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SOME INCLUSION RELATIONS BETWEEN VARIOUS SUBCLASSES OF PLANAR HARMONIC MAPPINGS INVOLVING CONFLUENT HYPERGEOMETRIC DISTRIBUTION SERIES

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Abstract. The purpose of the present paper is to establish connections between various subclasses of harmonic univalent functions by applying certain convolution operator involving Confluent Hypergeometric distribution series. To be more precise, we investigate such connections with Goodman-Rønning-type harmonic univalent functions in the open unit disc $\mathbb{U}$.

Keywords: Harmonic, Univalent functions, Confluent Hypergeometric Distribution Series.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Recently, Porwal [12] introduced a Poisson distribution series as

$$K(m, z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n.$$

The Poisson distribution series is a recent topic of study in Geometric Function Theory and co-relates G.F.T. with Probability distribution. It opened up a new and interesting direction of research in G.F.T. After the appearence of this paper some researchers ([1], [2], [15]) investigated hypergeometric distribution series, hypergeometric-type distribution series and confluent hypergeometric distribution series and give some beautiful applications on various subclasses of analytic and harmonic univalent functions. Noteworthy contribution in this direction are given in ([5], [6], [10], [14] and [16]). Very Recently, Porwal and Kumar [15] generalized the Poisson distribution series by introducing confluent

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hypergeometric distribution series in the following way. For this purpose first we recall the following definitions.

The confluent hypergeometric function is given by the power series

$$F(a;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n(1)_n}z^n,$$

where a,c are complex numbers such that $c\neq 0,-1,-2,\dots$ and $(a)_n$ is the Pochhammer symbol defined in terms of the Gamma function, by

$$\begin{aligned}(a)_n=&\frac{\Gamma(a+n)}{\Gamma(a)}\\ =&\begin{cases}1, & \text{if } n=0\\ a(a+1)\dots(a+n-1), & \text{if } n\in N=\{1,2,3,\dots\}\end{cases}\end{aligned}$$

is convergent for all finite value of z.

Now we define for $a,c,m>0$ such that the series

$$F(a;c;m)=\sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n(1)_n}m^n$$

is convergent.

Porwal and Kumar [15] introduce the confluent hypergeometric distribution whose probability mass function is

$$\frac{(a)_n m^n}{(c)_n n! F(a;c;m)},$$

$n=0,1,2,\dots$.

If we put $a=c$ then it reduce to the Poisson distribution.

Now, we introduce a new series $I(a;c;m;z)$ whose coefficients are probabilities of confluent hypergeometric distribution

$$I(a;c;m;z)=z+\sum_{n=2}^{\infty}\frac{(a)_{n-1}m^{n-1}}{(c)_{n-1}(n-1)!F(a;c;m)}z^n,$$

where $a,c,m>0$.

Denote by S_H the class of functions $f=h+\overline{g}$ that are harmonic univalent and sense-preserving in the open unit disc $\mathbb{U}$ for which $f(0)=f_z(0)-1=0$. Then for $f=h+\overline{g}\in S_H$, we may express the analytic functions h and g as

$$h(z)=z+\sum_{n=2}^{\infty}A_nz^n,\ g(z)=\sum_{n=1}^{\infty}B_nz^n,\ |B_1|<1. \tag{2}$$

We also let the subclass S_H^0 of S_H

$$S_H^0=\left\{f=h+\overline{g}\in S_H : g'(0)=B_1=0\right\}.$$

The classes S_H^0 and S_H were first studied in [7]. Also, we let K_H^0, $S_H^{*,0}$ and C_H^0 denote the subclasses of S_H^0 of harmonic functions which are, respectively, convex, starlike and close-to-convex in $\mathbb{U}$. Also, let T_H^0 be the class of sense-preserving, typically real harmonic functions $f=h+\overline{g}$ in S_H. For definitions and properties of these classes, one may refer to [7] or [8], (see also [19]).

