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ON A DISTRIBUTION OF THE PROCESS DESCRIBING A SERVICE SYSTEM WITH UNRELIABLE DEVICES

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Abstract. In the paper, the distribution is found for the process $\{\eta_t, \xi_t\}, t \geq 0$, in the terms of Laplace transformation. The considered process describes the queuing system with nonhomogeneous Poisson stream of demands and n unreliable devices. It is essential that the process $\{\eta_t, \xi_t\}, t \geq 0$, for $\xi_t \geq n$ is a homogeneous with respect to the second component Markov process. The results obtained in the paper are based on the theory of matrices and solution of the system of linear integral equations.

Keywords: Poisson process, Laplace transformation, Generating function, Markov chain, Homogeneous with respect to the second component.

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1. Introduction

In solution of many problems of the theory of queuing systems, the principles of the theory of controlled Poisson processes with or without boundaries are often used. This principle indeed is:
Given controlled Poisson process with boundary i.e. the Markov chain $\{\beta_t, m_t\}, t \geq 0$ in the phase space $T \times N^+$, where $T = \{\alpha, \beta, ...\}-$ is finite set, $N^+ = \{0, 1, ...\}$ and having known transmission probability on the small interval $(t, t + \Delta)$.

As a controlled Poisson process without boundary is understood as homogeneous with respect to the second component Markov process $\{\alpha_t, n_t, \}, \ t \geq 0$, in the phase space $T \times N, \ \ N = \{0, \pm 1, \pm 2, ...\}$..

Our aim is finding a relationship between the processes $\{\beta_t, m_t\}$ and $\{\alpha_t, n_t\}$.

Suppose that the local transition probabilities of the process $\{\beta_t, m_t\}, \ t \geq 0$ depend on such natural number c that for $m_t \geq c$ the increment of the process $\{\beta_t, m_t\}$ stochastically equivalent to the increment of the process $\{\alpha_t, n_t\}$. If $m_t \in [0, c-1]$, then evolution of the process $\{\beta_t, m_t\}$ is described by some auxiliary Markov chain with known local transition probabilities.

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We assume that $n_{t+0} - n_{t-0} \geq -2,\ \ t \geq 0$ with probability 1. It means that with the probability 1, the process n_t does not have negative jumps different from -1, and for the local characteristics, it is true that

$$q^k_{\alpha\beta}(t) = 0 \quad (t \geq 0; \quad \alpha, \beta \in T; \quad k \leq -2).$$

Such processes in the case of integer phase is natural to be called as "lower continuous" processes [1].

From the theoretical point of view, the problem is completely solved in [2]. In the present work following to [2] in the terms of Laplace transformation the distribution of the process $\{\eta_t, \xi_t\},\ \ t \geq 0$, is found describing the queuing system with nonhomogeneous Poisson stream and no reliable devices. The similar problems are considered in [3], [4]. In differ from those works, we use the principles of theory of controllable Poisson processes with and without boundaries.

2. Main results

Let the queuing system gets the waiting nonhomogeneous Poisson stream of demands with intensity.

$$\lambda = \sum_{k=1}^{\infty} \lambda_k < \infty.$$

Then, by $\Delta \downarrow 0$ the probability that in the interval $(t, t+\Delta)$ the system will get k demands is equal to $\lambda_k \Delta + o(\Delta),\ \ k \geq 1$..

The service time has exponential distribution with parameter μ. Each device can be broken with probability $\nu\Delta + o(\Delta)$, during service in the interval $(t, t+\Delta)$ and then repaired. The repairing time is exponential function with parameter π.

It is assumed that getting and service of the demand and breaking and repairing of the devices are independent from each other.

Consider two dimensional random process

$$\xi_t = \{\eta_t, \xi_t\},\ \ t \geq 0,$$

where $\eta_t \in \{0, 1, ..., n\}-$ is a number of broken devices at the time t, $\xi_t \in \{0, 1, 2, ..., \}-$ is a number of demands in the system at the moment t, $n$is a number of service devices.

