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TAUBERIAN THEOREMS FOR THE PRODUCT OF BOREL AND LOGARITHMIC METHODS OF SUMMABILITY

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Abstract. In this paper, we show that if a sequence is summable by the product method $(B)(\ell, k)$, then it is also summable by the logarithmic method (ℓ, k), provided two-sided conditions of Hardy-type are satisfied. We also obtain some classical Tauberian theorems and their generalizations as special cases of our main theorems.

Keywords: Tauberian theorems, Borel summability, logarithmic summability, Hardy-type conditions.

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1. Introduction and Background

Suppose throughout that (s_n) is a sequence of complex numbers and k is any nonnegative integer, unless indicated otherwise. We say that a sequence (s_n) is summable to ξ by the Borel method (B), briefly $s_n \to \xi(B)$, if

$$\sum_{n=0}^{\infty} \frac{s_n}{n!} x^n \text{ converges for all } x \in \mathbb{R},$$

and

$$\lim_{x\to\infty} e^{-x} \sum_{n=0}^{\infty} \frac{s_n}{n!} x^n = \xi.$$

The sequences of logarithmic means of (s_n) are defined by

$$t_n^{(1)} = t_n^{(1)}(s_n) = \frac{1}{\ell_n} \sum_{j=0}^{n} \frac{s_j}{j+1},$$

and for $k = 2, 3, \ldots$,

$$t_n^{(k)} = t_n^{(k)}(s_n) = \frac{1}{\ell_n} \sum_{j=0}^{n} \frac{t_j^{(k-1)}}{j+1},$$

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where

$$\ell_n = \sum_{k=0}^{n} \frac{1}{k+1} \sim \log n.$$

Here, we use the symbol $f_n \sim g_n$ to mean that $f_n/g_n \to 1$ as $n \to \infty$.

We say that (s_n) is summable to ξ by the $k-$times iterated logarithmic method (ℓ, k), briefly $s_n \to \xi(\ell, k)$, if

$$\lim_{n\to\infty} t_n^{(k)} = \xi.$$

Evidently, the summability $(\ell, 0)$ is the same as usual convergence. Note that, trivially, (ℓ, k) summability of a sequence implies its (ℓ, k') summability to the same number, whenever $k \geq 0$ and $k' > k$. However, the converse is not necessarily true provided by the example (see [20]):

$$s_n = ((-1)^n (n \log n + (n+1)\log(n+1))).$$

Here, (s_n) is summable $(\ell, 2)$ to 0, but not summable $(\ell, 1)$.

The product summability method $(B)(\ell, k)$ is obtained by superimposing (ℓ, k) summability on (B) summability.

We say that (s_n) is summable to ξ by the product method $(B)(\ell, k)$, and write $s_n \to \xi(B)(\ell, k)$, if $(t_n^{(k)})$ is Borel summable to ξ. $(B)(\ell, k)$ method reduces to Borel summability if $k = 0$.

A method of summability is called regular if it sums each convergent sequence or series to its usual sum. Since, Borel and logarithmic methods are regular (see [7]), $(B)(\ell, k)$ summability is also regular.

Product summability methods have a long history; see for example the papers [1, 17, 19] and they found new attention recently in [2, 3, 4] and [5, 6, 12].

Throughout this study, we write $\theta_n = O(1)$ to mean that (θ_n) is a bounded sequence and $\theta_n = o(1)$ to mean that (θ_n) converges to zero. Moreover, $f_n = O(g_n)$ means that $f_n/g_n = O(1)$ and $f_n = o(g_n)$ means that $f_n/g_n = o(1)$.

The backward difference of (s_n) is defined as

$$\Delta s_n = \begin{cases} s_n - s_{n-1} & , n \geq 1 \\ s_0 & , n = 0 \end{cases}.$$

The difference of a sequence and its logarithmic mean is represented by

$$s_n - t_n^{(1)} = v_n^{(0)} \tag{1}$$

where

$$v_n^{(0)} = v_n^{(0)}(s_n) = \frac{1}{\ell_n} \sum_{j=1}^{n} \ell_{j-1} \Delta s_j.$$

The identity (1) is called the logarithmic Kronecker identity. Besides, for all $k \geq 1$, we introduce k-th order iterated logarithmic means of $(v_n^{(0)})$ by

$$v_n^{(k)} = v_n^{(k)}(s_n) = \frac{1}{\ell_n} \sum_{j=0}^{n} \frac{v_j^{(k-1)}}{j+1}.$$

