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NUMERICAL SOLUTION OF AN INVERSE PROBLEM FOR THE LIOUVILLE EQUATION

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ABSTRACT. We consider an inverse problem for the Liouville Equation. We present the solvability conditions and obtain numerical solution of the problem based on the finite difference approximation.

Keywords: Inverse problem, Liouville equation, finite difference method.

AMS Subject Classification: 35R30, 65N21

1. INTRODUCTION

In this work, we consider the Liouville equation

$$Lu \equiv \frac{\partial u}{\partial t} + \{H, u\} = \lambda(x, v, t) \tag{1}$$

in the domain $\Omega = \{(x, v, t) : x \in D \subset R^n,\ v \in G \subset R^n,\ t \in (0, T)\}$, where

$$\{H, u\} = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial v_i} \frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial v_i} \frac{\partial H}{\partial x_i} \right)$$

and the boundary $\partial\Omega$ is sufficiently smooth.

In applications, $u(x, v, t)$ is the density of distribution of the number of the particles in the phase space, $H(x, v, t)$ is the Hamiltonian, $\lambda(x, v, t)$ is a source function, x is the space coordinate vector, v and t denote the velocity and time, respectively. The Liouville equation characterizes the continuity of the motion of a substance with phase volume conservation. It is used for quantitative and qualitative description of many physical, chemical, biological, social and other processes [3].

There have been many works on the direct problems for the Liouville equation, [See, e. g. 9]. As for the inverse problems, we refer to [1, 3], where the uniqueness of the solution was investigated. Numerical solution of some inverse problems for the stationary kinetic and transport equations were studied in [6-8]. To the best of our knowledge, there has been

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no study devoted to numerical solution of such inverse problems for the non-stationary Liouville equation.

In this paper, we investigate the solvability conditions and numerical solution of the following inverse problem:

Problem 1. Determine the functions $u(x,v,t)$ and $\lambda(x,v,t)$ that satisfy equation (1) provided that the trace of the solution $u(x,v,t)$ on the boundary $\partial\Omega$ is known:

$$u|_{\partial\Omega} = u_0. \tag{2}$$

The uniqueness of the solution of Problem 1 will be proved in the same way as in [1, p. 86], [3, p. 43].

Theorem 1. *Let the Hamiltonian* $H(x,v,t) \in C^2\left(\overline{\Omega}\right)$ *satisfy the conditions*

$$\sum_{i,j=1}^{n} \frac{\partial^2 H}{\partial v_i \partial v_j}\xi^i\xi^j \geq \alpha_1 |\xi|^2, \quad \sum_{i,j=1}^{n} \frac{\partial^2 H}{\partial x_i \partial x_j}\xi^i\xi^j \leq -\alpha_2 |\xi|^2, \tag{3}$$

for all $(x,v,t) \in \overline{\Omega}$, $\xi \in R^n$, where α_1, α_2 *are positive numbers. We assume that the function* $\lambda(x,v,t)$ *satisfies the equation*

$$\widehat{L}\lambda \equiv \sum_{j=1}^{n} \frac{\partial^2 \lambda}{\partial v_j \partial x_j} = 0. \tag{4}$$

Then Problem 1 has at most one solution (u,λ) *such that* $u \in C^2\left(\overline{\Omega}\right), \lambda \in C^2(\Omega)$.

Proof. Let (u,λ) *be a solution to Problem 1 such that* $u = 0$ *on* $\partial\Omega$ *and* $u \in C^2\left(\overline{\Omega}\right)$, $\lambda \in C^2(\Omega)$. *Since (4) holds for the function* λ, *we have*

$$\sum_{i=1}^{n} \frac{\partial u}{\partial v_i}\frac{\partial \lambda}{\partial x_i} = \sum_{i=1}^{n} \frac{\partial}{\partial v_i}\left(u\frac{\partial \lambda}{\partial x_i}\right). \tag{5}$$

On the other hand, we can write

$$\begin{aligned} &2\sum_{i=1}^{n} \frac{\partial u}{\partial v_i}\frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial t} + \{H,u\}\right) \\ = \; &\sum_{i,j=1}^{n}\left(\frac{\partial^2 H}{\partial v_i \partial v_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - \frac{\partial^2 H}{\partial x_i \partial x_j}\frac{\partial u}{\partial v_i}\frac{\partial u}{\partial v_j}\right) \\ &+\sum_{i=1}^{n}\left[\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial v_i}\frac{\partial u}{\partial x_i}\right) + \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial v_i}\left(\frac{\partial u}{\partial t} + \{H,u\}\right)\right)\right. \\ &\left.-\frac{\partial}{\partial v_i}\left(\frac{\partial u}{\partial x_i}\left(\frac{\partial u}{\partial t} + \{H,u\}\right)\right)\right] \\ &+\sum_{i,j=1}^{n}\left[\frac{\partial}{\partial x_j}\left(\frac{\partial H}{\partial v_j}\frac{\partial u}{\partial v_i}\frac{\partial u}{\partial x_i}\right) - \frac{\partial}{\partial v_j}\left(\frac{\partial H}{\partial x_j}\frac{\partial u}{\partial v_i}\frac{\partial u}{\partial x_i}\right)\right]. \end{aligned} \tag{6}$$

