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HARMONIOUS COLORING OF MULTICOPY OF COMPLETE GRAPHS

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Abstract. In this paper, we find the harmonious chromatic number of multicopy of complete graphs K_n. We generalize the result $\chi_H((n+2)K_n) > \binom{n+1}{2}$ given in [8] and also further improve the result to $\chi_H((n+2)K_n) \geq \binom{n+1}{2} + 3, \ \forall\ n > 8$.

Keywords: Harmonious coloring, Corona product, Pigeonhole Principle.

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1. Introduction

A harmonious coloring [2, 3, 4, 5, 6, 7] of a simple graph G is a proper vertex coloring such that each pair of colors appears together on at most one edge. The harmonious chromatic number $\chi_H(G)$ is the least number of colors in such a coloring. It was shown by Hopcroft and Krishnamoorthy [5] that the problem of determining the harmonious chromatic number of a graph is NP-hard.

The concept of harmonious coloring of graphs has been studied extensively by several authors; see [9] for surveys. If G has m edges and G has a harmonious coloring with k colors, then clearly, $\binom{k}{2} \geq m$. Let $k(G)$ be the smallest integer satisfying the inequality. This number can be expressed as a function of m, namely

$$k(G) = \left\lceil \frac{1+\sqrt{8m+1}}{2} \right\rceil.$$

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The problem of harmonious coloring can be applied in such diverse areas as radio navigation, data acquisition and image compression.

Marek Kubale suggest the following application of harmonious coloring in [7]. Besides the satellite navigation system, one of frequently used aviation guiding systems in bad weather conditions or in the case of invisibility of ground objects is the radio navigation system. This system is based on a network of *Very high frequency Omnidirectional Range* (VOR) radio beacons. To identify the current position of a plane one has to measure the signals of two VOR radio beacons. For safety reason two proximate beacons cannot be sending signals of the same frequency and no two airways can have radio beacons sending the same signals on their terminal nodes.

Let us hypothetically assume that state authorities decided to modernize the existing network of radio beacons. In order to reduce the cost of this enterprise they decided to install as few types of beacons as possible. The following two questions arise: How many beacons of each type are needed? Where should the beacons be located at ground level? Perhaps every country in the world has a formal network (system) of airways. Two adjacent nodes of the network determine precisely one airway. By associating a vertex of graph G with each such node, and modeling each airway by an edge of G, we obtain a graph corresponding to the network of airways. Now, if we find $\chi_H(G)$ of G and its optimal coloring then the harmonious chromatic number will be exactly the number of various types of beacons requested by state authorities.

We introduce some definitions and results necessary in order to understand this paper. First of all we introduce the corona product of two graphs, that was originally introduced by Frucht and Harary in 1970 [1].

The *corona* of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2.
We state the well known pigeonhole principle next.

Pigeonhole Principle : Assume that p items are put into q boxes, where $p > q; p, q \in \mathbb{N}$. Then, there is at least one box containing more than one object.

In this paper, we collect the results found in [8] concerning the harmonious chromatic number of the corona product of any graph G of order l with the complete graph K_n for $l \leq n$, as a matter of completeness. Motivated by this work, we obtain the harmonious chromatic number of t copies of K_n where $t \leq n+2$. We believe that an interesting line for future research is to investigate the harmonious chromatic number for the corona graph product in general.

We start with the following results, found in [8], that we state without proof.

Lemma 1.1. [8] $\chi_H(lK_n) \geq \sum\limits_{i=n+1-l}^{n} i \ (1 \leq l \leq n)$.

The following proposition is an easy fact about harmonious chromatic number.

Proposition 1.1. [8] $G \subseteq H \Rightarrow \chi_H(G) \leq \chi_H(H)$.

Proposition 1.2. [8] *Let G be a graph of order n. Then* $\chi_H(G \circ K_n) \geq \dfrac{n^2+3n}{2}$.

Lemma 1.2. [8] *Let $l \in [n]$ and let G be a graph of order l. Then*

$$\chi_H(G \circ K_n) \leq \sum_{i=n+2-l}^{n+1} i.$$

Then, we have the following nice corollary.

