TWMS J. App. and Eng. Math. V.9, No.1, Special Issue, 2019, pp. 30-37

ON SOME PROPERTIES OF HYPER-BESSEL AND RELATED FUNCTIONS

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Abstract. In this study, by using the Hadamard product representation of the hyper-Bessel function and basic concepts in mathematics we investigate the sign of the hyper-Bessel function $x \mapsto \mathcal{J}_{\alpha_d}(x)$ on some sets. Also, we show that the function $x \mapsto \mathcal{J}_{\alpha_d}(x)$ is a decreasing function on $[0, j_{\alpha_d,1})$, and the function $x \mapsto \frac{x\mathbb{I}'_{\alpha_d}\left(\sqrt[d+1]{x}\right)}{\mathbb{I}_{\alpha_d}\left(\sqrt[d+1]{x}\right)}$ is an increasing function on $(0,\infty)$, where $j_{\alpha_d,1}$ and $\mathbb{I}_{\alpha_d}$ denote the first positive zero of the function $\mathcal{J}_{\alpha_d}(x)$ and modified hyper-Bessel function, respectively. In addition, we prove the strictly log-concavity of the functions $\mathcal{J}_{\alpha_d}(x)$ and $J_{\alpha_d}(x)$ on some sets. Moreover, we give some illustrative examples regarding our main results.

Keywords: Decreasing and increasing functions, Hadamard product representation, hyper-Bessel function, log-concavity, modified hyper-Bessel function.

AMS Subject Classification: 33C45, 26A48, 33E20

1. Introduction

Special functions are intensively used in engineering and mathematical physics. Especially, Bessel and modified Bessel functions are some of the most important of them. The Bessel and modified Bessel functions are defined the following infinite series representations (see [13]), respectively,

$$J_\nu(x) = \sum_{n\geq 0} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{x}{2}\right)^{2n+\nu}, x \in \mathbb{R} \tag{1}$$

and

$$I_\nu(x) = \sum_{n\geq 0} \frac{1}{n!\Gamma(\nu+n+1)} \left(\frac{x}{2}\right)^{2n+\nu}, x \in \mathbb{R}, \tag{2}$$

where $\Gamma(x)$ denotes the familiar gamma function. Actually, these functions are solutions of the following homogeneous differential equations,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \tag{3}$$

and

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0, \tag{4}$$

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§ Selected papers of International Conference on Life and Engineering Sciences (ICOLES 2018), Kyrenia, Cyprus, 2-6 September, 2018.

which are known the Bessel and modified Bessel differential equations, respectively. There are some extensions of the above mentioned functions in the literature. It is important mentioning here that numerous investigations have been done on the Bessel and related functions in the recent years. Especially, the authors in the papers [1–4, 7, 8] investigated some geometric properties including univalence, starlikeness and convexity connected with the zeros of the special functions like Bessel, Struve, Lommel, Wright, q-Bessel and hyper-Bessel functions. In addition, in [5] the author gave Turán type inequalities for modified Bessel functions. On the other hand, in [10] the authors determined the location and reality of the zeros of the hyper-Bessel function when $\alpha_i > -1, i \in \{1, 2, \ldots, d\}$. Motivated by earlier works in this field our main aim is to present some new properties of hyper-Bessel and related functions.

2. Basic Concepts and Preliminaries

The hyper-Bessel and modified hyper-Bessel functions are defined by (see [11]):

$$J_{\alpha_d}(x) = \sum_{n\geq 0} \frac{(-1)^n \left(\frac{x}{d+1}\right)^{n(d+1)+\alpha_1+\cdots+\alpha_d}}{n!\Gamma\left(\alpha_1+n+1\right)\ldots\Gamma\left(\alpha_d+n+1\right)} \tag{5}$$

and

$$I_{\alpha_d}(x) = \sum_{n\geq 0} \frac{\left(\frac{x}{d+1}\right)^{n(d+1)+\alpha_1+\cdots+\alpha_d}}{n!\Gamma\left(\alpha_1+n+1\right)\ldots\Gamma\left(\alpha_d+n+1\right)}, \tag{6}$$

respectively. Note that, by putting $d = 1$ and $\alpha_1 = \nu$ in expressions (5) and (6) we have the classical Bessel and modified Bessel functions which are given by (1) and (2), respectively.

