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INEQUALITIES VIA STRONGLY (p, h)-HARMONIC CONVEX FUNCTIONS

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Abstract. The main aim of this paper is to consider a new class of harmonic convex functions with respect to an arbitrary non-negative function, which is called strongly (p, h)-harmonic convex function. We establish Hermite-Hadamard like integral inequalities via these new classes of convex functions. Some special cases are discussed, which can be obtained from our main results. The ideas and techniques of this paper may stimulate further research.

Keywords: Harmonic convex functions, p-convex functions, Hermite-Hadamard type inequalities.

AMS Subject Classification (2000): 26D15, 26D10, 90C23

1. Introduction

The theory of convexity has been subject to extensive research during the past few years due to its utility in various branches of pure and applied mathematics. It is known that the function is a convex function, if and only if, it satisfies the integral inequality, which is known as the Hermite-Hadamard inequality. The Hermite-Hadamard type integral inequalities are useful in physics, where upper and lower bounds for natural phenomena are described by integrals. For recent developments and generalizations, see [6, 9, 10, 11, 12, 13, 18]. Varosanec [20] introduced the notion of h-convexity, which unifies various classes of convex functions. With appropriate and suitable choice of the arbitrary function, one can obtain a wide class of convex functions, which are being investigated. Polyak [19] considered strongly convex functions which include the convex functions as special cases. In recent years, strongly convex are being used to discuss the convergence analysis of the iterative methods for solving variational inequalities and related optimization problems. The harmonic convex functions were introduced and studied by Anderson et al. [1] and Iscan [6]. Noor et al[11] introduced the concept of p-harmonic

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means, which includes the harmonic means and arithmetic means as special cases. Using this concept, they introduced and investigated the properties of p-harmonic convex sets and the p- harmonic convex functions. It have been proved that the p-hramonic convex functions include the harmonic convex functions and convex functions as special cases.

The concepts of strongly convex functions have been generalized in various directions using novel and innovative ideas. Noor et. al [14] have introduced a class of strongly harmonic convex functions and established some Hermite-Hadamard type integral inequalities. For recent developments, see [14, 15, 16]. Inspired and motivated by the ongoing research, we introduce the concept of strongly (p, h)-harmonic convex functions with respect to an arbitrary non-negative function h. This class is more general and contains several known and new classes of convex functions as special cases. We discuss some properties of strongly (p, h)-harmonic convex functions. We also derive several new Hermite-Hadamard inequalities. Ideas and techniques of this paper may motivate further research in this field.

2. Preliminaries

In this section, we introduce some new classes of harmonic convex functions.

Definition 2.1. [11]. *A set $I = [a, b] \subseteq \mathbb{R} \setminus \{0\}$ is said to be a p-harmonic convex set, if*

$$\left[\frac{x^p y^p}{tx^p + (1-t)y^p}\right]^{\frac{1}{p}} \in I, \qquad \forall x, y \in I,\ t \in [0,1],\ p \neq 0.$$

We would like to point out that if $p = 1$, then p-harmonic convex set becomes harmonic convex set and if If $p = -1$, then p-harmonic convex set becomes convex set. This shows that the concept of p-harmonic convex set is quite general and unifying one.

Definition 2.2. [11]. *A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to p-harmonic convex, where $p \neq 0$, if*

$$f\left(\left[\frac{x^p y^p}{tx^p + (1-t)y^p}\right]^{\frac{1}{p}}\right) \leq (1-t)f(x) + tf(y), \qquad \forall x, y \in I, t \in [0,1].$$

Noor et. al [11] have obtained the Hermite-Hadamard inequality for p-harmonic convex functions, which may be regarded as a refinement of the concept of convexity, see [16, 12].

We now consider a new class of harmonic convex functions, which was introduced in [16].

Definition 2.3. [16]. *Let $h : J = [0, 1] \to \mathbb{R}$ be an arbitrary nonnegative function. A function $f : I = [a, b] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be strongly (p, h)-harmonic convex function with respect to an arbitrary non-negative function h with modulus $c > 0$, if*

$$f\left(\left[\frac{x^p y^p}{tx^p + (1-t)y^p}\right]^{\frac{1}{p}}\right) \leq h(1-t)f(x) + h(t)f(y) - ct(1-t)\left(\frac{x^p - y^p}{x^p y^p}\right)^2. \tag{1}$$

The function f is said to be strongly (p, h)-harmonic concave function, if and only if, -f is strongly (p, h)-harmonic convex function.

For $t = \frac{1}{2}$ in (1), we have

$$f\left(\frac{2x^p y^p}{x^p + y^p}\right)^{\frac{1}{p}} \leq h\left(\frac{1}{2}\right)[f(x) + f(y)] - \frac{c}{4}\left(\frac{x^p - y^p}{x^p y^p}\right)^2, \quad x, y \in I. \tag{2}$$

The function f is called strongly Jensen (p,h)-harmonic convex function.

Remark 2.1. (i). *If $p=1$ in Definition* 2.3, *then it reduces to strongly harmonic h-convex functions introduced by Noor et. al* [17].
(ii). *If $p=-1$ in Definition 2.3, then it reduces to strongly h-convex functions* [2].