Taking into account the geometry of the domain Ω *and condition* $u = 0$ *on* $\partial\Omega$, from (5)-(6), *we get*

$$\sum_{i,j=1}^{n}\int_\Omega \left(\frac{\partial^2 H}{\partial v_i \partial v_j}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - \frac{\partial^2 H}{\partial x_i \partial x_j}\frac{\partial u}{\partial v_i}\frac{\partial u}{\partial v_j}\right)d\Omega = 2\sum_{i=1}^{n}\int_\Omega \frac{\partial u}{\partial v_i}\frac{\partial \lambda}{\partial x_i}d\Omega = 0. \tag{7}$$

By condition (3), we have

$$\alpha_1 \int_\Omega |\nabla_x u|^2 \, d\Omega + \alpha_2 \int_\Omega |\nabla_v u|^2 \, d\Omega \leq 0. \tag{8}$$

Since Ω is bounded and $u = 0$ on $\partial\Omega$, inequality (8) implies $u = 0$ in Ω. Hence, by equation (1), we have $\lambda = 0$ in Ω, which completes the proof of the theorem. □

It is easy to check that condition (4) holds, for example, for any function λ of the form $\lambda = \lambda_1(x,t) + \lambda_2(v,t)$, where λ_1 and λ_2 are continuously differentiable functions.

As for the existence of the solution of Problem 1, we reduce the problem to the following one with homogeneous boundary data:

Problem 2. Determine (u, λ) from the relations

$$\begin{aligned} Lu &= \lambda(x,v,t) + F(x,v,t), & (9)\\ u|_{\partial\Omega} &= 0, \ \widehat{L}\lambda = 0, & (10) \end{aligned}$$

provided that the boundary $\partial\Omega$ is sufficiently smooth and the concordance conditions for the data are satisfied. In (9), F is a known function in $H_2(\Omega)$.

Then the following theorem can be proven by the same method presented in [1] which is based on the Galerkin method and we will omit the proof here.

Theorem 2. *Under the hypothesis of Theorem 1, there exists a solution (u, λ) of Problem (2) in $H_1(\Omega)\times$ $L_2(\Omega)$.*

We note that the solvability of Problem 1 depends on the geometry of the domain Ω. Namely, it is necessary that Ω can be represented in the form of the direct product of the three domains D, G and $(0,T)$.

Next, we give another problem where the geometry of the domain is not essential for the solvability.

Problem 3. Find a pair of functions $(u(x,v), \lambda(x,v))$ defined in $D \times G$ that satisfy the stationary form of equation (1) and the conditions

$$\nabla u|_{\partial(D\times G)} = u_0, \ \ \nabla_v \lambda|_{\partial(D\times G)} = \lambda_0, \ u(x_0, v_0) = u_1, \tag{11}$$

where (x_0, v_0) is a point in $D \times G$.

Theorem 3. *Under the hypothesis of Theorem 1, Problem 3 has at most one solution (u, λ) such that $u \in C^2\left(\overline{D \times G}\right), \lambda \in C^2\left(\overline{D \times G}\right)$.*

Proof. Suppose that (u, λ) is a solution to problem (3) such that $u_0 = \lambda_0 = u_1 = 0$. Using relations (5)-(8) in the stationary case and by the fact that $\nabla_v \lambda$ is given on the entire boundary we obtain $u_{x_i} = u_{v_i} = 0$, $i = 1, 2, ..., n$. Then equation (1) in the stationary case implies $\lambda = 0$ in $D \times G$. Since $u(x_0, v_0) = u_1 = 0$, it follows that $u \equiv 0$ in $D \times G$, which completes the proof. □

It is worth noting here that, by using the semi-group theory, we can have a more general result devoted to solvability of Problem 1. Since the operator $L := \{H, \cdot\}$ is the infinitesimal generator of a contractive operator semigroup, see [4], we can prove the existence, uniqueness and stability of the solution of Problem 1 based on the method used in [5, p. 489].