The normalized hyper-Bessel function $\mathcal{J}_{\alpha_d}(x)$ and modified hyper-Bessel function $\mathbb{I}_{\alpha_d}(x)$ are defined by

$$J_{\alpha_d}(x) = \frac{\left(\frac{x}{d+1}\right)^{\alpha_1+\cdots+\alpha_d}}{\Gamma(\alpha_1+1)\ldots\Gamma(\alpha_d+1)}\mathcal{J}_{\alpha_d}(x) \tag{7}$$

and

$$I_{\alpha_d}(x) = \frac{\left(\frac{x}{d+1}\right)^{\alpha_1+\cdots+\alpha_d}}{\Gamma(\alpha_1+1)\ldots\Gamma(\alpha_d+1)}\mathbb{I}_{\alpha_d}(x). \tag{8}$$

By combining the equations (5), (7) and (6), (8) we get the following infinite series representations:

$$\mathcal{J}_{\alpha_d}(x) = \sum_{n\geq 0} \frac{(-1)^n \left(\frac{x}{d+1}\right)^{n(d+1)}}{n!(\alpha_1+1)_n\ldots(\alpha_d+1)_n} \tag{9}$$

and

$$\mathbb{I}_{\alpha_d}(x) = \sum_{n\geq 0} \frac{\left(\frac{x}{d+1}\right)^{n(d+1)}}{n!(\alpha_1+1)_n\ldots(\alpha_d+1)_n}, \tag{10}$$

where $(\beta)_n$ is the known Pochhammer symbol and it is defined by $(\beta)_0 = 1$ and $(\beta)_n = \beta(\beta+1)\ldots(\beta+n-1)$for $n \geq 1$.

In [10], Chaggara and Romdhane gave an infinite product representation of the hyper-Bessel function $J_{\alpha_d}(x)$ as follow:

$$J_{\alpha_d}(x) = \frac{\left(\frac{x}{d+1}\right)^{\alpha_1+\cdots+\alpha_d}}{\Gamma(\alpha_1+1)\ldots\Gamma(\alpha_d+1)} \prod_{n\geq 1}\left(1-\frac{x^{d+1}}{j_{\alpha_d,n}^{d+1}}\right), \tag{11}$$

where $j_{\alpha_d,n}$ denotes the nth positive zero of the function $\mathcal{J}_{\alpha_d}(x)$. As a result, the function $\mathcal{J}_{\alpha_d}(x)$ has the following infinite product representation:

$$\mathcal{J}_{\alpha_d}(x) = \prod_{n\geq 1}\left(1-\frac{x^{d+1}}{j_{\alpha_d,n}^{d+1}}\right). \tag{12}$$

It is worth to mention here that the above Hadamard product representation of the hyper-Bessel function has been intensively used to derive our main results. Also, we take advantage of some basic concepts and arithmetic operations in mathematics.

Now, we would like to remind the definition of logarithmic concavity of a function.

Definition 2.1 ([12]). *A function f is said to be log-concave on interval (a, b) if the function $\log f$ is a concave function on (a, b).*

For log-concavity of the function f on (a, b), it is enough to show that one of the following two conditions are true:
i. $\frac{f'}{f}$ monotone decreasing on (a, b).
ii. $\log f'' < 0$.

Also the following lemma due to Biernacki and Krzyż (see [9]) will be required to prove our second theorem.

Lemma 2.1. *Consider the power series $f(x) = \sum_{n\geq 0} a_n x^n$ and $g(x) = \sum_{n\geq 0} b_n x^n$, where $a_n \in \mathbb{R}$ and $b_n > 0$ for all $n \in \{0, 1, \ldots\}$, and suppose that both converge on $(-r, r), r > 0$. If the sequence $\{\frac{a_n}{b_n}\}_{n\geq 0}$ is increasing(decreasing), then the function $x \mapsto \left(\frac{f(x)}{g(x)}\right)$ is also increasing(decreasing) on $(0, r)$.*

3. Main Results

In this section we present our main results.