The sequence (s_n) is called slowly oscillating with respect to summability $(\ell, 1)$ if

$$\lim_{\lambda\to 1^+} \limsup_{n\to\infty} \max_{n<m\leq[n^\lambda]} |s_m - s_n| = 0 \tag{2}$$

or equivalently

$$\lim_{\lambda\to 1^-} \limsup_{n\to\infty} \max_{[n^\lambda]<m\leq n} |s_n - s_m| = 0, \tag{3}$$

where $[n^\lambda]$ denotes the integer part of n^λ.

The interest in summability methods is that they provide a way to understand sequences or series which are divergent. There are, for instance, important summability methods due to Abel, Euler, Cesàro, Borel and others.

Given two summability methods (X) and (Y), we write $(X) \subseteq (Y)$ if a sequence summable (X) is summable (Y) to the same number. Also, if there is a sequence summable (Y) but not summable (X), we write $(X) \subset (Y)$.

For any two regular summability methods (X) and (Y) with $(X) \subset (Y)$, the sufficient conditions under which $s_n \to \xi(Y)$ implies $s_n \to \xi(X)$ are called Tauberian conditions and the corresponding theorems are called Tauberian theorems. Most frequently in the theory of summability, the case in which method X is equivalent to the usual convergence is considered.

The main object of the present paper is to prove Tauberian theorems for the $(B)(\ell, k)$ summability.

2. Lemmas

In this section, we shall give the following lemmas required in the proofs of our main theorems.

The identities below were obtained in the study of Sezer and Çanak [18].

Lemma 2.1. *For every integer $k \geq 0$, the assertions*

(i) $t_n^{(k)} - t_n^{(k+1)} = v_n^{(k)}$,

(ii) $v_n^{(k)} - v_n^{(k+1)} = (n+1)\ell_{n-1}\Delta v_n^{(k+1)}$,

(iii) $v_n^{(k)} = (n+1)\ell_{n-1}\Delta t_n^{(k+1)}$

are valid.

The following lemma is a corollary of the theorem given by Kwee (Theorem A, [9]).

Lemma 2.2. *If $s_n \to \xi(\ell, 1)$ and (s_n) is slowly oscillating with respect to summability $(\ell, 1)$, then (s_n) is convergent to ξ.*

Lemma 2.3. *If $s_n \to \xi(B)$, then, for each positive integer k, $t_n^{(k)} \to \xi(B)$.*

The above lemma due to Kwee [11] indicates that $(B)(\ell, k)$ method includes (B) method. Also, it can be easily seen via Lemma 4 of Parameswaran [15] that this inclusion is strict.

Lemma 2.4. *If (s_n) is slowly oscillating with respect to summability $(\ell, 1)$, then $(v_n^{(0)})$ is bounded and slowly oscillating with respect to summability $(\ell, 1)$.*

Proof. Since we have the identity

$$\begin{aligned} \frac{1}{\ell_n}\sum_{j=0}^{n}\frac{s_n - s_j}{j+1} &= \frac{1}{\ell_n}\sum_{j=0}^{n}\sum_{i=j+1}^{n}\frac{\Delta s_i}{j+1} \\ &= \frac{1}{\ell_n}\sum_{j=1}^{n}\ell_{j-1}\Delta s_j = v_n^{(0)}, \end{aligned}$$

slow oscillation of (s_n) implies boundedness of $(v_n^{(0)})$ from the proof of Theorem 2 in [14]. Now, we need to show that $(v_n^{(0)})$ is slowly oscillaiting with respect to summability $(\ell, 1)$.