For different and appropriate choice of non-negative function h, one can obtain several new classes of harmonic convex functions and their variant forms.

The Euler Beta function is a special function defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \forall x,y>0,$$

where $\Gamma(\cdot)$ is a Gamma function.

3. Main Results

In this section, we obtain Hermite-Hadamard inequalities for strongly (p,h)-harmonic convex functions.

Theorem 3.1. *Let $f: I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a strongly (p,h)-harmonic convex function with modulus $c>0$. If $f\in L[a,b]$, then*

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)}\left[f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)+\frac{c}{12}\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right] &\leq \frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\\ &\leq [f(a)+f(b)]\int_0^1 h(t)\mathrm{d}t-\frac{c}{6}\left(\frac{a^p-b^p}{a^pb^p}\right)^2. \quad (3)\end{aligned}$$

Proof. Let f be strongly (p,h)-harmonic convex function with modulus $c>0$. Let $x=\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}$ and $y=\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}$ in (2). Then

$$\begin{aligned}f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right) &\leq h\left(\frac{1}{2}\right)\left[f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)+f\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\right]\\ &\quad -\frac{c}{4}(1-2t)^2\left(\frac{a^p-b^p}{a^pb^p}\right)^2\\ &= h\left(\frac{1}{2}\right)\left[\int_0^1 f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t+\int_0^1 f\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\right]\\ &\quad -\frac{c}{4}\left(\frac{a^p-b^p}{a^pb^p}\right)^2\int_0^1(1-2t)^2\mathrm{d}t\\ &= 2h\left(\frac{1}{2}\right)\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x-\frac{c}{12}\left(\frac{a^p-b^p}{a^pb^p}\right)^2,\end{aligned}$$

from which, we have

$$\begin{aligned}
\frac{1}{2h\left(\frac{1}{2}\right)}\left[f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)+\frac{c}{12}\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right] &\leq \frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\\
&= \int_0^1 f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\
&\leq [h(1-t)f(a)+h(t)f(b)]\\
&\quad -ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\\
&= [f(a)+f(b)]\int_0^1 h(t)\mathrm{d}t-\frac{c}{6}\left(\frac{a^p-b^p}{a^pb^p}\right)^2.
\end{aligned}$$

This completes the proof. □

Corollary 3.1. *Under the assumptions of Theorem 3.1 and $h(t)=t$, we have*

$$\begin{aligned}
f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)+\frac{c}{12}\left(\frac{a^p-b^p}{a^pb^p}\right)^2 &\leq \frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\\
&\leq \frac{f(a)+f(b)}{2}-\frac{c}{6}\left(\frac{a^p-b^p}{a^pb^p}\right)^2. \qquad (4)
\end{aligned}$$

The above inequality (4) is the Hermite-Hadamard inequality for strongly p-harmonic convex functions, which appears to be a new one.

Theorem 3.2. *Let $f: I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a strongly (p,h)-harmonic convex function with modulus $c>0$ on the interval $[a,b]$. Then*

$$\begin{aligned}
&\frac{1}{4h\left(\frac{1}{2}\right)}\left[f\left(\left[\frac{4a^pb^p}{a^p+3b^p}\right]^{\frac{1}{p}}+f\left(\left[\frac{4a^pb^p}{3a^p+b^p}\right]^{\frac{1}{p}}\right)\right)\right]+\frac{c}{48}\left(\frac{a^p-b^p}{a^pb^p}\right)^2\\
\leq\ & \frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\\
\leq\ & \left[\frac{f(a)+f(b)}{2}+f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)\right]\int_0^1 h(t)\mathrm{d}t-\frac{c}{24}\left(\frac{a^p-b^p}{a^pb^p}\right)^2. \qquad (5)
\end{aligned}$$

Proof. By applying (3) on each of the interval $[a,[\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}]$ and $[[\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}},b]$, we have

$$\begin{aligned}
\frac{1}{2h\left(\frac{1}{2}\right)}\left[f\left(\left[\frac{4a^pb^p}{a^p+3b^p}\right]^{\frac{1}{p}}\right)\right. &+ \left.\frac{c}{48}\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right]\leq\frac{2pa^pb^p}{b^p-a^p}\int_a^{[\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}}\frac{f(x)}{x^{1+p}}\mathrm{d}x\\
&\leq \left[f(a)+f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)\right]\int_0^1 h(t)\mathrm{d}t-\frac{c}{24}\left(\frac{a^p-b^p}{a^pb^p}\right)^2 \qquad (6)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2h\left(\frac{1}{2}\right)}\left[f\left(\left[\frac{4a^pb^p}{3a^p+b^p}\right]^{\frac{1}{p}}\right)\right. &+ \left.\frac{c}{48}\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right]\leq\frac{2pa^pb^p}{b^p-a^p}\int_{[\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}}^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\\
&\leq \left[f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}+f(b)\right)\right]\int_0^1 h(t)\mathrm{d}t-\frac{c}{24}\left(\frac{a^p-b^p}{a^pb^p}\right)^2 \qquad (7)
\end{aligned}$$