Theorem 3.1. *Let $\alpha_i > -1$ for $i \in \{1, 2, \ldots, d\}$, $n \in \mathbb{N}$ and $j_{\alpha_d,n}$ be nth positive zero of hyper-Bessel function $\mathcal{J}_{\alpha_d}(x)$. Further, consider the sets*

$$\Delta_1 = \bigcup_{n\geq 1}\left(j_{\alpha_d,2n-1}, j_{\alpha_d,2n}\right), \Delta_2 = \bigcup_{n\geq 1}\left(j_{\alpha_d,2n}, j_{\alpha_d,2n+1}\right) \text{ and } \Delta_3 = \bigcup_{n\geq 1}\left(0, j_{\alpha_d,n}\right).$$

The following statements are true:
a. *the function $x \mapsto \mathcal{J}_{\alpha_d}(x)$ is negative on Δ_1 and positive on $\Delta_2 \cup [0, j_{\alpha_d,1})$,*
b. *the function $x \mapsto \mathcal{J}_{\alpha_d}(x)$ is decreasing on $[0, j_{\alpha_d,1})$,*
c. *the function $x \mapsto \mathcal{J}_{\alpha_d}(x)$ is strictly log-concave on Δ_3,*
d. *the function $x \mapsto J_{\alpha_d}(x)$ is strictly log-concave on Δ_3, provided $\sum_{i=1}^{d} \alpha_i > 0$.*

Proof. **a.** From the infinite product representation of the hyper-Bessel function $\mathcal{J}_{\alpha_d}(x)$ which is given by (12), we can write that

$$\mathcal{J}_{\alpha_d}(x) = \prod_{n\geq 1}\left(1-\frac{x^{d+1}}{j_{\alpha_d,n}^{d+1}}\right) = \prod_{n\geq 1}\left(\frac{j_{\alpha_d,n}^{d+1} - x^{d+1}}{j_{\alpha_d,n}^{d+1}}\right) = A_n B_n,$$

where

$$A_n = \prod_{n\geq 1} (j_{\alpha_d,n} - x)$$

and

$$B_n = \prod_{n\geq 1} \frac{j^d_{\alpha_d,n} + j^{d-1}_{\alpha_d,n}x + \cdots + x^d}{j^{d+1}_{\alpha_d,n}}$$

for $n \geq 1$. It is clear that $B_n > 0$ for all $x \in \mathbb{R}^+ \cup \{0\}$. On the other hand, since

$$0 < j_{\alpha_d,1} < j_{\alpha_d,2} < \cdots < j_{\alpha_d,n} < \cdots$$

we have that if $x \in (j_{\alpha_d,2n-1}, j_{\alpha_d,2n})$, then the first $(2n-1)$ terms of A_n are strictly negative, and remained terms are strictly positive. Similarly, if $x \in (j_{\alpha_d,2n}, j_{\alpha_d,2n+1})$, then the first $(2n)$ terms of A_n are strictly negative, and the rest is strictly positive. Morever, if $x \in [0, j_{\alpha_d,1})$, then all the terms of A_n are strictly positive. Hence the function $\mathcal{J}_{\alpha_d}(x)$ is negative on Δ_1, while it becomes positive on $\Delta_2 \cup [0, j_{\alpha_d,1})$.

b. To prove this part, it is enough to show that $\mathcal{J}'_{\alpha_d}(x) < 0$ for all $x \in [0, j_{\alpha_d,1})$. Now consider infinite product representation of the function $\mathcal{J}_{\alpha_d}(x)$ which is given by (12). If we take the logarithmic derivative both sides of the equality (12), then we get

$$\begin{aligned}
\frac{d}{dx}\left[\log \mathcal{J}_{\alpha_d}(x)\right] &= \frac{\mathcal{J}'_{\alpha_d}(x)}{\mathcal{J}_{\alpha_d}(x)} \\
&= \frac{d}{dx}\left[\log \prod_{n\geq 1}\left(1 - \frac{x^{d+1}}{j^{d+1}_{\alpha_d,n}}\right)\right] \\
&= \frac{d}{dx}\left[\sum_{n\geq 1} \log\left(1 - \frac{x^{d+1}}{j^{d+1}_{\alpha_d,n}}\right)\right] \\
&= -\sum_{n\geq 1} \frac{(d+1)x^d}{j^{d+1}_{\alpha_d,n} - x^{d+1}}.
\end{aligned}$$

So, we have that

$$\frac{\mathcal{J}'_{\alpha_d}(x)}{\mathcal{J}_{\alpha_d}(x)} = -\sum_{n\geq 1} \frac{(d+1)x^d}{j^{d+1}_{\alpha_d,n} - x^{d+1}} < 0$$

for all $x \in [0, j_{\alpha_d,1})$. Here it is important to remind from part **a.** that the function $x \mapsto \mathcal{J}_{\alpha_d}(x)$ is positive on $[0, j_{\alpha_d,1})$. As a result, the function $\mathcal{J}'_{\alpha_d}(x)$ is negative on $[0, j_{\alpha_d,1})$. Therefore, the function $x \mapsto \mathcal{J}_{\alpha_d}(x)$ is decreasing on $[0, j_{\alpha_d,1})$.