For $0\le\gamma<1$, let

$$N_H(\gamma)=\left\{f\in S_H : \Re\left(\frac{f'(z)}{z'}\right)\ge\gamma,\ z=re^{i\theta}\in\mathbb{U}\right\},$$

and

$$G_H(\gamma)=\left\{f\in S_H:\Re\left\{\left(1+e^{i\alpha}\right)\frac{zf'(z)}{f(z)}-e^{i\alpha}\right\}\geq\gamma,\ \alpha\in R,\ z\in\mathbb{U}\right\},$$

where

$$z'=\frac{\partial}{\partial\theta}\left(z=re^{i\theta}\right),\ f'(z)=\frac{\partial}{\partial\theta}f\left(re^{i\theta}\right).$$

Define

$$TN_H(\gamma)\equiv N_H(\gamma)\cap T \text{ and } TG_H(\gamma)\equiv G_H(\gamma)\cap T,$$

where T consists of the functions $f=h+\overline{g}$ in S_H so that h and g are of the form

$$h(z)=z-\sum_{n=2}^{\infty}|A_n|\,z^n,\ g(z)=\sum_{n=1}^{\infty}|B_n|\,z^n. \tag{3}$$

The classes $N_H(\gamma)$, $TN_H(\gamma)$, $G_H(\gamma)$ and $TG_H(\gamma)$ were initially introduced and studied, respectively, in ([4], [17]). A function f in $G_H(\gamma)$ is called Goodman-Rønning-type harmonic univalent functions in $\mathbb{U}$.

Now, for $a_1,c_1,a_2,c_2,m_1,m_2>0$, we introduce the operator $\Omega(f)=\Omega\begin{pmatrix}a_1, & c_1, & m_1\\ a_2, & c_2, & m_2\end{pmatrix}f(z)$ for $f(z)\in S_H$ as

$$\Omega\begin{pmatrix}a_1, & c_1, & m_1\\ a_2, & c_2, & m_2\end{pmatrix}f(z)=I(a_1;c_1;m_1;z)*h(z)+\overline{I(a_2;c_2;m_2;z)*g(z)}=H(z)+\overline{G(z)},$$

where

$$H(z)=z+\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}(n-1)!F(a_1;c_1;m_1)}A_nz^n,\ G(z)=B_1z+\sum_{n=2}^{\infty}\frac{(a_2)_{n-1}m_2^{n-1}}{(c_2)_{n-1}(n-1)!F(a_2;c_2;m_2)}B_nz^n. \tag{4}$$

Throughout this paper, we will frequently use the notation

$$\Omega(f)=\Omega\begin{pmatrix}a_1, & c_1, & m_1\\ a_2, & c_2, & m_2\end{pmatrix}f,$$

The purpose of the present paper is an attempt to interconnect between distribution function and Geometric Function Theory. Motivated by results on connections between various subclasses of analytic and harmonic univalent functions by using hypergeometric functions, generalized Bessel functions, Poisson distribution series, hypergeometric distribution series, hypergeometric-type distribution series and confluent hypergeometric distribution series (see [1], [2], [3], [9], [11], [13] and [18]), we establish a number of connections between the classes $G_H(\gamma)$, K_H^0, $S_H^{*,0}$, C_H^0 and $N_H(\beta)$ by applying the convolution operator Ω.

2. Main Results

In order to establish connections between harmonic convex functions and Goodman-Rønning-type harmonic univalent functions, we need following results in Lemma 2.1 and Lemma 2.2.

Lemma 2.1. *(see [7], [8]). If $f=h+\overline{g}\in K_H^0$ where h and g are given by (2) with $B_1=0$, then*

$$|A_n|\leq\frac{n+1}{2},\ |B_n|\leq\frac{n-1}{2}.$$

Lemma 2.2. *(See [17]). Let $f = h + \overline{g}$ be given by (2). If $0 \leq \gamma < 1$ and*

$$\sum_{n=2}^{\infty} (2n-1-\gamma)\,|A_n| + \sum_{n=1}^{\infty} (2n+1+\gamma)\,|B_n| \leq 1-\gamma, \tag{5}$$

then f is sense-preserving, Goodman-Rønning-type harmonic univalent functions in $\mathbb{U}$ and $f \in G_H(\gamma)$.