Theorem 1.1. [8] *Let $l \in [n]$ and let G be a graph of order l. Then,*

$$\chi_H\left(G \circ K_n\right) = \sum_{i=n+2-l}^{n+1} i.$$

In particular when $l = n$, we obtain the following corollary.

Corollary 1.1. [8] *Let G be a graph of order n then, $\chi_H\left(G \circ K_n\right) = \frac{n^2+3n}{2}$.*

Theorem 1.2. [8] $\chi\left(nK_n\right) = \chi\left((n+1)K_n\right) = \binom{n+1}{2}$.

It is interesting to notice, as a matter of curiosity, that the harmonious coloring used on $(n+1)K_n$, repeats each color from 1 up to $\binom{n+1}{2}$ on exactly two different copies of K_n.

We will show that $\chi_H\left((n+2)K_n\right) > \binom{n+1}{2}$.

We have proved the following lemma in [8].

Lemma 1.3. [8] *Let C be any harmonious coloring of $(n+1)K_n$ with exactly $\binom{n+1}{2}$ colors. Then each one of the $\binom{n+1}{2}$ colors appears in exactly two components of $(n+1)K_n$.*

Therefore, we have the following result.

Proposition 1.3. [8] $\chi_H\left((n+2)K_n\right) > \binom{n+1}{2}$.

2. Main Results

The following theorem is a generalization of Proposition 1.3.

Theorem 2.1. $\chi_H((n+2)K_n) \geq \binom{n+1}{2} + 2$.

Proof. By contradiction. Assume to the contrary that $\chi_H((n+2)K_n) < \binom{n+1}{2} + 2$. It is clear that $\chi_H((n+2)K_n) \geq \binom{n+1}{2} + 1$. Thus, let us assume that $\chi_H((n+2)K_n) = \binom{n+1}{2} + 1$. Let C be a harmonious coloring of $(n+2)K_n$ with exactly $\binom{n+1}{2} + 1$ colors. For every color $i \in \left\{1, 2, \dots, \binom{n+1}{2} + 1\right\}$, let $\#(i)$ denote the number of components that have i assigned to some vertex in the graph $(n+2)K_n$. First of all, we show that $\max\left\{\#(i) : 1 \leq i \leq \binom{n+1}{2} + 1\right\} \geq 3$. Assume to the contrary that $\max\left\{\#(i) : 1 \leq i \leq \binom{n+1}{2} + 1\right\} \leq 2$. This implies that the number of vertices colored is at most $2\left(\binom{n+1}{2} + 1\right) = (n+1)n+2$. On the other hand $|V((n+2)K_n)| = n(n+2)$

and

$$\begin{aligned}(n+1)n+2 &\geq (n+2)n\\ \Rightarrow n &\leq 2.\end{aligned}$$

For any $n \in \mathbb{N}\backslash\{1,2\}$, we have $\max\left\{\#(i) : 1 \leq i \leq \binom{n+1}{2}+1\right\} \geq 3$. At this point, we show that $\max\left\{\#(i) : 1 \leq i \leq \binom{n+1}{2}+1\right\} \leq 3$.

Assume to the contrary that $\exists i \in \left\{1,2,\ldots,\binom{n+1}{2}+1\right\}$ such that $\#(i) \geq 4$.

Let $K_n^{(1)} \cup K_n^{(2)} \cup \cdots \cup K_n^{(n+2)}$ be the components of $(n+2)K_n$. Also, without loss of generality let 1 be a color that has been assigned to one vertex in each of the components $K_n^{(1)}$, $K_n^{(2)}$, $K_n^{(3)}$, $K_n^{(4)}$. Then, it is clear that there are $1+4(n-1) = 4n-3$ different colors used on the vertices of the components $K_n^{(1)}$,$K_n^{(2)}$, $K_n^{(3)}$, $K_n^{(4)}$. At this point let us color the vertices of the component $K_n^{(5)}$. By the pigeonhole principle we need at least $(n-4)$ new colors to color the vertices of $K_n^{(5)}$. If we keep reasoning in this same way, it is clear that to color the vertices of $K_n^{(n)}$ we need at least 1 new color. Thus if we add up all the colors needed, we get that we need at least