c. To show strictly log-concavity of the function $\mathcal{J}_{\alpha_d}(x)$ on Δ_3, it is enough to show that

$$\frac{d^2}{dx^2}\left[\log \mathcal{J}_{\alpha_d}(x)\right] < 0$$

for all $x \in \Delta_3$. By using the infinite product representation of the function $\mathcal{J}_{\alpha_d}(x)$ we conclude that

$$\begin{aligned}
\frac{d^2}{dx^2}\left[\log \mathcal{J}_{\alpha_d}(x)\right] &= \frac{d^2}{dx^2}\left[\log \prod_{n\geq 1}\left(1-\frac{x^{d+1}}{j_{\alpha_d,n}^{d+1}}\right)\right] \\
&= \frac{d}{dx}\left[\frac{d}{dx}\sum_{n\geq 1}\log\left(1-\frac{x^{d+1}}{j_{\alpha_d,n}^{d+1}}\right)\right] \\
&= \frac{d}{dx}\left[-\sum_{n\geq 1}\frac{(d+1)x^d}{j_{\alpha_d,n}^{d+1}-x^{d+1}}\right] \\
&= -(d+1)\sum_{n\geq 1}\frac{dx^{d-1}j_{\alpha_d,n}^{d+1}+x^{2d}}{\left(j_{\alpha_d,n}^{d+1}-x^{d+1}\right)^2} \\
&< 0
\end{aligned}$$

for all $x \in \Delta_3$, and consequently the function $\mathcal{J}_{\alpha_d}(x)$ is strictly log-concave on Δ_3.

d. To prove this assertion rewrite (3) as follow:

$$J_{\alpha_d}(x) = \frac{x^{\alpha_1+\cdots+\alpha_d}}{(d+1)^{\alpha_1+\cdots+\alpha_d}\Gamma(\alpha_1+1)\ldots\Gamma(\alpha_d+1)}\mathcal{J}_{\alpha_d}(x).$$

From part **c.**, the strict log-concavitiy of the function $J_{\alpha_d}(x)$ can be obtained easily. Since

$$\frac{d^2}{dx^2}\left[\log x^{\alpha_1+\cdots+\alpha_d}\right] = -\frac{x^{\alpha_1+\cdots+\alpha_d}}{x^2} < 0$$

for all $x \in \Delta_3$ when $\sum_{i=1}^{d}\alpha_i > 0$, the function $x \mapsto x^{\alpha_1+\cdots+\alpha_d}$ is strictly log-concave on Δ_3. In addition, it is known from part **c.** that the function $\mathcal{J}_{\alpha_d}(x)$ is strictly log-concave on Δ_3. As a consequence, $J_{\alpha_d}(x)$ is strictly log-concave on Δ_3 as a product of two strictly log-concave functions. □

Theorem 3.2. *The function*

$$x \mapsto \frac{x\mathbb{I}'_{\alpha_d}\left(\sqrt[d+1]{x}\right)}{\mathbb{I}_{\alpha_d}\left(\sqrt[d+1]{x}\right)}$$

is increasing on $(0,\infty)$ *for all* $\alpha_i > -1, i = 1, 2, \ldots, d.$

Proof. By using the infinite sum representation of the modified hyper-Bessel function $\mathbb{I}_{\alpha_d}$ which is given by (10), one can easily obtain the following:

$$\frac{x\mathbb{I}'_{\alpha_d}\left(\sqrt[d+1]{x}\right)}{\mathbb{I}_{\alpha_d}\left(\sqrt[d+1]{x}\right)} = \frac{\sum_{n\geq 0}A_{\alpha_d,n}x^n}{\sum_{n\geq 0}B_{\alpha_d,n}x^n}$$

where

$$A_{\alpha_d,n} = \frac{n}{n!(d+1)^{(n+1)(d+1)}\left(\alpha_1+1\right)_n\ldots\left(\alpha_d+1\right)_n}$$

and

$$B_{\alpha_d,n} = \frac{1}{n!(d+1)^{n(d+1)}\left(\alpha_1+1\right)_n\ldots\left(\alpha_d+1\right)_n}.$$