From Lemma 2.1, for $k=0$ we have $\Delta t_n^{(1)} = \dfrac{v_n^{(0)}}{(n+1)\ell_{n-1}}$. Then, for some number $H>0$

$$\left|\Delta t_n^{(1)}\right| = \left|t_n^{(1)} - t_{n-1}^{(1)}\right| \leq \frac{H}{(n+1)\ell_{n-1}}.$$

Hence, we find

$$\begin{aligned} \left|t_m^{(1)} - t_n^{(1)}\right| &\leq \sum_{j=n+1}^{m} \left|t_j^{(1)} - t_{j-1}^{(1)}\right| \\ &\leq \left(\frac{1}{(m+1)\ell_{m-1}} + \frac{1}{m\ell_{m-2}} + \cdots + \frac{1}{(n+2)\ell_n}\right) H \\ &\leq \left(\frac{\ell_m - \ell_n}{\ell_n}\right) H. \end{aligned}$$

It follows from the last inequality above that

$$\max_{n<m\leq[n^\lambda]} \left|t_m^{(1)} - t_n^{(1)}\right| \leq \left(\frac{\ell_{[n^\lambda]} - \ell_n}{\ell_n}\right) H. \tag{4}$$

Next, taking lim sup of both sides of the inequality (4) as n tends to ∞ yields

$$\limsup_{n\to\infty} \max_{n<m\leq[n^\lambda]} \left|t_m^{(1)} - t_n^{(1)}\right| \leq \limsup_{n\to\infty} \left(\frac{\ell_{[n^\lambda]} - \ell_n}{\ell_n}\right) H. \tag{5}$$

Now, letting $\lambda \to 1^+$ from the both sides of (5) we obtain

$$\lim_{\lambda\to1^+} \limsup_{n\to\infty} \max_{n<m\leq[n^\lambda]} \left|t_m^{(1)} - t_n^{(1)}\right| \leq 0,$$

which means $(t_n^{(1)})$ is slowly oscillating with respect to summability $(\ell,1)$. Therefore, considering the identity (1) slow oscillation of $(v_n^{(0)})$ follows. □

The following lemma due to Kwee [10] indicates that convergence of $(v_n^{(0)})$ to zero is a Tauberian condition for the $(\ell,1)$ summability.

Lemma 2.5. *If $s_n \to \xi(\ell,1)$ and $v_n^{(0)} = o(1)$, then (s_n) is convergent to ξ.*

Our main theorems based on the following well-known Tauberian results given for the Borel summability in Hardy's famous book of Divergent Series [7].

Lemma 2.6. *If $s_n \to \xi(B)$ and $\sqrt{n}\Delta s_n = O(1)$, then (s_n) is convergent to ξ.*

Lemma 2.7. *If $s_n \to \xi(B)$ and $\sqrt{n}\Delta s_n = o(1)$, then (s_n) is convergent to ξ.*

3. Main Results

In this section, we state and prove following Tauberian theorems for the $(B)(\ell,k)$ summability.

Theorem 3.1. *If $s_n \to \xi(B)(\ell,k)$ and*

$$v_n^{(k)} = O(\sqrt{n}\log n), \tag{6}$$

then $s_n \to \xi(\ell,k+1)$.

Proof. By the hypothesis of the theorem, we have $t_n^{(k)} \to \xi(B)$, which also implies from Lemma 2.3
$$t_n^{(k+1)} \to \xi(B). \tag{7}$$
Then, since
$$v_n^{(k)} = (n+1)\ell_{n-1}\Delta t_n^{(k+1)} = O(\sqrt{n}\log n),$$
we get
$$\sqrt{n}\Delta t_n^{(k+1)} = O(1). \tag{8}$$
Thus, it follows from (7), (8) and Lemma 2.6 that $s_n \to \xi(\ell, k+1)$. □

Corollary 3.1. ([11]) *If $s_n \to \xi(B)$ and*
$$v_n^{(0)} = O(\sqrt{n}\log n), \tag{9}$$
then $s_n \to \xi(\ell, 1)$.

Proof. Take $k = 0$ in Theorem 3.1. □

Corollary 3.2. ([16]) *If $s_n \to \xi(B)$ and*
$$s_n = O(\sqrt{n}\log n), \tag{10}$$
then $s_n \to \xi(\ell, 1)$.

