
In the same way, we have

$$\begin{aligned} H(xyz) = H(x(yz)) &= H(x) \cup H(yz) \\ &\subseteq H(xy) \cup H(yz) \\ &= H(x) \cup H(y) \cup H(z) \\ &= H(xyz) \end{aligned}$$

Therefore, $H(xyz) = H(xy) \cup H(yz) = H(yx) \cup H(yz)$.
Since H is prime, we have $H(xyz) = H(yzx)$

$$\begin{aligned} H(xyz) = H(yzx) = H(y(zx)) &= H(y) \cup H(zx) \\ &\subseteq H(zy) \cup H(zx) \\ &= H(x) \cup H(y) \cup H(z) \\ &= H(xyz) \end{aligned}$$

Therefore, $H(xyz) = H(zy) \cup H(zx)$.
Hence H is 3-prime. □

In general the 3-prime hesitant fuzzy ideal need not necessarily hesitant fuzzy prime ideal as shown in the following example:

Example 4.1. Let $S = \{0, 1, 2\}$ be a po-semigroup with the following cayley table and the order below:

.	0	1	2
0	0	0	0
1	0	1	0
2	0	0	2

$\leq := \{(0,0); (0,1); (1,1); (2,2)\}$
Let H be a hesitant fuzzy subset of S such that

$$H(x) = \begin{cases} [0,1] & \text{if } x = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

For any $x, y \in S$. Let one of x, y be 0 then we have
$H(xy) = H(0) \supseteq H(x)$ and $H(xy) = H(0) \supseteq H(y)$.

In case x, y be different from 0. Then$H(xy)$ has the following cases:

$$\begin{aligned} H(11) &= H(1) \supseteq H(1) \\ H(12) &= H(0) \supseteq H(1) \\ H(21) &= H(0) \supseteq H(1) \\ H(22) &= H(2) \end{aligned}$$

Thus $H(xy) \supseteq H(x)$ and $H(xy) \supseteq H(y)$ $\forall x, y \in S$
Moreover for all $x, y \in S, x \leq y$ $H(x) \supseteq H(y)$. Hence H is a hesitant fuzzy ideal of S.
For any x, y and $z \in S$, let one of x, y and z be 0 then we have

$$\begin{aligned} H(xyz) = H(0) = [0,1] &= H(xy) \cup H(xz) \\ &= H(yx) \cup H(yz) \\ &= H(zx) \cup H(zy) \end{aligned}$$

In case x, y and z be different from 0. Then $H(xyz)$ has the following cases:

$$\begin{aligned} H(111) &= H(1) = H(11) \cup H(11) \\ H(112) &= H(0) = H(11) \cup H(12) = H(12) \cup H(21) \\ H(222) &= H(2) = H(22) \cup H(22) \\ H(221) &= H(0) = H(22) \cup H(21) = H(21) \cup H(12) \end{aligned}$$

Hence H is a hesitant fuzzy 3-prime ideal of S.
But $H(12) := [0,1] \neq H(1) \cup H(2) := \emptyset$
Therefore, H is not hesitant fuzzy prime.

Theorem 4.2. *Let S be a commutative po-semigroup and H a hesitant fuzzy subset of S. Then H is 3-prime if and only if for each $x \in S$, $< x, H >$ is prime.*

Proof. Let H be 3-prime. Then for any $y, z \in S$, we have

$$\begin{aligned} < x, H > (yz) &= H(xyz) \\ &= H(xy) \cup H(xz) \\ &= < x, H > (y) \cup < x, H > (z) \end{aligned}$$

Hence $< x, H >$ is prime.

Conversely, Let S be a commutative po-semigroup and for each $x \in S, < x, H >$ is prime. Then $\forall$ $y, z \in S$.

$$\begin{aligned} H(xyz) &= < x, H > (yz) \\ &= < x, H > (y) \cup < x, H > (z) \\ &= H(xy) \cup H(xz) \end{aligned}$$

Since S is commutative, $H(xyz) = H(yxz) = H(zxy)$ we have

$$\begin{aligned} H(xyz) &= < y, H > (xz) \\ &= < y, H > (x) \cup < y, H > (z) \\ &= H(yx) \cup H(yz) \end{aligned}$$

In the same way we have

$$\begin{aligned} H(xyz) &= <z, H>(xy) \\ &= <z, H>(x) \cup <z, H>(y) \\ &= H(zx) \cup H(zy) \end{aligned}$$

Hence H is 3-prime. □

Remark 4.1. *Commutativity of a po-semigroup S is not necessary to prove that any extension of H is prime, when H is 3-prime.*

Theorem 4.2 is illustrated by the following example.