Remark 2.1. *In [17], it is also shown that $f = h+\overline{g}$ given by (3) is in the family $TG_H(\gamma)$, if and only if the coefficient condition (5) holds. Moreover, if $f \in TG_H(\gamma)$, then*

$$|A_n| \leq \frac{1-\gamma}{2n-1-\gamma},\ n \geq 2,$$

$$|B_n| \leq \frac{1-\gamma}{2n+1+\gamma},\ n \geq 1.$$

Theorem 2.1. *Let $a_1, c_1, a_2, c_2, m_1, m_2 > 0$. If for some $\gamma (0 \leq \gamma < 1)$, the inequality*

$$\begin{aligned}
&\frac{1}{F(a_1;c_1;m_1)}\left\{2\frac{a_1(a_1+1)}{c_1(c_1+1)}m_1^2F(a_1+2;c_1+2;m_1) + (7-\gamma)\frac{a_1}{c_1}m_1F(a_1+1;c_1+1;m_1)\right.\\
&\left.+2(1-\gamma)\left(F(a_1;c_1;m_1)-1\right)\right\} + \frac{1}{F(a_2;c_2;m_2)}\left\{2\frac{a_2(a_2+1)}{c_2(c_2+1)}m_2^2F(a_2+2;c_2+2;m_2)\right.\\
&\left.+\,(5+\gamma)\,\frac{a_2}{c_2}m_2F(a_2+1;c_2+1;m_2)\right\}\\
&\leq 2(1-\gamma),
\end{aligned}$$

is satisfied then $\Omega\left(K_H^0\right) \subset G_H(\gamma)$.

Proof. Let $f = h + \overline{g} \in K_H^0$ where h and g are of the form (2) with $B_1 = 0$. We need to show that $\Omega\,(f) = H + \overline{G} \in G_H\,(\gamma)$, where H and G defined by (4) with $B_1 = 0$ are analytic functions in $\mathbb{U}$.

In view of Lemma 2.2, we need to prove that

$$P_1 \leq 1-\gamma,$$

where

$$P_1 = \sum_{n=2}^{\infty} (2n-1-\gamma)\left|\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}(n-1)!F(a_1;c_1;m_1)}A_n\right| + \sum_{n=2}^{\infty} (2n+1+\gamma)\left|\frac{(a_2)_{n-1}m_2^{n-1}}{(c_2)_{n-1}(n-1)!F(a_2;c_2;m_2)}B_n\right|. \tag{6}$$