2. The Finite Difference Method

Now we concern with the construction of finite difference approximation for the following inverse problem:

Problem 4. Find (u, λ) from the relations

$$u_t(x,v,t) + H_v(x,v,t)\,u_x(x,v,t) - H_x(x,v,t)\,u_v(x,v,t) = \lambda(x,v,t) + F(x,v,t), \quad (12)$$

$$u(x,v,t)|_{\partial\Omega} = 0,$$

$$\widehat{L}\lambda = 0,$$

where $\Omega = \{(x,v,t)|\ x \in (a,b) \subset \mathbb{R},\ v \in (c,d) \subset \mathbb{R},\ t \in (e,f) \subset \mathbb{R}\}$.

By applying the operator $\widehat{L}$ to both sides of equation (12), we get an auxiliary Dirichlet boundary value problem for a third order partial differential equation:

$$u_{txv}+u_{xvx}H_v-u_{vvx}H_x+u_{xx}H_{vv}-u_{vv}H_{xx}+u_{xv}H_{vx}-u_{vx}H_{xv}+u_xH_{vvx}-u_vH_{xvx} = \mathcal{F}(x,v,t), \quad (13)$$

$$u|_{\partial\Omega} = 0. \quad (14)$$

By using the central finite difference formulas in (13)-(14), we obtain the following discrete version of the previous problem:

$$\begin{aligned}
&(-k_2+k_1)\,\tilde{u}^k_{i-1,j-1} + (2k_2 - k3 + k_5)\,\tilde{u}^k_{i,j-1} + (-k_1-k_2)\,\tilde{u}^k_{i+1,j-1}\\
&+(-2k_1+k_4-k_6)\,\tilde{u}^k_{i-1,j} + (-2k_4+2k_3)\,\tilde{u}^k_{i,j} + (2k_1+k_4+k_6)\,\tilde{u}^k_{i+1,j}\\
&+(k_1+k_2)\,\tilde{u}^k_{i-1,j+1} + (-2k_2-k_3-k_5)\,\tilde{u}^k_{i,j+1} + (k_2-k_1)\,\tilde{u}^k_{i+1,j+1}\\
&+(k_7)(\tilde{u}^{k+1}_{i+1,j+1} - \tilde{u}^{k+1}_{i-1,j+1} - \tilde{u}^{k+1}_{i+1,j-1} + \tilde{u}^{k+1}_{i-1,j-1} - \tilde{u}^{k-1}_{i+1,j+1} + \tilde{u}^{k-1}_{i-1,j+1}\\
&+\tilde{u}^{k-1}_{i+1,j-1} - \tilde{u}^{k-1}_{i-1,j-1}) = \widetilde{f}^k_{i,j},\ i=1,...,I,\ j=1,...,J,\ k=1,...,K; \quad (15)
\end{aligned}$$

$$\begin{aligned}
&\tilde{u}^k_{0,j} = \tilde{u}^k_{I+1,j} = \tilde{u}^k_{i,0} = \tilde{u}^k_{i,J+1} = \tilde{u}^0_{i,j} = \tilde{u}^{K+1}_{i,j} = 0,\\
&i = 0,1,...,I+1,\ j=0,1,...,J+1,\ k=0,1,...,K+1, \quad (16)
\end{aligned}$$

where $I,\ J,\ K$ are positive integers, $\Delta x = \frac{(b-a)}{(I+1)}$, $\Delta v = \frac{(d-c)}{(J+1)}$ and $\Delta t = \frac{(f-e)}{(K+1)}$ are step sizes in the directions $x,\ v,\ t$, respectively. In (15), $\tilde{u}^k_{i,j}$ is the finite difference approximation for the solution $u(x_i, v_j, t_k) = u(a+i\Delta x,\ c+j\Delta v,\ e+k\Delta t)$, $h^k_{i,j}$ is the finite difference approximation for the function $H(x_i,v_j,t_k) = H(a+i\Delta x,\ c+j\Delta v,\ e+k\Delta t)$, $\widetilde{f}^k_{i,j}$ is the approximation to the function $\mathcal{F}(x_i,v_j,t_k) = \mathcal{F}(a+i\Delta x,\ c+j\Delta v,\ e+k\Delta t)$ and