Proof. Considering the assumption (10) together with Stolz-Cesàro theorem ([13], p.85), we obtain $t_n = O(\sqrt{n}\log n)$. Hence, the identity (1) implies $v_n^{(0)} = O(\sqrt{n}\log n)$, which completes the proof. □

Corollary 3.3. *If $s_n \to \xi(B)(\ell, k)$ and (s_n) is slowly oscillating with respect to summability $(\ell, 1)$, then (s_n) is convergent to ξ.*

Proof. Taking Lemma 2.4 and the slow oscillation of (s_n) into account, we have $v_n^{(k)} = O(1)$ and the slow oscillation of $(t_n^{(k)})$ for each integer $k > 0$. Then, we also have $v_n^{(k)} = O(\sqrt{n}\log n)$. This yields
$$\lim_{n\to\infty} t_n^{(k+1)} = \xi \tag{11}$$
from Theorem 3.1. Now, combining Lemma 2.2 and the slow oscillation of $(t_n^{(k)})$, it follows
$$\lim_{n\to\infty} t_n^{(k)} = \xi. \tag{12}$$
Considering (11) and (12) and arguing in the same way, we observe that (s_n) is convergent to ξ. □

Corollary 3.4. *If $s_n \to \xi(B)(\ell, k)$ and*
$$n\log n\Delta s_n = O(1), \tag{13}$$
then (s_n) is convergent to ξ.

Proof. The proof is completed by the fact that the assumption (13) implies the slow oscillation of (s_n). □

Corollary 3.5. *If $s_n \to \xi(B)(\ell, k)$ and*
$$n\log n\Delta s_n = o(1), \tag{14}$$
then (s_n) is convergent to ξ.

Proof. Condition (14) is sufficient for (13). □

Note that, the above corollary is a generalization of a classical Tauberian theorem due to Ishiguro [8] given for the $(\ell, 1)$ summability.

Theorem 3.2. *If $s_n \to \xi(B)(\ell, k+1)$ and*

$$n \log n \Delta v_n^{(k+1)} = o(1), \tag{15}$$

then $s_n \to \xi(\ell, k)$.

Proof. Since $t_n^{(k+1)} \to \xi(B)$, we easily get

$$t_n^{(k+2)} \to \xi(B) \tag{16}$$

by using Lemma 2.3. From the identity

$$t_n^{(k+1)} - t_n^{(k+2)} = v_n^{(k+1)} \tag{17}$$

we find

$$v_n^{(k+1)} \to 0(B). \tag{18}$$

Moreover, it is obvious from (15) that

$$\sqrt{n} \Delta v_n^{(k+1)} = o(1). \tag{19}$$

Considering (18) and (19) together with Lemma 2.7 we obtain

$$v_n^{(k+1)} = (n+1)\ell_{n-1} \Delta t_n^{(k+2)} = o(1), \tag{20}$$

that implies

$$\sqrt{n} \Delta t_n^{(k+2)} = o(1).$$

Now, applying Lemma 2.7 to the sequence $(t_n^{(k+2)})$ we conclude

$$\lim_{n\to\infty} t_n^{(k+2)} = \xi. \tag{21}$$

Using (20) and (21), we get via identity (17) that

$$\lim_{n\to\infty} t_n^{(k+1)} = \xi. \tag{22}$$

By Lemma 2.1, we also have

$$v_n^{(k)} - v_n^{(k+1)} = (n+1)\ell_{n-1} \Delta v_n^{(k+1)},$$

which necessiates

$$v_n^{(k)} = o(1) \tag{23}$$

from (15) and (20). Therefore, since

$$t_n^{(k)} - t_n^{(k+1)} = v_n^{(k)},$$

we conclude $s_n \to \xi(\ell, k)$, which completes the proof. □

Corollary 3.6. *If $s_n \to \xi(B)(\ell, k+1)$ and*

$$v_n^{(0)} = o(1), \tag{24}$$

then (s_n) is convergent to ξ.

Proof. Suppose $v_n^{(0)} = o(1)$, then for all integer $k \geq 1$, we get $v_n^{(k)} = o(1)$. Hence, by the identity $v_n^{(k)} - v_n^{(k+1)} = (n+1)\ell_{n-1} \Delta v_n^{(k+1)}$, it follows

$$n \log n \Delta v_n^{(k+1)} = o(1).$$

Thus, from Theorem 3.2 we obtain $s_n \to \xi(\ell, k)$ which is equivalent to $t_n^{(k-1)} \to \xi(\ell, 1)$. Since,

$$v_n^{(k-1)} = t_n^{(k-1)}(v_n^{(0)}) = v_n^{(0)}(t_n^{(k-1)}) = o(1)$$

we get $s_n \to \xi(\ell, k-1)$ by using Lemma 2.5. Continuing in the same fashion, we conclude $\lim\limits_{n\to\infty} s_n = \xi$. This completes the proof. $\square$

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