$$(4n-3) + \sum_{i=1}^{n-4} i = \frac{n^2+n+6}{2}$$

different colors. However, we only have $\binom{n+1}{2}+1 = \frac{n^2+n+2}{2}$ colors. This implies we do not have enough colors. Contradiction. Hence, any harmonious coloring of $(n+2)K_n$ has the property that $\max\left\{\#(i) : 1 \leq i \leq \binom{n+1}{2}+1\right\} = 3$.

Next we show that in any such harmonious coloring $\exists!\, i : \left(1 \leq i \leq \binom{n+1}{2}+1\right)$ such that $\#(i) = 3$.

We again proceed by contradiction. Let C be a harmonious coloring of $(n+2)K_n$ with $\binom{n+1}{2}+1$ different colors and such that

(1) $\max\left\{\#(i) : 1 \leq i \leq \binom{n+1}{2}+1\right\} = 3$.

(2) $\exists\, k \neq l : \left(1 \leq l \leq \binom{n+1}{2}+1\right)$ such that $\#(k) = \#(l) = 3$.

We consider two cases:

Case 1:: Let 1 be a color of 3 components of $(n+2)K_n$. Without loss of generality, let these components be $K_n^{(1)}$, $K_n^{(2)}$, $K_n^{(3)}$. Also let 2 be a color of one of these components. Without loss of generality let 2 be a color of $K_n^{(1)}$ and assume that $\#(2) = 3$. Again, without loss of generality assume that 2 is a color on $K_n^{(4)}$, $K_n^{(5)}$.

Now, in order to color the vertices of $K_n^{(1)}$, $K_n^{(2)}$ and $K_n^{(3)}$, we need n different colors for $K_n^{(1)}$, $(n-1)$ new colors for $K_n^{(2)}$ and $(n-1)$ new colors for $K_n^{(3)}$. Hence, so far we need $(3n-2)$ different colors.

Next, we cannot use the colors used on $K_n^{(1)}$ in order to color $K_n^{(4)}$, $K_n^{(5)}$ since we assumed that color 2 is also on $K_n^{(1)}$. Therefore on $K_n^{(4)}$ we can only repeat one color of $K_n^{(2)}$ and one color of $K_n^{(3)}$. The other $(n-3)$ colors must be new. A similar reasoning shows that we also need $(n-3)$ new colors in order to color the vertices of $K_n^{(5)}$. Therefore in order to color the vertices of the set $\bigcup_{i=1}^{6} V(K_n^{(i)})$ we need at least $(3n-2)+2(n-3)$ different colors. By the pigeonhole principle, we need at least $(n-5)$ colors to color the vertices of $K_n^{(6)}$. If we keep reasoning in this way, we will finally conclude that we need 1 new color in order to color the vertices of $K_n^{(n)}$. Therefore, adding up all the colors needed, we need

$$\begin{aligned} (3n-2)+2(n-3)+\sum_{i=1}^{n-5} i &= \frac{n^2+n+4}{2} \text{ colors} \\ \text{and } \frac{n^2+n+4}{2} > \frac{n^2+n+2}{2} &= \binom{n+1}{2}+1. \quad \text{Contradiction.} \end{aligned}$$