$$\begin{aligned}
k_1 &= \frac{h^k_{i+1,j} - h^k_{i-1,j}}{4(\Delta x)^2(\Delta v)^2},\ k_2 = \frac{h^k_{i,j+1} - h^k_{i,j-1}}{4(\Delta x)^2(\Delta v)^2},\\
k_3 &= \frac{h^k_{i+1,j} - 2h^k_{i,j} + h^k_{i-1,j}}{(\Delta x)^2(\Delta v)^2},\ k_4 = \frac{h^k_{i,j+1} - 2h^k_{i,j} + h^k_{i,j-1}}{(\Delta x)^2(\Delta v)^2},\\
k_5 &= \frac{h^k_{i+1,j+1} - 2h^k_{i,j+1} + h^k_{i-1,j+1} - h^k_{i+1,j-1} + 2h^k_{i,j-1} - h^k_{i-1,j-1}}{4(\Delta x)^2(\Delta v)^2},\\
k_6 &= \frac{h^k_{i+1,j+1} - 2h^k_{i+1,j} + h^k_{i+1,j-1} - h^k_{i-1,j+1} + 2h^k_{i-1,j} - h^k_{i-1,j-1}}{4(\Delta x)^2(\Delta v)^2},\\
k_7 &= \frac{1}{8(\Delta x)(\Delta v)(\Delta t)}.
\end{aligned}$$

The approximate solution $\tilde{u}^k_{i,j}$ of Problem 4 is obtained at $I \times J \times K$ mesh points of Ω by solving the matrix equation

$$\tilde{A}\widetilde{\mathbf{u}} = \widetilde{\mathcal{F}}, \tag{17}$$

where $\tilde{A}$ is a block tridiagonal banded matrix of the form

$$\tilde{A} = \begin{bmatrix} \mathcal{A}^{(1)} & \mathcal{B}^{(1)} & 0 & \cdots & 0 \\ \mathcal{C}^{(2)} & \mathcal{A}^{(2)} & \mathcal{B}^{(2)} & \ddots & \vdots \\ 0 & \mathcal{C}^{(3)} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \mathcal{B}^{(I-1)} \\ 0 & \cdots & 0 & \mathcal{C}^{(I)} & \mathcal{A}^{(I)} \end{bmatrix}_{IJK\times\, IJK} . \tag{18}$$

In (18), the matrices $\mathcal{A}^{(i)}$, $\mathcal{B}^{(i)}$, $\mathcal{C}^{(i)}$ are defined as follows

$$\mathcal{A}^{(i)} = \begin{bmatrix} A_1^{(i,1)} & A_2^{(i,1)} & 0 & \cdots & 0 \\ A_3^{(i,2)} & A_1^{(i,2)} & A_2^{(i,2)} & \ddots & \vdots \\ 0 & A_3^{(i,3)} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & A_2^{(i,J-1)} \\ 0 & \cdots & 0 & A_3^{(i,J)} & A_1^{(i,J)} \end{bmatrix}_{JK\,\times\,JK}, \ i = 1, 2, \ldots, I;$$

$$\mathcal{B}^{(i)} = \begin{bmatrix} B_1^{(i,1)} & B_2^{(i,1)} & 0 & \cdots & 0 \\ B_3^{(i,2)} & B_1^{(i,2)} & B_2^{(i,2)} & \ddots & \vdots \\ 0 & B_3^{(i,3)} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & B_2^{(i,J-1)} \\ 0 & \cdots & 0 & B_3^{(i,J)} & B_1^{(i,J)} \end{bmatrix}_{JK\,\times JK}, \ \ i = 1, 2, ..., I-1;$$

$$\mathcal{C}^{(i)} = \begin{bmatrix} C_1^{(i,1)} & C_2^{(i,1)} & 0 & \cdots & 0 \\ C_3^{(i,2)} & C_1^{(i,2)} & C_2^{(i,2)} & \ddots & \vdots \\ 0 & C_3^{(i,3)} & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & C_2^{(i,J-1)} \\ 0 & \cdots & 0 & C_3^{(i,J)} & C_1^{(i,J)} \end{bmatrix}_{JK\,\times\,JK}, \ \ i = 2, 3, ..., I;$$

where

$$A_s^{(i,j)} = \begin{bmatrix} f_s(i,j) & 0 & \cdots & 0 \\ 0 & f_s(i,j) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & f_s(i,j) \end{bmatrix}_{K \text{ x } K},$$

$$B_s^{(i,j)} = \begin{bmatrix} g_s(i,j) & \frac{(-1)^s(s-1)}{(s-1)!}a & 0 & \cdots & 0 \\ \frac{(-1)^{(s+1)}(s-1)}{(s-1)!}a & g_s(i,j) & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{(-1)^s(s-1)}{(s-1)!}a \\ 0 & \cdots & 0 & \frac{(-1)^{(s+1)}(s-1)}{(s-1)!}a & g_s(i,j) \end{bmatrix}_{K\times K},$$

$$C_s^{(i,j)} = \begin{bmatrix} h_s(i,j) & \frac{(-1)^{(s+1)}(s-1)}{(s-1)!}a & 0 & \cdots & 0 \\ \frac{(-1)^s(s-1)}{(s-1)!}a & h_s(i,j) & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{(-1)^{(s+1)}(s-1)}{(s-1)!}a \\ 0 & \cdots & 0 & \frac{(-1)^s(s-1)}{(s-1)!}a & h_s(i,j) \end{bmatrix}_{K\times K},$$