Example 4.2. Let $S = \{x, y, z, e\}$ be commutative po-semigroup with the following multiplication table and the order below:

.	x	y	z	e
x	y	y	e	e
y	y	y	e	e
z	e	e	z	e
e	e	e	e	e

$\leq := \{(x, x); (y, y); (z, z); (e, e)\}$.

Define a hesitant fuzzy subset H on S as follows:

$$H(a) = \begin{cases} \{0.1\} \text{ if } a = e \\ \emptyset \quad \text{otherwise} \end{cases}$$

For any $a, b \in S$, we have $H(ab) \supseteq H(a)$ and $H(ab) \supseteq H(b)$. Moreover for all $a, b \in S$, $a \leq b$ we have $H(a) \supseteq H(b)$. Hence H is a hesitant fuzzy ideal of S.

Now we will show for each $a \in S, < a, H >$ is prime; i,e for any $a, b, c \in S$

$$< a, H > (bc) =< a, H > (b) \cup < a, H > (c).$$

If any of a, b and c is e. Then we have

$$< a, H > (bc) = H(e) = \{0.1\} =< a, H > (b) \cup < a, H > (c).$$

Now, assume that a, b and c be different from e. Then we have the following cases:

$$\begin{aligned} < x, H > (xx) &= \emptyset = < x, H > (x) \cup < x, H > (x); \\ < x, H > (xy) &= \emptyset = < x, H > (x) \cup < x, H > (y); \\ < y, H > (yx) &= \emptyset = < y, H > (y) \cup < y, H > (x); \\ < y, H > (yy) &= \emptyset = < y, H > (y) \cup < y, H > (y); \\ < z, H > (zz) &= \emptyset = < z, H > (z) \cup < z, H > (z); \\ < x, H > (xz) &= \{0.1\} =< x, H > (x) \cup < x, H > (z); \\ < x, H > (yz) &= \{0.1\} =< x, H > (y) \cup < x, H > (z); \\ < y, H > (yz) &= \{0.1\} =< y, H > (y) \cup < y, H > (z); \\ < z, H > (zx) &= \{0.1\} =< z, H > (z) \cup < z, H > (x); \\ < z, H > (zy) &= \{0.1\} =< z, H > (z) \cup < z, H > (y). \end{aligned}$$

Hence, in any case

$$< a, H > (bc) =< a, H > (b) \cup \ < a, H > (c) \ \forall \ a, b \in S.$$

Finally, we will show H is 3-prime; i,e for any $a, b, c \in S$

$$H(abc) = H(ab) \cup H(ac).$$

If any of of a, b and c is e. Since S is commutative and $ae = ea = e$, we have

$$H(abc) = H(e) = \{0.1\} = H(ab) \cup H(ac).$$

In case, a, b and c be different from e. Then $H(abc)$ has the following cases:

$$\begin{aligned}
H(x^3) &= \emptyset = H(x^2) \cup H(x^2);\\
H(y^3) &= \emptyset = H(y^2) \cup H(y^2);\\
H(z^3) &= \emptyset = H(z^2) \cup H(z^2);\\
H(x^2y) &= \emptyset = H(xy) \cup H(xy);\\
H(y^2x) &= \emptyset = H(yx) \cup H(yx);\\
H(x^2z) &= \{0.1\} = H(xz) \cup H(xz);\\
H(y^2z) &= \{0.1\} = H(yz) \cup H(yz);\\
H(z^2x) &= \{0.1\} = H(zx) \cup H(zx);\\
H(z^2y) &= \{0.1\} = H(zy) \cup H(zy);\\
H(xyz) &= \{0.1\} = H(xy) \cup H(xz).
\end{aligned}$$

Hence, in any case $H(abc) = H(ab) \cup H(ac) \ \forall \ a, b, c \in S$.

Theorem 4.3. *Let S be a po-semigroup with an identity e and H be any hesitant fuzzy set of S. Then H is 3-prime if and only if H is prime.*

Proof. The proof follows from Theorem 4.1 and Theorem 4.2. □

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