Case 2:: Assume that the components $K_n^{(1)}, K_n^{(2)}, K_n^{(3)}$ are colored with the numbers from 1 up to $(3n-2)$ with $\#(1)=3$. Also assume that $\#(3n-1)=3$ and that $3n-1$ has been assigned to the components $K_n^{(4)}$, $K_n^{(5)}$, $K_n^{(6)}$. First of all, notice that just like in the previous case, we need $(3n-2)$ colors in order to color all the vertices of the set $\bigcup_{i=1}^{3} V(K_n^{(i)})$. Next, let us concentrate on component $K_n^{(4)}$. Since $(3n-1)$ is not a color of any component $K_n^{(1)}$, $K_n^{(2)}$, $K_n^{(3)}$, it follows that we can repeat three colors, one of $K_n^{(1)}$, one of $K_n^{(2)}$ and one of $K_n^{(3)}$. The rest of the colors are new. Thus, we have $(n-3)$ new colors on $K_n^{(4)}$. Now on $K_n^{(5)}$ and $K_n^{(6)}$ we cannot use any color used on $K_n^{(4)}$ except for color $(3n-1)$. Hence, we can only repeat 3 colors of the components $K_n^{(1)}$, $K_n^{(2)}$, $K_n^{(3)}$. That is to say, we need $(n-4)$ new colors on $K_n^{(5)}$ and $(n-4)$ new colors on $K_n^{(6)}$. Therefore, we need a total of $(3n-2)+(n-3)+2(n-4)=6n-13$ colors in order to color the vertices of the set $\bigcup_{i=1}^{6} V(K_n^{(i)})$.

Once again, applying the pigeonhole principle, on the components $K_n^{(7)}$, $K_n^{(8)}$, …, $K_n^{(n)}$, we conclude that we need

$$\left(\sum_{i=1}^{n-6} i\right)+(6n-13)=\frac{n^2+n+4}{2}$$

different colors. Again a contradiction. From this, we can conclude that if C is a harmonious coloring of $(n+2)K_n$ with exactly $\binom{n+2}{2}+1$ colors, then $\exists!\, i : \left(1 \le i \le \binom{n+2}{2}+1\right)$ such that $\#(i)=3\,(\text{if } k \ne i \text{ then } \#(k) \le 2)$.

Finally, we will see that such a coloring cannot exist. One more time, we proceed by contradiction. Assume to the contrary that such a coloring exists.

Now, $\left|V\Big((n+2)K_n\Big)\right| = (n+2)n$. Also, we have $\binom{n+1}{2}+1$ colors avaliable. One of them is repeated three times, and the other colors have been repeated at most two times. Hence we have colored at most $3+2\binom{n+1}{2}$ vertices. Now, $3+2\binom{n+1}{2} = 3+n^2+n < n^2+2n$ $(n>3)$. Contradiction.

□

We can further improve our previous result as follows:

Theorem 2.2. $\chi_H((n+2)K_n) \geq \binom{n+1}{2}+3,\ \forall\ n>8.$

Proof. For this proof, we assume that $n>8$. We proceed to prove this result by contradiction. Assume to the contrary that there exists a harmonious coloring C of the set $V((n+2)K_n)$ with colors in the set $\left\{1,2,\dots,\binom{n+1}{2}+2\right\}$. From now on, we will assume that $(n+2)K_n \cong \bigcup\limits_{(i=1)}^{(n+2)} K_n^{(i)}$ where $K_n^{(i)} \cong K_n \quad \forall i:\ (1\leq i\leq n+2)$. Also, for any color $i \in \left\{1,2,\dots,\binom{n+1}{2}+2\right\}$ we will use again the notation $\#(i)$ in order to denote the number of vertices that receive color i under the coloring C. At this point, notice that we have exactly $\binom{n+1}{2}+2 = \dfrac{n^2+n+4}{2}$ colors available, and with them we have to be able to obtain a harmonious coloring of $(n+2)K_n$, which has order $(n+2)n = n^2+2n$.

Now, assume that $\exists i \in \left\{1,2,\dots,\binom{n+1}{2}+2\right\}$ such that $\#(i)\geq 4$ under our coloring C. Without loss of generality assume that $i=1$ and let 1 be assigned to some vertex of $K_n^{(1)}, K_n^{(2)}, K_n^{(3)}, K_n^{(4)}$ under coloring C.

At this point, let us see how the colors must be distributed on the vertices of $(n+2)K_n$, starting with component $K_n^{(1)}$ and keep going in super index increasing order.