$$s = 1,2,3 \ , \quad j = 1,2,...,J \ ,$$

and

$$\begin{aligned} f_1(i,j) &= -2k_4 + 2k_3, f_2(i,j) = -2k_2 - k_3 - k_5, f_3(i,j) = 2k_2 - k_3 + k_5, \\ h_1(i,j) &= -2k_1 + k_4 - k_6, h_2(i,j) = k_1 + k_2, h_3(i,j) = k_1 - k_2, \\ g_1(i,j) &= 2k_1 + k_4 + k_6, g_2(i,j) = -k_1 + k_2, g_3(i,j) = -k_1 - k_2, a = k_7. \end{aligned}$$

In (17), $\widetilde{\mathcal{F}}$ is a column matrix which consists of

$$\widetilde{\mathcal{F}} = \left[\widetilde{f}_{1,1}^1, \widetilde{f}_{1,1,}^2, ..., \widetilde{f}_{1,1}^K, \widetilde{f}_{1,2}^1, \widetilde{f}_{1,2}^2, ..., \widetilde{f}_{1,2}^K, ..., \widetilde{f}_{1,J}^1, \widetilde{f}_{1,J}^2, ..., \widetilde{f}_{1,J}^K, ..., \widetilde{f}_{I,J}^K\right]^T$$

and $\widetilde{\mathbf{u}}$ is the solution vector:

$$\widetilde{\mathbf{u}} = \left[\tilde{u}_{1,1}^1, \tilde{u}_{1,1,}^2, ..., \tilde{u}_{1,1}^K, \tilde{u}_{1,2}^1, \tilde{u}_{1,2}^2, ..., \tilde{u}_{1,2}^K, ..., \tilde{u}_{1,J}^1, \tilde{u}_{1,J}^2, ..., \tilde{u}_{1,J}^K, ..., \tilde{u}_{I,J}^K\right]^T.$$

Finally, we obtain λ numerically from the difference equation

$$\frac{(\tilde{u}_{i,j}^{k+1} - \tilde{u}_{i,j}^{k-1})}{2\Delta t} + k_2 \frac{(\tilde{u}_{i+1,j}^k - \tilde{u}_{i-1,j}^k)}{2\Delta x} - k_1 \frac{(\tilde{u}_{i,j+1}^k + \tilde{u}_{i,j-1}^k)}{2\Delta v} - \widetilde{f}_{i,j}^k = \tilde{\lambda}_{i,j}^k,$$

$i = 1,2,...,I;\ j = 1,2,...,J;\ k = 1,2,...,K$, where $\tilde{\lambda}_{i,j}^k$ is the approximation to the function $\lambda(x_i, v_j, t_k) = \lambda(a + i\Delta x,\ c + j\Delta v,\ e + k\Delta t)$. The last relation is derived from (12) by using the central-difference formulas.

3. Numerical Experiments

In this section, we present numerical solution of three inverse problems of the form (9)-(10) by using the method developed in the previous section. The computations are performed using MATLAB R2014a program on a PC with Intel(R) Core(TM) i7-7700HQ CPU 2.80 GHz, 16 Gb memory RAM, running under Windows 10.

Example 1. Let us consider the problem of finding (u, λ) in $\Omega = (2,3)\times(-1,1)\times(0,1)$ from relations (9)-(10) provided that

$$\begin{aligned} H(x,v,t) &= v^2 + \log(x), \\ F(x,v,t) &= x^3(v - 2tv + 2tv^3 - v^3) + x^2(6t^2v^4 - 9t^2v^2 - 6tv^4 - 10tv^3 + 9tv^2 \\ &\quad +10tv + 5v^3 - 5v) + x(-20t^2v^4 + 35t^2v^2 + 20tv^4 + 12tv^3 \\ &\quad -35tv^2 - 12tv - 6v^3 + 6v). \end{aligned}$$

It is known that the exact solution of the inverse problem is

$$\begin{aligned} u(x,v,t) &= xv(x-2)(x-3)(v^2-1)(t^2-t), \\ \lambda(x,v,t) &= t(t-1)(12v^4 - 30v^2 + x^2 - 5x + 6). \end{aligned}$$

In the following figures, we compare the exact solution with the calculated finite difference solution of the problem for $I = 80, J = 200, K = 2$, that is we consider 32000 mesh points.