According to Cauchy-Hadamard Theorem, both the series $\sum_{n\geq 0} A_{\alpha_d,n}x^n$ and $\sum_{n\geq 0} B_{\alpha_d,n}x^n$ are convergent on $(-\infty,\infty)$, since

$$\lim_{n\to\infty}\left|\frac{A_{\alpha_d,n}}{A_{\alpha_d,n+1}}\right| = \lim_{n\to\infty}\left|\frac{B_{\alpha_d,n}}{B_{\alpha_d,n+1}}\right| = \infty.$$

Also, it is clear that $A_{\alpha_d,n}\in\mathbb{R}$ and $B_{\alpha_d,n}>0$ for all $n\in\{0,1,\dots\}$. If we consider

$$h_n = \frac{A_{\alpha_d,n}}{B_{\alpha_d,n}} = \frac{n}{d+1},$$

then we have

$$\frac{h_{n+1}}{h_n} = \frac{n+1}{n} > 1.$$

Therefore, the sequence $\{h_n\}_{n\geq 0}$ is increasing. By applying the Lemma (2.1), the proof is completed. □

Now, we would like present some applications of our main results. To do this, we consider the relationships between Bessel and modified Bessel functions and their extensions. As we mentioned before, for $d=1$ and $\alpha_1=\nu$, the hyper-Bessel and modified hyper-Bessel functions reduce to classical Bessel function J_ν and modified Bessel function I_ν which are given (1) and (2), respectively. More precisely, we have the following equalities for $d=1$ and $\alpha_1=\nu$:

$$\mathcal{J}_\nu(x) = 2^\nu x^{-\nu}\Gamma(\nu+1)J_\nu(x) \tag{13}$$

and

$$\mathbb{I}_\nu(x) = 2^\nu x^{-\nu}\Gamma(\nu+1)I_\nu(x). \tag{14}$$

On the other hand, it is well-known from [6] that the Bessel and modified Bessel functions reduce to some basic trigonometric and hyperbolic functions for some special values of ν, respectively. Especially, for $\nu=-\frac{1}{2},\nu=\frac{1}{2}$ and $\nu=\frac{3}{2}$ we have the following basic trigonometric and hyperbolic functions:

$$J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x, J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x, J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x}-\cos x\right)$$

and

$$I_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cosh x, I_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sinh x, I_{\frac{3}{2}}(x) = -\sqrt{\frac{2}{\pi x}}\left(\frac{\sinh x}{x}-\cosh x\right).$$

By using above relationships in (13) and (14), we have the followings:

$$\mathcal{J}_{-\frac{1}{2}}(x) = \cos x, \mathcal{J}_{\frac{1}{2}}(x) = \frac{\sin x}{x}, \mathcal{J}_{\frac{3}{2}}(x) = 3\left(\frac{\sin x - x\cos x}{x^3}\right)$$

and

$$\mathbb{I}_{-\frac{1}{2}}(x) = \cosh x, \mathbb{I}_{\frac{1}{2}}(x) = \frac{\sinh x}{x}, \mathbb{I}_{\frac{3}{2}}(x) = 3\left(\frac{x\cosh x - \sinh x}{x^3}\right),$$

respectively.

If we set $d=1$ in Theorem(3.1), then for the special values of $\alpha_1=\nu$ we have the following:

Corollary 3.1. *The following assertions hold true.*

i. *The function* $x\mapsto\mathcal{J}_{-\frac{1}{2}}(x)=\cos x$ *is decreasing on* $\left[0,j_{-\frac{1}{2},1}\right)$, *where* $j_{-\frac{1}{2},1}\approx 1.5708$ *is the first positive root of the equation* $\cos x=0$.

ii. *The function* $x\mapsto\mathcal{J}_{\frac{1}{2}}(x)=\frac{\sin x}{x}$ *is decreasing on* $\left[0,j_{\frac{1}{2},1}\right)$, *where* $j_{\frac{1}{2},1}\approx 3.1415$ *is the first positive root of the equation* $\sin x=0$.