To study ξ_t, we consider the random process $\{\eta^*_t, \xi^{|*}_t\}$ in the phase space $T \times N$, where $T = \{0, 1, ..., n\},\ \ N = \{0, \pm 1, \pm 2, ...\}$, and with the following transition probabilities by $\Delta \downarrow 0:$

$$(m, r) \xrightarrow{\Delta} \begin{cases} (m, r-1) = (n - m\mu\Delta + o(\Delta) \\ (m-1, r) = m\pi\Delta + o(\Delta) \\ (m, r) = 1 - [\lambda + m\pi + (n-m)(\mu+\nu)]\Delta + o(\Delta) \\ (m, r+k) = \lambda_k\Delta + o(\Delta),\ \ k \geq 1 \\ (m+1, r) = (n-m)\nu\Delta + o(\Delta) \end{cases} \tag{1}$$

$$(m = 0, 1, ..., n; \quad r = 0, \pm 1, ...)$$

As one can see from (1) the process $\{\eta^*_t, \xi^*_t\}$ is homogeneous with respect to the second component Markov process [5].

Introduce the denotations

$$P^*_{lm}(t, r) = P\{\eta^*_t = m, \quad \xi^*_t = r/\eta^*_0 = l, \quad \xi^*_0 = 0\}$$

Using (1) we get

$$P^*_{lm}(t+\Delta,r) = P^*_{lm}(t,r)[1-(\lambda+m\pi+(\mu+\nu)(n-m))\Delta]+$$
$$+(n-m)\mu P^*_{lm}(t,r+1)\Delta + \sum_{j=-\infty}^{r-1} P^*_{lm}(t,j)\lambda_{r-j}+$$
$$(1-\delta_{mn})P^*_{l,m+1}(t,r)(m+1)\pi\Delta + (1-\delta_{mo})(n-m+1)\nu P^*_{l,m-1}(t,r)\Delta + o(\Delta).$$

From the last we obtain

$$\frac{dP^*_{lm}(t,r)}{dt} = -\left[\lambda+m\pi+(n-m)(\mu+\nu)\right]P^*_{lm}(t,r)+(n-m)\mu P^*_{lm}(t,r+1)+$$
$$+\sum_{j=-\infty}^{r-1}\lambda_{r-j}P^*_{lm}(t,j)+(1-\delta_{mn})(m+1)\pi P^*_{l,m+1}(t,r)+ \qquad (2)$$
$$+(1-\delta_{mo})(n-m+1)\nu P^*_{l,m-1}(t,r),$$

where δ_{ij} is Kronecker's symbol, $l,m=0,1,...,n;\quad r=0,\pm1,\pm2,...$
Let us introduce the generating function

$$\varphi^*_{lm}(t,\theta) = \sum_{r=-\infty}^{+\infty} P^*_{lm}(t,r)\theta^r \qquad |\theta|=1$$
$$l,m=0,1,...,n$$

Then the system (2) on the generating functions takes the form

$$\frac{\partial\varphi^*_{lm}(t,\Delta)}{\partial t} = \left[\lambda(\theta)-\lambda+(n-m)\mu\left(\frac{1}{\theta}-1\right)-m\pi-(n-m)\nu\right]\varphi^*_{lm}(t,\theta)+$$
$$+(1-\delta_{mn})(m+1)\pi\varphi^*_{l,m+1}(t,\theta)+(1-\delta_{mo})(n-m+1)\nu\varphi^*_{l,m-1}(t,\theta) \qquad (3)$$

Introducing the function

$$\psi_{lm}(t,\theta) = e^{[\lambda-\lambda(\theta)]t}\varphi^*_{lm}(t,\theta)$$

one can write the system (3) in the following form

$$\begin{aligned} &\frac{\partial\psi_{lm}(t,\theta)}{\partial t} = \left[(n-m)\mu\left(\frac{1}{\theta}-1\right)-m\pi-(n-m)\nu\right]\psi_{lm}(t,\theta)+\\ &+(1-\delta_{mn})(m+1)\pi\psi_{l,m+1}(t,\theta)+\\ &+(1-\delta_{mo})(n-m+1)\nu\psi_{l,m-1}(t,\theta)\\ &l,m=0,1,...,n \end{aligned} \qquad (4)$$