In view of Lemma 2.1, we have

$$
\begin{aligned}
P_1 &\leq \frac{1}{2}\left[\sum_{n=2}^{\infty}(n+1)(2n-1-\gamma)\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}(n-1)!F(a_1;c_1;m_1)}\right.\\
&\left.+\sum_{n=2}^{\infty}(n-1)(2n+1+\gamma)\frac{(a_2)_{n-1}m_2^{n-1}}{(c_2)_{n-1}(n-1)!F(a_2;c_2;m_2)}\right]\\
&=\frac{1}{2}\left[\sum_{n=2}^{\infty}\{2(n-1)(n-2)+(7-\gamma)(n-1)+2(1-\gamma)\}\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}(n-1)!F(a_1;c_1;m_1)}\right.\\
&\left.+\sum_{n=2}^{\infty}\{2(n-2)+(5+\gamma)\}\frac{(a_2)_{n-1}m_2^{n-1}}{(c_2)_{n-1}(n-2)!F(a_2;c_2;m_2)}\right]\\
&=\frac{1}{2}\left[\frac{1}{F(a_1;c_1;m_1)}\left\{2\sum_{n=3}^{\infty}\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}(n-3)!}+(7-\gamma)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}(n-2)!}+2(1-\gamma)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}(n-1)!}\right\}\right.\\
&\left.+\frac{1}{F(a_2;c_2;m_2)}\left\{2\sum_{n=3}^{\infty}\frac{(a_2)_{n-1}m_2^{n-1}}{(c_2)_{n-1}(n-3)!}+(5+\gamma)\sum_{n=2}^{\infty}\frac{(a_2)_{n-1}m_2^{n-1}}{(c_2)_{n-1}(n-2)!}\right\}\right]\\
&=\frac{1}{2}\left[\frac{1}{F(a_1;c_1;m_1)}\left\{2\frac{a_1(a_1+1)}{c_1(c_1+1)}m_1^2F(a_1+2;c_1+2;m_1)+(7-\gamma)\frac{a_1}{c_1}m_1F(a_1+1;c_1+1;m_1)\right.\right.\\
&\left.+2(1-\gamma)\left(F(a_1;c_1;m_1)-1\right)\right\}+\frac{1}{F(a_2;c_2;m_2)}\left\{2\frac{a_2(a_2+1)}{c_2(c_2+1)}m_2^2F(a_2+2;c_2+2;m_2)\right.\\
&\left.\left.+(5+\gamma)\frac{a_2}{c_2}m_2F(a_2+1;c_2+1;m_2)\right\}\right]\\
&\leq 1-\gamma,
\end{aligned}
$$

by given hypothesis.

This completes the proof of Theorem 2.1. □

Analogous to Theorem 2.1, we next find conditions of the classes $S_H^{*,0}$, C_H^0 with $G_H(\gamma)$. However we first need the following result which may be found in [7], [8].

Lemma 2.3. *If $f=h+\overline{g}\in S_H^{*,0}$ or C_H^0, where h and g are given by (2) with $B_1=0$, then*

$$|A_n|\leq\frac{(2n+1)(n+1)}{6},\ |B_n|\leq\frac{(2n-1)(n-1)}{6}.$$

Theorem 2.2. *Let $a_1,c_1,a_2,c_2,m_1,m_2>0$. If for some $\gamma(0\leq\gamma<1)$ and the inequality*

$$
\begin{aligned}
&\frac{1}{F(a_1;c_1;m_1)}\left\{4\frac{a_1(a_1+1)(a_1+2)}{c_1(c_1+1)(c_1+2)}m_1^3F(a_1+3;c_1+3;m_1)+(28-2\gamma)\frac{a_1(a_1+1)}{c_1(c_1+1)}m_1^2F(a_1+2;c_1+2;m_1)\right.\\
&\qquad\left.+(39-9\gamma)\frac{a_1}{c_1}F(a_1+1;c_1+1;m_1)m_1+6(1-\gamma)\left(F(a_1;c_1;m_1)-1\right)\right\}\\
&\qquad+\frac{1}{F(a_2;c_2;m_2)}\left\{4\frac{a_2(a_2+1)(a_2+2)}{c_2(c_2+1)(c_2+2)}m_2^3F(a_2+3;c_2+3;m_2)\right.\\
&\qquad\qquad+(20+2\gamma)\frac{a_2(a_2+1)}{c_2(c_2+1)}m_2^2F(a_2+2;c_2+2;m_2)\\
&\qquad\qquad\left.+(15+3\gamma)\frac{a_2}{c_2}m_2F(a_2+1;c_2+1;m_2)\right\}
\end{aligned}
$$

$$\leq 6(1-\gamma),$$

is satisfied, then

$$\Omega(S_H^{*,0}) \subset G_H(\gamma) \text{ and } \Omega(C_H^0) \subset G_H(\gamma).$$

Proof. Let $f = h + \overline{g} \in S_H^{*,0}$ (C_H^0 where h and g are given by (2) with $B_1 = 0$). We need to show that $\Omega(f) = H + \overline{G} \in G_H(\gamma)$, where H and G defined by (4) with $B_1 = 0$ are analytic functions in $\mathbb{U}$. In view of Lemma 2.2, it is enough to show that $P_1 \leq 1-\gamma$, where P_1 is given by (6).