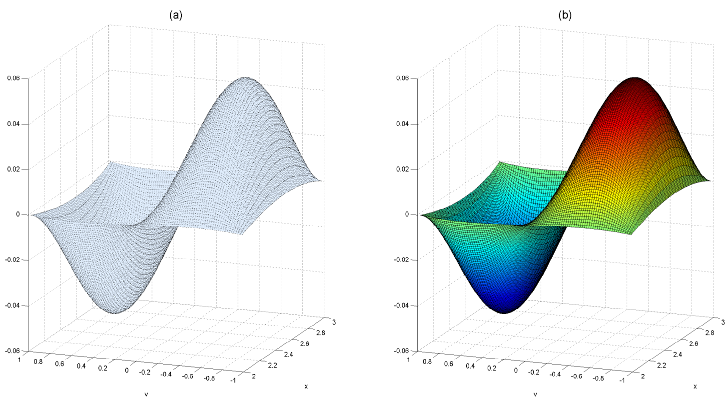

Figure 1. (a) Computed values and (b) Exact values of u for t=0.5.

Table 1. Errors in the computation for Example 1

Number of mesh points	$I=50,\ J=100,$ $K=2$ 10000	$I=50,\ J=200,$ $K=2$ 20000	$I=80,\ J=200,$ $K=2$ 32000
Elapsed time	$90.42\ s$	$183.49\ s$	$622.27\ s$
Maximum error for u	$3.8445e-04$	$8.9666e-06$	$8.4668e-06$
Maximum error for λ	$3.9840e-04$	$8.5537e-05$	$8.2733e-05$

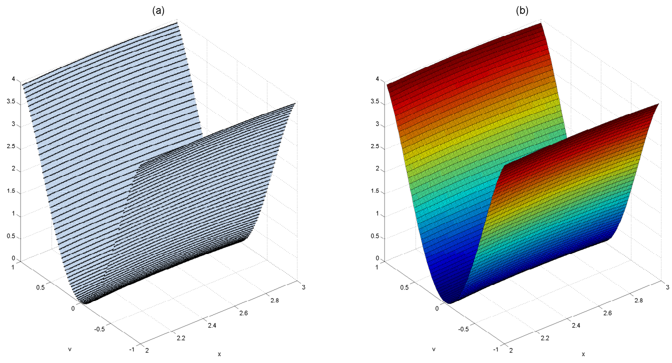

FIGURE 2. (a) Computed values and (b) Exact values of λ for t=0.5.

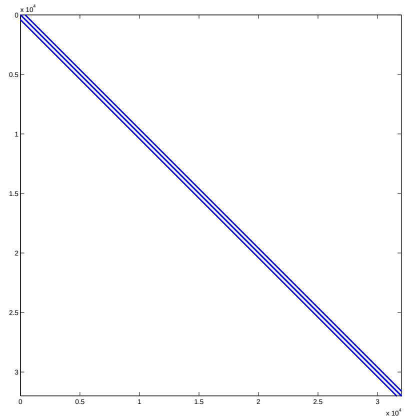

FIGURE 3. The structure of the matrix $\tilde{\mathrm{A}}$ (32000×32000) in Example 1.

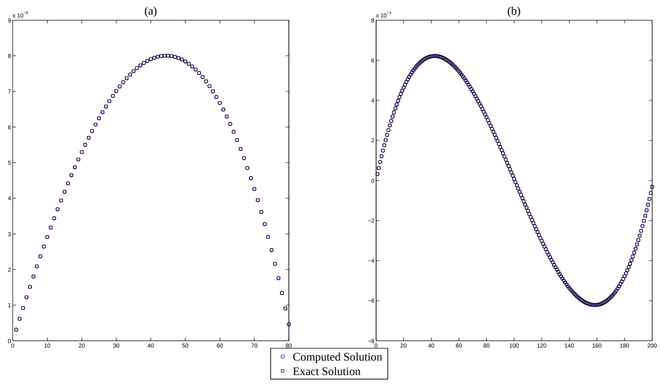

FIGURE 4. A comparison of computed and exact solutions for u:
(a) fixed v, t; (b) fixed x, t.