From (4) follows that if $\eta^*_t = m$, then $\xi^*_t = \xi^*_t(m)$ is a Poisson process with parameter $\tilde{\lambda} = (n-m)\mu$. Denoting

$$\gamma_m(\theta) = (n-m)\mu\left(\frac{1}{\theta}-1\right)-m\pi-(n-m)\nu, \qquad m=0,1,...,n$$

$$\vec{\psi}(t,\theta) = \{\psi_{l0}(t,\theta),\psi_{l1}(t,\theta),...,\psi_{\ln}(t,\theta)\},$$

$$\Gamma(\theta) = \begin{pmatrix} \gamma_0(\theta) & \pi & 0 & ... & 0 \\ n\nu & \gamma_1(\theta) & 2\pi & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & ... & \nu\gamma_n(\theta) \end{pmatrix}$$

we can write the system (4) in the following matrix form

$$\frac{\partial\vec{\psi}(t,\theta)}{\partial t}=\Gamma(\theta)\vec{\psi}(t,\theta).$$

from this

$$\vec{\psi}(t,\theta)=\vec{\psi}(0,\theta)\exp\{t,\Gamma(\theta)\},$$

where $\vec{\psi}(0,\theta)$is a known vector.
Finding $\vec{\psi}(t,\theta)$ makes possible to find transition probabilities $\{P^*_{lm}(t,r)\}$ of the process $\{\eta^*_t,\xi^*_t\}$..

Now let us investigate the main process $\{\eta_t,\xi_t\},\ \ t\geq 0$..
By the description of the process $\{\eta_t,\xi_t\},\ \ t\geq 0$ it is easy to see that by $\Delta\downarrow 0$ it has the following transition probabilities

$$(m,r)\xrightarrow{\Delta}\begin{cases}(m,r-1)=\min(r,n-m)\mu\Delta+o(\Delta)\\(m+1,r)=\min(r,n-m)\nu\Delta+o(\Delta)\\(m,r)=1-[\lambda+\min(r,n-m)(\mu+\nu)+m\pi]\Delta+o(\Delta)\\(m,r+k)=\lambda_k\Delta+o(\Delta),\ \ k\geq 1\\(m-1,r)=m\pi\Delta+o(\Delta)\end{cases}\tag{5}$$

$$m=0,1,..,n;\quad r=0,1,2,...$$

As one can see from (5) the process $\{\eta_t,\xi_t\},\ \ t\geq 0$ for $\xi_t\geq n$ is homogeneous with respect to the second component Markov process.
Let us denote

$$\begin{array}{l}P_{m,r}(t)=P\{\eta_t=m,\ \ \xi_t=r\},\\m=0,1,...,n;\quad r=0,1,2,...\end{array}$$

Then using (5) we obtain

$$\begin{array}{l}\frac{dP_{mr}(t)}{dt}=-\left[\lambda+m\pi+\min(r,n-m)\mu+\min(r,n-m)\nu\right]P_{m,r}(t)+\\+\min(r,n-m+1)\nu P_{m-1,r}(t)+(m+1)\pi P_{m+1,r}(t)+\\+\min(r+1,n-m)\mu P_{m,r+1}(t)+\sum\limits_{i=1}^{r}P_{m,r-i}(t)-\lambda_i,\\m=0,1,...,n-1;\ \ r=0,1,2,...\end{array}\tag{6}$$

$$\frac{dP_{n0}(t)}{dt}=-(\lambda+n\pi)P_{n0}(t),\tag{7}$$

$$\frac{dP_{nr}(t)}{dt}=-(\lambda+n\pi)P_{nr}(t)+\nu P_{n-1,r}(t)+\sum_{i=1}^{r}\lambda_iP_{n,r-i}(t).\tag{8}$$

Considering transition function

$$\varphi_m(t,\theta)=\sum_{r=0}^{\infty}P_{m,r}(t)\theta^r\qquad|\theta|\leq 1$$

Multiplying both sides of (6) by $\theta^r$and taking a sum from 0 to ∞we get

$$\begin{aligned}&\sum_{r=0}^{\infty}\tfrac{dP_{mr}(t)}{dt}\theta^r=-(\lambda+m\pi)\sum_{r=0}^{\infty}P_{mr}(t)\theta^r-\\&-(\nu+\mu)\sum_{r=0}^{\infty}\min(r,n-m)P_{m,r}(t)\theta^r+\nu\sum_{r=0}^{\infty}\min(r,n-m+1)P_{m-1,r}(t)\theta^r+\\&+(m+1)\pi\sum_{r=0}^{\infty}P_{m+1,r}(t)\theta^r+\mu\sum_{r=0}^{\infty}\min(r+1,n-m)P_{m,r+1}(t)\theta^r+\\&+\sum_{r=0}^{\infty}\sum_{i=1}^{r}P_{m,r-i}(t)\lambda_i\theta^r,\quad m=0,1,...,n-1.\end{aligned}\tag{9}$$