In view of Lemma 2.3, we have

$$\begin{aligned}
P_1 &\leq \frac{1}{6}\left[\sum_{n=2}^{\infty}(2n+1)(n+1)(2n-1-\gamma)\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}(n-1)!F(a_1;c_1;m_1)}\right.\\
&\left.+\sum_{n=2}^{\infty}(2n-1)(n-1)(2n+1+\gamma)\frac{(a_2)_{n-1}m_2^{n-1}}{(c_2)_{n-1}(n-1)!F(a_2;c_2;m_2)}\right]\\
&= \frac{1}{6}\left[\frac{1}{F(a_1;c_1;m_1)}\left\{\sum_{n=2}^{\infty}\{4(n-1)(n-2)(n-3)+(28-2\gamma)(n-1)(n-2)\right.\right.\\
&\left.+(39-9\gamma)(n-1)+6(1-\gamma)\}\right\}\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}(n-1)!}\\
&\left.+\frac{1}{F(a_2;c_2;m_2)}\left\{\sum_{n=2}^{\infty}\{4(n-2)(n-3)+(20+2\gamma)(n-2)+(15+3\gamma)\}\right\}\frac{(a_2)_{n-1}m_2^{n-1}}{(c_2)_{n-1}(n-2)!}\right]\\
&= \frac{1}{6}\left[\frac{1}{F(a_1;c_1;m_1)}\left\{4\sum_{n=4}^{\infty}\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}(n-4)!}+(28-2\gamma)\sum_{n=3}^{\infty}\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}(n-3)!}\right.\right.\\
&\left.+(39-9\gamma)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}(n-2)!}+6(1-\gamma)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}(n-1)!}\right\}+\frac{1}{F(a_2;c_2;m_2)}\\
&\left.\left\{4\sum_{n=4}^{\infty}\frac{(a_2)_{n-1}m_2^{n-1}}{(c_2)_{n-1}(n-4)!}+(20+2\gamma)\sum_{n=3}^{\infty}\frac{(a_2)_{n-1}m_2^{n-1}}{(c_2)_{n-1}(n-3)!}+(15+3\gamma)\sum_{n=2}^{\infty}\frac{(a_2)_{n-1}m_2^{n-1}}{(c_2)_{n-1}(n-2)!}\right\}\right]\\
&= \frac{1}{6}\left[\frac{1}{F(a_1;c_1;m_1)}\left\{4\frac{a_1(a_1+1)(a_1+2)}{c_1(c_1+1)(c_1+2)}m_1^3F(a_1+3;c_1+3;m_1)\right.\right.\\
&+(28-2\gamma)\frac{a_1(a_1+1)}{c_1(c_1+1)}m_1^2F(a_1+2;c_1+2;m_1)\\
&\left.+(39-9\gamma)\frac{a_1}{c_1}F(a_1+1;c_1+1;m_1)m_1+6(1-\gamma)\left(F(a_1;c_1;m_1)-1\right)\right\}\\
&+\frac{1}{F(a_2;c_2;m_2)}\left\{4\frac{a_2(a_2+1)(a_2+2)}{c_2(c_2+1)(c_2+2)}m_2^3F(a_2+3;c_2+3;m_2)\right.\\
&\left.\left.+(20+2\gamma)\frac{a_2(a_2+1)}{c_2(c_2+1)}m_2^2F(a_2+2;c_2+2;m_2)+(15+3\gamma)\frac{a_2}{c_2}m_2F(a_2+1;c_2+1;m_2)\right\}\right]\\
&\leq 1-\gamma,
\end{aligned}$$

follows from the given condition. □

In order to determine connection between $TN_H(\beta)$ and $G_H(\gamma)$, we need the following result in Lemma 2.4.