It is clear that we need exactly n colors in order to color the vertices of $K_n^{(1)}$. It is also clear that we need $(n-1)$ new colors in order to color the vertices of $K_n^{(2)}$, $(n-1)$ new colors in order to color the vertices of $K_n^{(3)}$ and $(n-1)$ new colors in order to color the vertices of $K_n^{(4)}$. By the pigeonhole principle it is clear that we need at least $(n-4)$ new colors in order to color the vertices of $K_n^{(5)}$, at least $(n-5)$ new colors in order to color the vertices of $K_6^{(6)}$ and so on. Therefore, adding up all the colors needed, we conclude that, we need at least $(n+3(n-1)) + \sum\limits_{i=1}^{n-4} i = \dfrac{n^2+n+6}{2}$ different colors in order to get our harmonious coloring of $(n+2)K_n$. But we only have $\dfrac{n^2+n+4}{2}$ colors available. This is a contradiction. Hence, we can conclude that in order to obtain a harmonious coloring of $(n+2)K_n$ with at most $\binom{n+1}{2}+2$ colors, the coloring C used must have the property that $\#(i)\leq 3\,\forall i$ in the coloring. At this point assume that there are at most two distinct colors in $\left\{1,2,\dots,\binom{n+1}{2}+2\right\}$, namely colors i and j such that $\#(i)=\#(j)=3$.

Then, the following must be true $\#(k) \leq 2 \,\forall\, k \in \{1, 2, \ldots, \binom{n+1}{2} + 2\} \backslash \{i, j\}$. Thus, we have been able to color at most $\left(\frac{n^2+n+4}{2} - 2\right) 2 + 6 = n^2 + n + 6$ vertices. However we need to be able to color exactly $n^2 + 2n$ vertices, and $n^2 + n + 6 < n^2 + 2n$ for $n > 6$. Therefore, we cannot color all the vertices. Thus, we have to assume that $\exists \{i, j, k\} \subseteq \left\{1, 2, \ldots, \binom{n+1}{2} + 2\right\}$ such that $\#(i) = \#(j) = \#(k) = 3$ under C.

We consider 4 different cases next.

Case 1:: Assume that colors i, j and k are not adjacent under C. Without loss of generality assume that $i = 1, j = 2$ and $k = 3$, and that 1 has been assigned to $K_n^{(1)}, K_n^{(2)}$ and $K_n^{(3)}$, 2 has been assigned to $K_n^{(4)}, K_n^{(5)}$ and $K_n^{(6)}$ and 3 has been assigned to $K_n^{(7)}, K_n^{(8)}$ and $K_n^{(9)}$. Then we need exactly n different colors to color the vertices of $K_n^{(1)}$, and since color 1 is repeated on components $K_n^{(2)}$ and $K_n^{(3)}$, it follows that we need $(n-1)$ new colors on each one of these two components. Now, in order to color the vertices of $K_n^{(4)}$, we use color 2, which, at this point is a new color. Furthermore, by the pigeonhole principle, we can only use 3 of the colors used on the components $K_n^{(1)}, K_n^{(2)}$ and $K_n^{(3)}$, in order to color the vertices of $K_n^{(4)}$. Therefore, we used at least $(n-3)$ new colors in order to color the vertices of $K_n^{(4)}$. On components $K_n^{(5)}$ and $K_n^{(6)}$, we have color 2, which at this point is not a new color anymore. In addition, we cannot use any color appearing on any other component using color 2. Thus, by the pigeonhole principle we need $(n-4)$ new colors on each of these two components. Finally, we need to color the vertices of components $K_n^{(7)}, K_n^{(8)}$ and $K_n^{(9)}$. To color component $K_n^{(7)}$, we use color 3 for the first time, and hence, at this point color 3 is a new color. Also by the pigeonhole principle, we can repeat at most 6 colors among the colors appearing among the first six components. Then, we need at least $(n-6)$ colors in order to color the vertices of $K_n^{(7)}$. Repeating the previous reasoning, we get that we need at least $(n-7)$ new colors in order to color the vertices of components $K_n^{(8)}$ and $K_n^{(9)}$. Thus, we need at least $n + 2(n-1) + (n-3) + 2(n-4) + (n-6) + 2(n-7)$ different colors in order to color the vertices of the components $K_n^{(1)}$, $K_n^{(2)}, \ldots, K_n^{(9)}$. This adds up to $(9n-33)$ new colors.