iii. *The function* $x \mapsto \mathcal{J}_{\frac{3}{2}}(x) = 3\left(\frac{\sin x - x\cos x}{x^3}\right)$ *is decreasing on* $\left[0, j_{\frac{3}{2},1}\right)$, *where* $j_{\frac{3}{2},1} \approx 4.4934$ *is the first positive root of the equation* $\tan x = x$.
iv. *The function* $x \mapsto \mathcal{J}_{-\frac{1}{2}}(x) = \cos x$ *is strictly log-concave on* $\bigcup_{n\geq 1}\left(0, j_{-\frac{1}{2},n}\right)$, *where* $j_{-\frac{1}{2},n}$ *denotes nth positive zero of the equation* $\cos x = 0$.
v. *The function* $x \mapsto \mathcal{J}_{\frac{1}{2}}(x) = \frac{\sin x}{x}$ *is strictly log-concave on* $\bigcup_{n\geq 1}\left(0, j_{\frac{1}{2},n}\right)$, *where* $j_{\frac{1}{2},n}$ *denotes nth positive zero of the equation* $\sin x = 0$.
vi. *The function* $x \mapsto \mathcal{J}_{\frac{3}{2}}(x) = 3\left(\frac{\sin x - x\cos x}{x^3}\right)$ *is strictly log-concave on* $\bigcup_{n\geq 1}\left(0, j_{\frac{3}{2},n}\right)$, *where* $j_{\frac{3}{2},n}$ *denotes nth positive zero of the equation* $\tan x = x$.

Similarly, if we set $d = 1$ in Theorem(3.2), then for the special values of $\alpha_1 = \nu$ we have the following:

Corollary 3.2. *The following statements are valid.*
i. *The function* $x \mapsto p(x) = \frac{1}{2}\sqrt{x}\tanh\sqrt{x}$ *is increasing on* $(0, \infty)$.
ii. *The function* $x \mapsto q(x) = \frac{1}{2}\left(\sqrt{x}\coth\sqrt{x} - 1\right)$ *is increasing on* $(0, \infty)$.
iii. *The function* $x \mapsto r(x) = \frac{(3+x)\sinh\sqrt{x} - 3\sqrt{x}\cosh\sqrt{x}}{2\sqrt{x}\cosh\sqrt{x} - 2\sinh\sqrt{x}}$ *is increasing on* $(0, \infty)$.

Now, we would like to give the following graphs regarding our results. The following FIGURE 1 is related to Corollary (3.1), while FIGURE 2 is related to Corollary (3.2).

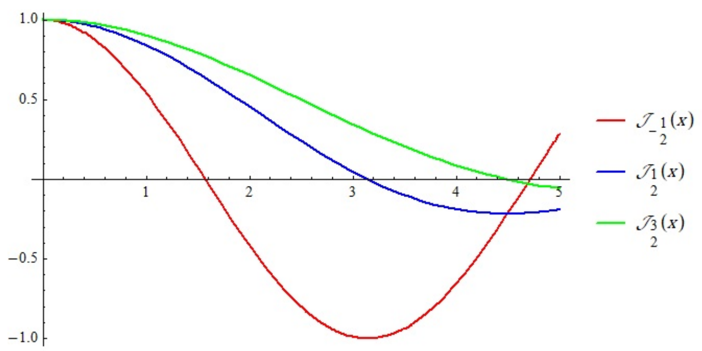

FIGURE 1.

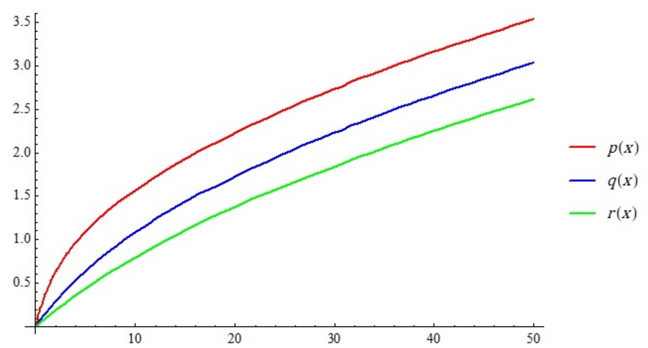

FIGURE 2.

4. Conclusions

In this study, we present some new properties of hyper-Bessel and related functions. Also, by using the modified hyper-Bessel function we define the function $x \mapsto \dfrac{x\mathbb{I}'_{\alpha_d}\left(\sqrt[d+1]{x}\right)}{\mathbb{I}_{\alpha_d}\left(\sqrt[d+1]{x}\right)}$ and we show that this function is an increasing function on $(0, \infty)$. In additon, we give some specific and illustrative examples about our main results.

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