Lemma 2.4. *(See [4]). Let $f = h + \overline{g}$ where h and g are given by (2) with $B_1 = 0$, and suppose that $0 \le \beta < 1$. Then*

$$f \in TN_H(\beta) \Leftrightarrow \sum_{n=2}^{\infty} n\,|A_n| + \sum_{n=2}^{\infty} n\,|B_n| \le 1 - \beta.$$

Remark 2.2. *If $f \in TN_H(\beta)$, then $|A_n| \le \frac{1-\beta}{n},\ n \ge 2$ and $|B_n| \le \frac{1-\beta}{n},\ n \ge 1$.*

Theorem 2.3. *If $a_1, c_1, a_2, c_2 > 1, m_1, m_2 > 0$. If for some $\beta(0 \le \beta < 1)$ and $\gamma(0 \le \gamma < 1)$ and the inequality*

$$(1-\beta)\left[\frac{1}{F(a_1;c_1;m_1)}\left\{2\left(F(a_1;c_1;m_1)-1\right) - \frac{(1+\gamma)(c_1-1)}{m_1(a_1-1)}\left(F(a_1-1;c_1-1;m_1) - 1 - \frac{(a_1-1)}{(c_1-1)}m_1\right)\right\}\right.$$
$$\left. + 2\left(F(a_2;c_2;m_2)-1\right) + \frac{(1+\gamma)(c_2-1)}{m_2(a_2-1)}\left(F(a_2-1;c_2-1;m_2) - 1 - \frac{(a_2-1)}{(c_2-1)}m_2\right)\right]$$
$$+ (3+\gamma)\,|B_1|$$
$$\le 1-\gamma$$

is satisfied then

$$\Omega\left(TN_H(\beta)\right) \subset G_H(\gamma).$$

Proof. Let $f = h + \overline{g} \in TN_H(\beta)$ where h and g are given by (2). In view of Lemma 2.2, it is enough to show that $P_2 \le 1 - \gamma$, where P_2 is given by the following expression

$$P_2 = \sum_{n=2}^{\infty}(2n-1-\gamma)\left|\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}(n-1)!F(a_1;c_1;m_1)}A_n\right| + \sum_{n=2}^{\infty}(2n+1+\gamma)\left|\frac{(a_2)_{n-1}m_2^{n-1}}{(c_2)_{n-1}(n-1)!F(a_2;c_2;m_2)}B_n\right| + (3+\gamma)\,|B_1|. \tag{7}$$

Using Remark 2.2, we have

$$P_2 \le (1-\beta)\left[\sum_{n=2}^{\infty}\left\{2 - \frac{(1+\gamma)}{n}\right\}\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}(n-1)!F(a_1;c_1;m_1)}\right.$$
$$\left. + \sum_{n=2}^{\infty}\left\{2 + \frac{(1+\gamma)}{n}\right\}\frac{(a_2)_{n-1}m_2^{n-1}}{(c_2)_{n-1}(n-1)!F(a_2;c_2;m_2)}\right] + (3+\gamma)\,|B_1|$$
$$= (1-\beta)\left[\frac{1}{F(a_1;c_1;m_1)}\left\{2\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}(n-1)!} - (1+\gamma)\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}n!}\right\}\right.$$
$$\left. + \frac{1}{F(a_2;c_2;m_2)}\left\{2\sum_{n=2}^{\infty}\frac{(a_2)_{n-1}m_2^{n-1}}{(c_2)_{n-1}(n-1)!} + (1+\gamma)\sum_{n=2}^{\infty}\frac{(a_2)_{n-1}m_2^{n-1}}{(c_2)_{n-1}n!}\right\}\right] + (3+\gamma)\,|B_1|$$
$$= (1-\beta)\left[\frac{1}{F(a_1;c_1;m_1)}\left\{2\left(F(a_1;c_1;m_1)-1\right)\right.\right.$$
$$\left. - \frac{(1+\gamma)(c_1-1)}{m_1(a_1-1)}\left(F(a_1-1;c_1-1;m_1) - 1 - \frac{(a_1-1)}{(c_1-1)}m_1\right)\right\}$$
$$\left. + 2\left(F(a_2;c_2;m_2)-1\right) + \frac{(1+\gamma)(c_2-1)}{m_2(a_2-1)}\left(F(a_2-1;c_2-1;m_2) - 1 - \frac{(a_2-1)}{(c_2-1)}m_2\right)\right]$$
$$+ (3+\gamma)\,|B_1|$$
$$\le 1-\gamma,$$