For the rest of the components, using the pigeonhole principle, we see that we need at least $(n-9)$ new colors in order to color $K_n^{(10)}$, $(n-10)$ new colors in order to color $K_n^{(11)}$ and so on. Then, we need at least $(9n-33) + \sum_{i=1}^{n-9} i = \frac{n^2+n+6}{2}$ different colors in order to color the vertices of $(n+2)K_n$. This is a contradiction, since we only have $\frac{n^2+n+4}{2}$ different colors available.

Case 2:: Assume that among the colors i, j and k there are two colors assigned to the same component by coloring C. Without loss of generality let $i = 1, j = 2$ and $k = 3$, and assume that 1 and 2 are assigned to $K_n^{(1)}$. Then, again without loss of generality, assume that color 1 has been assigned to $K_n^{(1)}$, $K_n^{(4)}$, and $K_n^{(5)}$, and let color 3 belong to $K_n^{(6)}$, $K_n^{(7)}$, $K_n^{(8)}$. Obviously, in order to color the vertices of $K_n^{(1)}$ we need to use n colors. Also in order to color the vertices of $K_n^{(2)}$ and $K_n^{(3)}$ we need at least $(n-1)$ colors on each component. Therefore so far, we have used $n+2(n-1)$ different colors, at least. In order to color the vertices of $K_n^{(4)}$ we cannot use any color already used on $K_n^{(1)}$, except of course, for color 2. We can only use one color used on $K_n^{(2)}$ and one color used on $K_n^{(3)}$, by the pigeonhole principle. Therefore, we need to use at least $(n-3)$ new colors in order to color the vertices of $K_n^{(4)}$. In order to color the vertices of $K_n^{(5)}$, we already used color 2. This eliminates any other color assigned to component $K_n^{(1)}$ by C. We can use one color used on $K_n^{(2)}$ and one color used on $K_n^{(3)}$, by the pigeonhole principle. We cannot use any color assigned to $K_n^{(4)}$, other than color 2 already used. Therefore, we need at least $(n-3)$ new colors in order to color the vertices of $K_n^{(5)}$. Now, in order to color the vertices of $K_n^{(6)}$, we use color 3 for first time and we can use, by the pigeonhole principle at most one color used in each of the components $K_n^{(i)}$; $i \in \{1, 2, \ldots, 5\}$. Therefore, we need at least $(n-5)$ new colors to color $K_n^{(6)}$. Now, in order to color $K_n^{(7)}$ we already used color 3 from $K_n^{(6)}$, and we can use exactly one color from each of the components $K_n^{(i)}$ for $1 \le i \le 5$. Thus, we need exactly $(n-6)$ new colors in order to be able to color the vertices of $K_n^{(7)}$. Next, to color the vertices of $K_n^{(8)}$, we use color 3 that appears on components $K_n^{(6)}$ and $K_n^{(7)}$ and one color from each of the components $K_n^{(i)}$; $1 \le i \le 5$. Thus, we need at least $(n-6)$ new colors again. In order to color $K_n^{(9)}$, we need at least $(n-8)$ new colors by the pigeonhole principle. In order to color $K_n^{(10)}$ we need at least $(n-9)$ new colors by the pigeonhole principle, and so on. Therefore, in total we need at least $n+2(n-1)+2(n-3)+(n-5)+2(n-6)+\sum_{i=1}^{n-8} i = \dfrac{n^2+n+6}{2}$ different colors in order to obtain our coloring C. But, we only have $\dfrac{n^2+n+4}{2}$ different colors available. This is a contradiction.