Example 2. Find a pair of functions (u, λ) defined in $\Omega = (2,3) \times (0,3) \times (0,1)$ that satisfies equation (9)-(10) with

$$\begin{aligned}
H\left(x,v,t\right) &= \frac{v^2}{2} - x^2, \\
F\left(x,v,t\right) &= x^3(4te^v - 4t^2e^v - 2tve^v + 2t^2ve^v) + x^2(v + 3e^v + 20t^2e^v - 2tv - 26te^v \\
&\quad -ve^v + 12tve^v - 10t^2ve^v) + x(4tv - 15e^v - 2t^2v^2 - 24t^2e^v - 5v \\
&\quad +2tv^2 + 6t^2v + 54te^v + 5ve^v - 16tve^v - 2tv^2e^v + 6t^2ve^v + 2t^2v^2e^v).
\end{aligned}$$

The exact solution of the problem is

$$
\begin{aligned}
u\,(x,v,t) &= (x-2)(x-3)(e^{v}-1)(v-3)(t^{2}-t),\\
\lambda\,(x,v,t) &= x^{2}(-2t^{2}+2t)x^{3}+(10t^{2}-4t-3)+x(-12t^{2}-18t+15)+36t+6v\\
&\quad +tv(5tv-5v-15t+3)+e^{v}(v-3)(12t+5tv-5t^{2}v-6)-18.
\end{aligned}
$$

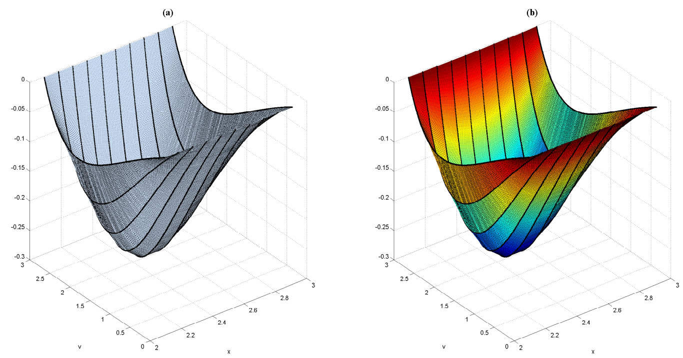

FIGURE 5. (a) Computed values and (b) Exact values of u for t=0.7.

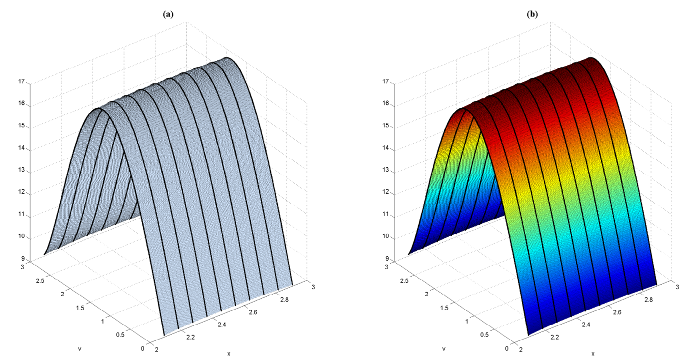

FIGURE 6. (a) Computed values and (b) Exact values of λ for t=0.7.

Table 2. Errors in the computation for Example 2

	$I=4,\ J=500,$ $K=4$	$I=10,\ J=700,$ $K=3$	$I=10,\ J=1300,$ $K=3$
Number of mesh points	8000	21000	39000
Elapsed time	39.11 s	214.51 s	1153.12 s
Maximum error for u	1.2796e-05	8.1005e-06	2.3519e-06
Maximum error for λ	4.4605e-04	2.8981e-04	8.4804e-05

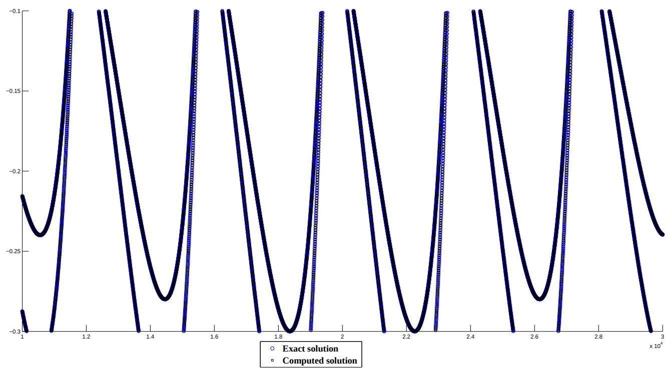

FIGURE 7. A comparison of computed and exact solutions for u (all values).

Example 3. Find a pair of functions (u, λ) defined in $\Omega = (-3,3) \times (0,1) \times (1,3)$ that satisfies relation (9)-(10) with

$$
\begin{aligned}
H(x,v,t) &= \frac{v^2}{2} - x^2, \\
F(x,v,t) &= \frac{1}{t^2}\Big[vx(3x-3vx) - tvx(-6v^2+6v-12x^2+108) \\
&\quad +3ve^{\cos(\frac{\pi v}{2}+x)})(x^2-9)(v-1)\Big] \\
&\quad +\frac{1}{t}\Big[6xe^{\cos(\frac{\pi v}{2}+x)})(-v^3+v^2-2vx^2+18v+x^2-9) \\
&\quad +e^{\cos(\frac{\pi v}{2}+x)}\sin(\frac{\pi v}{2}+x)(x^2-9)(v^2-v)(v+\pi x)(t^2-4t+3)\Big] \\
&\quad +vx(-8v^2+vx+8v-16x^2-x+144) - tvx(-2v^2+2v-4x^2+36) \\
&\quad -e^{\cos(\frac{\pi v}{2}+x)}\Big[x^3(4tv-16v-2t+8)+x^2(v^2-v) \\
&\quad +x(18t+144v-36tv-2tv^2+2tv^3+8v^2-8v^3-72)-9v^2+9v\Big].
\end{aligned}
$$