by the given hypothesis. □

In next theorem, we establish connections between $TG_H(\gamma)$ and $G_H(\gamma)$.

Theorem 2.4. *Let $a_1, c_1, a_2, c_2, m_1, m_2 > 0$. If for some $\gamma (0 \leq \gamma < 1)$ the inequality*

$$\frac{1}{F(a_1;c_1;m_1)}\left(F(a_1;c_1;m_1)-1\right)+\frac{1}{F(a_2;c_2;m_2)}\left(F(a_2;c_2;m_2)-1\right)\leq 1-\frac{3+\gamma}{1-\gamma}|B_1| \quad (8)$$

is satisfied, then $\Omega\left(TG_H(\gamma)\right) \subset G_H(\gamma)$.

Proof. Making use of Lemma 2.2, we only need to prove that $P_2 \leq 1-\gamma$, where P_2 is given by (7). Using Remark 2.1, it follows that

$$\begin{aligned}
P_2 = &\sum_{n=2}^{\infty}(2n-1-\gamma)\left|\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}(n-1)!F(a_1;c_1;m_1)}A_n\right| \\
&+\sum_{n=2}^{\infty}(2n+1+\gamma)\left|\frac{(a_2)_{n-1}m_2^{n-1}}{(c_2)_{n-1}(n-1)!F(a_2;c_2;m_2)}B_n\right|+(3+\gamma)\left|B_1\right| \\
&\leq (1-\gamma)\left[\sum_{n=2}^{\infty}\frac{(a_1)_{n-1}m_1^{n-1}}{(c_1)_{n-1}(n-1)!F(a_1;c_1;m_1)}+\sum_{n=2}^{\infty}\frac{(a_2)_{n-1}m_2^{n-1}}{(c_2)_{n-1}(n-1)!F(a_2;c_2;m_2)}\right]+(3+\gamma)\left|B_1\right| \\
&= (1-\gamma)\left[\frac{1}{F(a_1;c_1;m_1)}\left(F(a_1;c_1;m_1)-1\right)+\frac{1}{F(a_2;c_2;m_2)}\left(F(a_2;c_2;m_2)-1\right)\right]+(3+\gamma)\left|B_1\right| \\
&\leq (1-\gamma),
\end{aligned}$$

by the given condition and this completes the proof. □

In next theorem, we present conditions on the parameters $a_1, c_1, a_2, c_2, m_1, m_2$ and obtain a characterization for operator Ω which maps $TG_H(\gamma)$ on to itself.

Theorem 2.5. *If $a_1, c_1, a_2, c_2, m_1, m_2 > 0$ and $\gamma(0 \leq \gamma < 1)$. Then*

$$\Omega\left(TG_H(\gamma)\right) \subset TG_H(\gamma),$$

if and only if, the condition (8) is satisfied.

Proof. The proof of above theorem is similar to that of Theorem 2.4. Therefore we omits the details involved. □

Remark 2.3. *If we put $a_1 = c_1; a_2 = c_2$ then we obtain the corresponding results for the Poisson distribution series.*

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