Case 3:: Assume that colors i, j and k are assigned to the same component. Without loss of generality let $i=1, j=2$ and $k=3$, and let the component that contains 1,2 and 3 be $K_n^{(1)}$. Then, we need n different colors in order to color $K_n^{(1)}$. At this point and without loss of generality, assume that 1 is assigned to $K_n^{(2)}$ and $K_n^{(3)}$, 2 is assigned to $K_n^{(4)}$ and $K_n^{(5)}$ and 3 is assigned to $K_n^{(6)}$ and $K_n^{(7)}$. It is clear that in order to color $K_n^{(2)}$ and $K_n^{(3)}$ we need at least $(n-1)$ new colors on $K_n^{(2)}$ and $(n-1)$ new colors on $K_n^{(3)}$ since color 1 is shared by these three components and hence no other color can be repeated in these three components. In order to color component $K_n^{(4)}$, we already used color 2. Thus, we cannot use any other color from component $K_n^{(1)}$. By the pigeonhole principle, we can use at most one color from $K_n^{(2)}$ and at most one color from $K_n^{(3)}$. Therefore, we need at least $(n-3)$ new colors in order to color the vertices of $K_n^{(4)}$. To color the vertices of $K_n^{(5)}$, we already used color 2. Hence we cannot use any other color from component $K_n^{(1)}$. By the pigeonhole principle, we can use exactly one color from $K_n^{(2)}$ and exactly one color from $K_n^{(3)}$. But since 2 is a color of $K_n^{(4)}$, it follows that we cannot use any other color from $K_n^{(4)}$, and 2 has already been used. Then we need at least $(n-3)$ new colors in order to color the vertices of $K_n^{(5)}$. Color 3 is in $K_n^{(6)}$ and color 3 has been used in $K_n^{(1)}$. Hence by the pigeonhole principle we can use at most one color from each of $K_n^{(2)}$, $K_n^{(3)}$, $K_n^{(4)}$, $K_n^{(5)}$. Therefore we need at least

$(n-5)$ new colors in order to color $K_n^{(6)}$. The pigeonhole principle together with the fact that 3 is a color on both $K_n^{(6)}$ and $K_n^{(7)}$ allows us to conclude that we need at least $(n-5)$ new colors in order to color the vertices of $K_n^{(7)}$. For the rest of the components we need at least $\sum_{i=1}^{(n-7)} i$ new colors by the pigeonhole principle. Therefore we need at least $n + 2(n-1+n-3+n-5) + \sum_{i=1}^{(n-7)} i = \dfrac{n^2+n+6}{2}$ different colors in order to color the vertices of our graph. Contradicition since we only have $\dfrac{n^2+n+4}{2}$ colors available.

Case 4:: For this case we assume that there are 3 adjacencies among colors i, j and k and that these adjacencies appear in three different components. Without loss of generality assume the following facts.

(1) $i = 1, j = 2, k = 3$.
(2) Component $K_n^{(1)}$ contains colors 1 and 2. Component $K_n^{(2)}$ contains colors 1 and 3.
(3) Component $K_n^{(3)}$ contains colors 2 and 3.

At this point it is clear that we need n different colors in order to color the vertices of $K_n^{(1)}$ and at least $(n-1)$ new colors to color the vertices of $K_n^{(2)}$. Since 2 and 3 are colors on $K_n^{(1)}$ and $K_n^{(2)}$ respectively, it follows that we need at least $(n-2)$ new colors in order to color the vertices of $K_n^{(3)}$. Now, if 1 is a color of $K_n^{(4)}$ and 1 belongs to the components $K_n^{(1)}$ and $K_n^{(2)}$ we cannot use any other colors of these two components and we can choose one color from $K_n^{(3)}$. Hence we need at least $(n-2)$ new colors in order to color the vertices of $K_n^{(4)}$. If, without loss of generality, we assume that 2 is a color of $K_n^{(5)}$ and 3 is a color of $K_n^{(6)}$, we conclude , using similiar reasonings to the ones used so far that we need at least $(n-3)$ new colors to color the vertices of $K_n^{(5)}$ and $(n-4)$ new colors to color the vertices of $K_n^{(6)}$. Therefore, we need at least $n + (n-1) + 2(n-2) + (n-3) + (n-4) + \sum_{i=1}^{(n-6)} i = \dfrac{n^2+n+6}{2}$ different colors in order to color the vertices of our graph. Contradicition since we only have $\dfrac{n^2+n+4}{2}$ colors available. Therefore, at this point, we can conclude that $\chi_H((n+2)K_n) \geq \binom{n+1}{2} + 3, \ \forall\, n > 8$.