We assume that $P_{-1,r}(t)\equiv 0$ for any $r\geq 0$. Note that

$$\sum_{r=0}^{\infty}\min(r,n-m)P_{mr}(t)\theta^r=(n-m)\varphi_m(t,\theta)+\sum_{r=0}^{n-m-1}(r-n+m)P_{mr}(t)\theta^r.$$

Considering this from (9) we have

$$\begin{aligned}&\tfrac{\partial\varphi_m(t,\theta)}{\partial t}=(n-m+1)\nu\varphi_{m-1}(t,\theta)+\left[\lambda(\theta)+\tfrac{\mu}{\theta}(n-m)-(\lambda+m\pi)-(n-m)(\nu+\mu)\right]\varphi_m(t,\theta)+\\&(m+1)\pi\varphi_{m+1}(t,\theta)+\nu\sum_{r=0}^{n-m}(r-n+m-1)P_{m-1,r}(t)\theta^r-(\nu+\mu)\sum_{r=0}^{n-m-1}(r-n+m)P_{mr}(t)\theta^r+\\&+\tfrac{\mu}{\theta}\sum_{r=0}^{n-m-1}(r-n+m)P_{mr}(t)\theta^r,\quad(m=0,1,...,n-1)\end{aligned}$$

where $\lambda(\theta)=\sum\limits_{r=1}^{\infty}\lambda_k\theta^k \qquad |\theta|\leq 1$

By the similar way from the equations (7) and (8) one can get

$$\frac{\partial\varphi_m(t,\theta)}{\partial t}=\nu\varphi_{n-1}(t,\theta)+[\lambda(\theta)-\lambda-n\pi]\,\varphi_n(t,\theta)-\nu P_{n-1,0}(t).$$

Introduce

$$\begin{aligned}&q_m(\theta)=\lambda(\theta)+\tfrac{\mu}{\theta}(n-m)-(\lambda+m\pi)-(n-m)(\nu+\mu),\\&\psi_m(t,\theta)=\left(\tfrac{\mu}{\theta}-\mu-\nu\right)\sum_{r=0}^{n-m-1}(r-n+m)P_{mr}(t)\theta^r+\nu\sum_{r=0}^{n-m}(r-n+m-1)P_{m-1,r}(t)\theta^r,\\&(m=0,1,...,n)\end{aligned}$$

Suppose that

$$\sum_{r=0}^{-1}rP_{mr}(t)\theta^r\equiv 0.$$

As a result we get the following system of differential equations

$$\begin{cases}\tfrac{\partial\varphi_0(t,\theta)}{\partial t}=q_0(\theta)\varphi_0(t,\theta)+\pi\varphi_1(t,\theta)+\psi_0(t,\theta)\\ \tfrac{\partial\varphi_1(t,\theta)}{\partial t}=n\nu\varphi_0(t,\theta)+q_1(\theta)\varphi_1(t,\theta)+2\pi\varphi_2(t,\theta)+\psi_1(t,\theta)\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \tfrac{\partial\varphi_n(t,\theta)}{\partial t}=\nu\varphi_{n-1}(t,\theta)+q_n(\theta)\varphi_n(t,\theta)+\psi_n(t,\theta).\end{cases}\tag{10}$$

Introducing the denotations

$$\vec{\varphi}(t,\theta)=\begin{pmatrix}\varphi_0(t,\theta)\\ \varphi_1(t,\theta)\\ ...\\ \varphi_n(t,\theta)\end{pmatrix},\qquad \vec{\psi}(t,\theta)=\begin{pmatrix}\psi_0(t,\theta)\\ \psi_1(t,\theta)\\ ...\\ \psi_n(t,\theta),\end{pmatrix}$$

$$Q(\theta)=\begin{pmatrix} q_0(\theta) & \pi & 0 & ... & 0\\ n\nu & q_1(\theta) & 2\pi & ... & 0\\ ... & ... & ... & ... & ...\\ 0 & 0 & 0 & q_{n-1}(\theta) & n\pi\\ 0 & 0 & ... & \nu & q_n(\theta).\end{pmatrix}$$