The exact solution of the problem is

$$
\begin{aligned}
u(x,v,t) &= (t-3)(\frac{1}{t}-1)(e^{\cos(\frac{\pi v}{2}+x)}-1)(x^2-9)(v^2-v), \\
\lambda(x,v,t) &= x^3(8-\frac{6}{t}-2t)+x(18t+\frac{54}{t}-72)+9v+\frac{1}{t^2}(27v^2-27v)-9v^2.
\end{aligned}
$$

The finite difference and the exact solution of the inverse problem at $t=2$ are shown in Figures 8-9 for $I=100, J=100, K=3$.

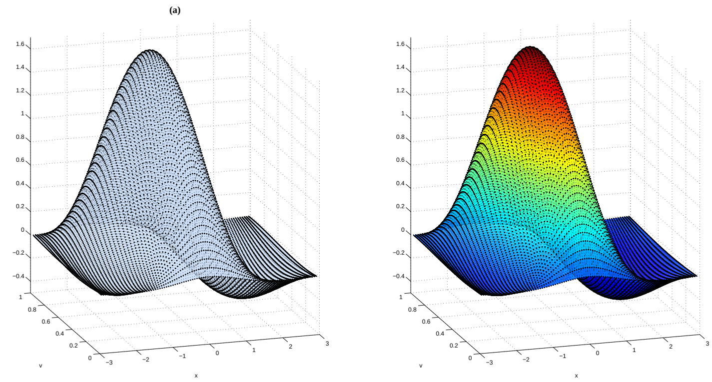

FIGURE 8. (a) Computed values and (b) Exact values of u for t=2.

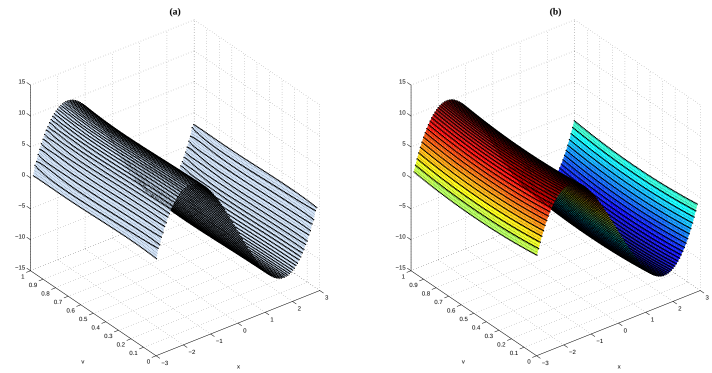

FIGURE 9. (a) Computed values and (b) Exact values of λ for t=2.

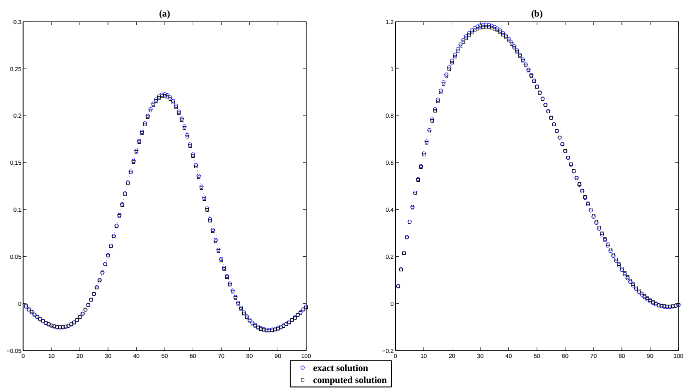

FIGURE 10. A comparison of computed and exact solutions for u:
(a) fixed v, t; (b) fixed x, t.

Consequently, numerical experiments have demonstrated that the proposed method provides highly accurate numerical solutions for the source inverse problems for the Liouville equation. It is worth to note that the method used for proving the solvability of the inverse problem paves the way of solving the problem numerically. Namely, applying the operator $\widehat{L}$, our problem is reduced to a direct problem for u. Then the finite difference approximation for the Dirichlet problem for a third order partial differential equation gives the result directly.

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