□

Observation 2.1. *The following matrices correspond to harmonious coloring of graphs isomorphic to $(n+2)K_n$, for different values of n.*

$$n = 3 \Rightarrow 5K_3$$

$$\begin{matrix} 1 & 1 & 2 & 3 & 1 \\ 2 & 4 & 4 & 5 & 6 \\ 3 & 5 & 6 & 6 & 7 \end{matrix}$$

$$\text{Number of colors needed } \binom{n+1}{2} + 1 = \binom{4}{2} + 1 = 7.$$

$$n = 4 \Rightarrow 6K_4$$

1	1	2	3	4	1
2	5	5	6	7	8
3	6	8	8	9	11
4	7	9	10	10	12

Number of colors needed $\binom{n+1}{2}+2=\binom{5}{2}+2=12.$

$$n=5\Rightarrow 7K_5$$

1	1	2	3	4	5	1
2	6	6	7	8	9	10
3	7	10	10	11	12	15
4	8	11	13	13	14	16
5	9	12	14	15	15	17

Number of colors needed $\binom{n+1}{2}+2=\binom{6}{2}+2=17.$

$$n=6\Rightarrow 8K_6$$

1	1	2	3	4	5	6	1
2	7	7	8	9	10	11	12
3	8	12	12	13	14	15	19
4	9	13	16	16	17	18	22
5	10	14	17	19	19	20	23
6	11	15	18	20	21	21	24

Number of colors needed $\binom{n+1}{2}+3=\binom{7}{2}+3=24.$

$$n=7\Rightarrow 9K_6$$

1	1	2	3	4	5	6	7	1
2	8	8	9	10	11	12	13	14
3	9	14	14	15	16	17	18	23
4	10	15	19	19	20	21	22	28
5	11	16	20	23	23	24	25	29
6	12	17	21	24	26	26	27	30
7	13	18	22	25	27	28	28	31

Number of colors needed $\binom{n+1}{2}+3=\binom{6}{2}+3=31.$

$$n=8\Rightarrow 10K_8$$

1	1	2	3	4	5	6	7	8	1
2	9	9	10	11	12	13	14	15	16
3	10	16	16	17	18	19	20	21	27
4	11	17	22	22	23	24	25	26	34
5	12	18	23	27	27	28	29	30	37
6	13	19	24	28	31	31	32	33	38
7	14	20	25	29	32	34	34	35	39
8	15	21	26	30	33	35	36	36	40

Number of colors needed $\binom{n+1}{2} + 4 = \binom{9}{2} + 4 = 40.$

$$n = 9 \Rightarrow 11K_9$$

1	1	2	3	4	5	6	7	8	9	1
2	10	10	11	12	13	14	15	16	17	18
3	11	18	18	19	20	21	22	23	24	31
4	12	19	25	25	26	27	28	29	30	40
5	13	20	26	31	31	32	33	34	35	45
6	14	21	27	32	36	36	37	38	39	46
7	15	22	28	33	37	40	40	41	42	47
8	16	23	29	34	38	41	43	43	44	48
9	17	24	30	35	39	42	44	45	45	49

Number of colors needed $\binom{n+1}{2} + 4 = \binom{10}{2} + 4 = 49.$

$$\vdots$$

which yields the following Conjecture

Conjecture 2.1. $\chi_H((n+2)K_n) \geq \binom{n+1}{2} + \left\lfloor \frac{n}{2} \right\rfloor \ \forall n \in \mathbb{N}\backslash\{1\}.$

This suggests us the following question.

Question 2.1. *Let $l, n \in \mathbb{N}$. Compute $\chi_H(lK_n)$.*

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