We can write the system (10) in the following matrix form

$$\frac{\partial\vec{\varphi}(t,\theta)}{\partial t}=Q(\theta)\vec{\varphi}(t,\theta)+\vec{\psi}(t,\theta),$$

from this we have [6]

$$\vec{\varphi}(t,\theta)=\vec{\varphi}(0,\theta)e^{tQ(\theta)}+\int_0^t e^{(t-\tau)Q(\theta)}\vec{\psi}(\tau,\theta)d\tau, \tag{11}$$

where $\vec{\varphi}(0,\theta)$ is an initial solution that is assumed known.
From the theory of homogeneous with respect to the second component Markov processes is known the following representation for $e^{tQ(\theta):}$

$$e^{tQ(\theta)}=\left\|r_{ij}(t,\theta)\right\|_{i,j=0}^{n},$$

where $r_{ij}(t,\theta)$ $(i,j=0,1,...,n)$–is a generating function for the Markov process $\{\eta_t^*,\xi_t^*\}$, that is homogeneous with respect to the second component.

Since the behavior of the process $\{\eta_t^*,\xi_t^*\}$ is as the process $\{\eta_t,\xi_t\}$ by $\xi_t\geq n$, then

$$r_{ij}(t,\theta)=\sum_{k=-\infty}^{+\infty}g_{ij}(t,k)\theta^k,\qquad |\theta|=1$$
$$g_{ij}(t,k)=P\{\eta_t^*=j,\quad \xi_t^*=k/\eta_0^*=i,\ \xi_0^*=0\}$$
$$(i,j=0,1,...,n)$$

Taking equal the corresponding components at the right and left sides of the equality (11) we obtain

$$\varphi_k(t,\theta)=\sum_{i=0}^{n}r_{ki}(t,\theta)\varphi_i(0,\theta)+\int_0^t\left[\sum_{i=0}^{n}r_{ki}(t-\tau,\theta)\psi_i(\tau,\theta)\right]d\tau$$

In the last passing to the expressions for φ_k,ψ_k and r_{ki}we get

$$\begin{aligned}&\sum_{j=0}^{\infty}P_{kj}(t)\theta^j=\sum_{i=0}^{n}\left(\sum_{j=0}^{\infty}P_{ij}(\theta)\theta^j\right)\left(\sum_{m=-\infty}^{+\infty}g_{ki}(t,m)\theta^m\right)+\\ &+\sum_{i=0}^{n}\int_0^t\left[\left(\tfrac{\mu}{\theta}-\mu-\nu\right)\sum_{r=0}^{n-i-1}(r-n+i)P_{ir}(\tau)\theta^r\right]\left[\sum_{m=-\infty}^{+\infty}g_{ki}(t-\tau,m)\theta^m\right]d\tau+\\ &+\sum_{i=0}^{n}\int_0^t\nu\left[\sum_{r=0}^{n-i}(r-n+i-1)P_{i-1,r}(\tau)\theta^r\right]\left[\sum_{m=-\infty}^{+\infty}g_{ki}(t-\tau,m)\theta^m\right]d\tau,\end{aligned} \tag{12}$$

where $P_{ij}(0)$ -is a known initial distribution. It is easy to check that

$$\begin{aligned}&\sum_{i=0}^{n}\left(\sum_{j=0}^{\infty}P_{ij}(\theta)\theta^{j}\right)\left(\sum_{m=-\infty}^{+\infty}g_{ki}(t,m)\theta^{m}\right)=\sum_{j=0}^{\infty}\left(\sum_{i=0}^{n}d_{k,j}(i,t)\right)\theta^{j}+\\&+\sum_{j=0}^{\infty}\sum_{r=1}^{\infty}\left(\sum_{i=0}^{n}P_{ij}(0)g_{ki}(t,-r)\right)\theta^{j-r},\end{aligned}\tag{13}$$

where

$$d_{k,j}(i,t)=\sum_{m=0}^{j}P_{im}(0)g_{ki}(t,j-m),\qquad (k=0,1,....n).$$

Denoting

$$a_{jr}(k,t)=\sum_{i=0}^{n}P_{ij}(0)g_{ki}(t,-r).$$

we have

$$\begin{aligned}&\sum_{j=0}^{\infty}\sum_{r=1}^{\infty}a_{jr}(k,t)\theta^{j-r}=\sum_{j=0}^{\infty}\sum_{m=1}^{\infty}a_{j+m,m}(k,t)\theta^{j}+\\&+\sum_{j=0}^{\infty}\sum_{r=j+1}^{\infty}a_{jr}(k,t)\theta^{j-r}.\end{aligned}\tag{14}$$

Now let's deal with the second expression under the sign of integral in the left hand side of (12). Using the representation

$$\sum_{m=-\infty}^{\infty}g_{ki}(t,m)\theta^{m}=\sum_{m=0}^{\infty}g_{ki}(t,m)\theta^{m}+\sum_{m=1}^{\infty}g_{ki}(t,-m)\theta^{-m},\quad (|\theta|=1)$$

we obtain

$$\left(\frac{\mu}{\theta}-\mu-\nu\right)\sum_{m=-\infty}^{\infty}g_{ki}(t,m)\theta^{m}=\sum_{m=0}^{\infty}q_{ki}(t,m)\theta^{m}+\sum_{m=1}^{\infty}q_{ki}(t,-m)\theta^{-m},$$

where

$$q_{ki}(t,m)=\mu g_{ki}(t,m+1)-(\mu+\nu)g_{ki}(t,m).$$

Now it is easy to show that

$$\begin{aligned}&\left[\left(\tfrac{\mu}{\theta}-\mu-\nu\right)\sum_{m=-\infty}^{\infty}g_{ki}(t-\tau,m)\theta^{m}\right]\left[\sum_{r=0}^{n-i-1}(r-n+i)P_{ir}(\tau)\theta^{r}\right]=\\&=\sum_{j=0}^{\infty}\alpha_{ki}(t-\tau,j)\theta^{j}+\sum_{j=1}^{\infty}\alpha_{ki}(t-\tau,-j)\theta^{-j},\end{aligned}\tag{15}$$

where

$$\alpha_{ki}(t-\tau,j)=\sum_{r=0}^{n-i-1}\varepsilon_{i,r}(\tau)q_{ki}(t-\tau,j-r),$$

$$\varepsilon_{i,r}(\tau)=(r-n+i)P_{ir}(\tau).$$

Similarly for the third expression under the sign of integral in the right hand side of (12) we get

$$\begin{array}{l}\left[\sum\limits_{r=0}^{n-i}(r-n+i-1)P_{i-1,r}(\tau)\theta^{r}\right]\left[\sum\limits_{m=-\infty}^{\infty}g_{ki}(t-\tau,m)\theta^{m}\right]=\\=\sum\limits_{j=0}^{\infty}\gamma_{ki}(t-\tau,j)\theta^{j}+\sum\limits_{j=-1}^{\infty}\gamma_{ki}(t-\tau,-j)\theta^{-j},\end{array} \tag{16}$$

where

$$\gamma_{ki}(t-\tau,j)=\sum_{r=0}^{n-i}\varepsilon_{i-1.r}(\tau)g_{ki}(t-\tau,j-r).$$

Denoting

$$\begin{array}{l}m_{k,j}(t)=\sum\limits_{i=0}^{n}d_{k,j}(i,t),\\b_{k,j}(t)=\sum\limits_{m=1}^{\infty}a_{j+m,m}(k,t).\end{array}$$

On the base of (13)-(16) we obtain

$$\begin{array}{l}\sum\limits_{j=0}^{\infty}P_{k,j}(t)\theta^{j}=\sum\limits_{j=0}^{\infty}m_{k,j}(t)\theta^{j}+\sum\limits_{j=0}^{\infty}b_{k,j}(t)\theta^{j}+\\+\sum\limits_{j=0}^{\infty}\sum\limits_{r=j+1}^{\infty}a_{j,r}(k,t)\theta^{j-r}+\sum\limits_{j=0}^{\infty}\left(\sum\limits_{i=0}^{n}\int_{0}^{t}\alpha_{ki}(t-\tau,j)d\tau\right)\theta^{j}+\\+\sum\limits_{j=1}^{\infty}\left(\sum\limits_{i=0}^{n}\int_{0}^{t}\alpha_{ki}(t-\tau,-j)d\tau\right)\theta^{-j}+\\+\sum\limits_{j=0}^{\infty}\left(\nu\sum\limits_{i=0}^{n}\int_{0}^{t}\gamma_{ki}(t-\tau,j)d\tau\right)\theta^{j}+\sum\limits_{i=0}^{n}\left(\sum\limits_{i=0}^{n}\int_{0}^{t}\gamma_{ki}(t-\tau,-j)d\theta\right)\theta^{-j}\\(k=0,1,...,n)\end{array}$$

Taking equal the coefficients at the same degrees of θin the last relations we get

$$P_{k,j}(t)=m_{k,j}(t)+b_{k,j}(t)+\sum_{i=0}^{n}\int_{0}^{t}\alpha_{ki}(t-\tau,j)d\tau+\nu\sum_{i=0}^{n}\int_{0}^{t}\gamma_{ki}(t-\tau,j)d\tau,$$

or

$$P_{k.j}(t)=f_{k,j}(t)+\sum_{i=0}^{n}\int_{0}^{t}\beta_{ki}(t-\tau,j)d\tau,$$

where

$$\begin{array}{l}f_{k,j}(t)=m_{k,j}(t)+b_{k,j}(t),\\\beta_{ki}(t,j)=\alpha_{ki}(t,j)+\nu\gamma_{ki}(t,j)\end{array}.$$

Considering the expressions for α_{ki} and γ_{ki} for β_{ki} we obtain

$$\begin{array}{l}\beta_{ki}(t-\tau,j)=\sum\limits_{r=0}^{n-i}\left[\nu\varepsilon_{i-1,r}(\tau)g_{ki}(t-\tau,j-r)+\varepsilon_{i,r}(\tau)q_{ki}(t-\tau,j-r)\right]=\\=\sum\limits_{r=0}^{n-i}\left[\nu(r-n+i-1)g_{ki}(t-\tau,j-r)P(\tau)+(r-n+i)q_{ki}(t-\tau,j-r)P_{ir}(\tau)\right].\end{array}$$

Finally we have

$$P_{k,j}(t) = f_{k,j}(t)+$$
$$+\sum_{i=0}^{n}\sum_{r=0}^{n-i}\int_0^t \left[\nu(r-n+i-1)g_{ki}(t-\tau,j-r)P_{i-1,r}(\tau)+(r-n+i)q_{ki}(t-\tau,j-r)P_{ir}(\tau)\right]d\tau$$
$$k=0,1,...,n, \quad j=0,1,2,.... \tag{17}$$

To find $P_{k,j}(t)$, $(k=0,1,...,n, \quad j=0,1,2,...)$ it is sufficient to solve the system (17) for given values of $k.j=0,1,...,n-1.$.
In the case of $n=1$ according to (17) we get the system

$$\begin{array}{l} P_{k,j}(t) = f_{k,j}(t)+\int_0^t F_{k,j}(t-\tau)P_{0,0}(\tau)d\tau, \\ (k=0,1; \quad j=0,1,...) \end{array} \tag{18}$$

where

$$F_{k,j}(t) = (\mu+\nu)g_{k0}(t,j)-\nu g_{k1}(t,j)-\mu g_{k0}(t.j+1).$$

Assuming $k=0$ and $j=0$ in (18) one can have

$$P_{0,0}(t) = f_{0,0}(t)+\int_0^t F_{0,0}(t-\tau)P_{0,0}(\tau)d\tau$$

or in the terms of Laplace transformation

$$\tilde{P}_{0,0}(t) = \tilde{f}_{0,0}(t)+\tilde{F}_{0,0}(t)\tilde{P}_{0,0}(t),$$

from which follows

$$\tilde{P}_{0,0}(t) = \frac{\tilde{f}_{0,0}(t)}{1-\tilde{F}_{0,0}(t)}.$$

Knowing $P_{0,0}(t)$ on the base of (18) it is possible to find the distribution $\{P_{k,j}(t)\}$ for the values $k=0,\ 1;\ \ j=0,\ 1,\ \dots$.

In the general case solution of the system of linear integral equations (17) does not meet any difficulties. [7].

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