Summing up (6) and (7) side by side, we obtain

$$\begin{aligned}
&\frac{1}{4h\left(\frac{1}{2}\right)}\left[f\left(\left[\frac{4a^pb^p}{a^p+3b^p}\right]^{\frac{1}{p}}+f\left(\left[\frac{4a^pb^p}{3a^p+b^p}\right]^{\frac{1}{p}}\right)\right]+\frac{c}{48}\left(\frac{a^p-b^p}{a^pb^p}\right)^2\\
&\leq\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\\
&\leq\left[\frac{f(a)+f(b)}{2}+f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)\right]\int_0^1 h(t)\mathrm{d}t-\frac{c}{24}\left(\frac{a^p-b^p}{a^pb^p}\right)^2,
\end{aligned}$$

which is the required result. $\square$

Theorem 3.3. *Let $f,g:I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be strongly (p,h)-harmonic convex functions with modulus $c>0$. If $f,g\in L[a,b]$, then*

$$\begin{aligned}
&\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)g\left(\frac{a^pb^px^p}{(a^p+b^p)x^p-a^pb^p}\right)}{x^{1+p}}\mathrm{d}x\\
\leq\ &M(a,b)\int_0^1 h(t)h(1-t)\mathrm{d}t+N(a,b)\int_0^1[h(t)]^2\mathrm{d}t\\
&-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2S(a,b)\int_0^1 t(1-t)h(t)\mathrm{d}t+\frac{c^2}{30}\left(\frac{a^p-b^p}{a^pb^p}\right)^4,
\end{aligned}$$

where

$$M(a,b)=f(a)g(a)+f(b)g(b) \tag{8}$$

$$N(a,b)=f(a)g(b)+f(b)g(a) \tag{9}$$

$$S(a,b)=f(a)+f(b)+g(a)+g(b). \tag{10}$$

Proof. Let f,g be strongly (p,h)-harmonic convex functions with modulus $c>0$. Then

$$\begin{aligned}
&\frac{pa^pb^p}{b^p-a^p}\int_a^b\frac{f(x)g\left(\frac{a^pb^px^p}{(a^p+b^p)x^p-a^pb^p}\right)}{x^{1+p}}\mathrm{d}x\\
&=\int_0^1 f\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)g\left(\left[\frac{a^pb^p}{(1-t)a^p+tb^p}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\
&\leq\int_0^1[h(1-t)f(a)+h(t)f(b)-ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2]\\
&\quad\times[h(t)g(a)+h(1-t)g(b)-ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2]\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
&= f(a)g(b)\int_0^1 [h(1-t)]^2\mathrm{d}t + f(b)g(a)\int_0^1 [h(t)]^2\mathrm{d}t \\
&\quad +[f(a)g(a)+f(b)g(b)]\int_0^1 h(t)h(1-t)\mathrm{d}t \\
&\quad -c\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2[f(a)+g(b)]\int_0^1 t(1-t)h(1-t)\mathrm{d}t \\
&\quad -c\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2[f(b)+g(a)]\int_0^1 t(1-t)h(t)\mathrm{d}t \\
&\quad +c^2\Big(\frac{a^p-b^p}{a^pb^p}\Big)^4\int_0^1 t^2(1-t)^2\mathrm{d}t \\
&= [f(a)g(b)+f(b)g(a)]\int_0^1 [h(t)]^2\mathrm{d}t + [f(a)g(a)+f(b)g(b)]\int_0^1 h(t)h(1-t)\mathrm{d}t \\
&\quad -c\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2[f(a)+f(b)+g(a)+g(b)]\int_0^1 t(1-t)h(t)\mathrm{d}t + \frac{c^2}{30}\Big(\frac{a^p-b^p}{a^pb^p}\Big)^4 \\
&= M(a,b)\int_0^1 h(t)h(1-t)\mathrm{d}t + N(a,b)\int_0^1 [h(t)]^2\mathrm{d}t \\
&\quad -c\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2 S(a,b)\int_0^1 t(1-t)h(t)\mathrm{d}t + \frac{c^2}{30}\Big(\frac{a^p-b^p}{a^pb^p}\Big)^4,
\end{aligned}$$

which is the required result. □

Theorem 3.4. *Let $f,g : I \subset \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be strongly (p,h)-harmonic convex functions with modulus $c>0$. If $f,g \in L[a,b]$, then*

$$\begin{aligned}
\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{1+p}}\mathrm{d}x &\le M(a,b)\int_0^1 [h(t)]^2\mathrm{d}t + N(a,b)\int_0^1 h(t)h(1-t)\mathrm{d}t \\
&\quad -c\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2 S(a,b)\int_0^1 t(1-t)h(t)\mathrm{d}t + \frac{c^2}{30}\Big(\frac{a^p-b^p}{a^pb^p}\Big)^4,
\end{aligned}$$

where $M(a,b)$, $N(a,b)$ and $S(a,b)$ are given by (8), (9) and (10), respectively.

Proof. Let f,g be strongly (p,h)-harmonic convex functions with modulus $c>0$. Then

$$\begin{aligned}
&\frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g(x)}{x^{1+p}}\mathrm{d}x \\
&= \int_0^1 f\Big(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac{1}{p}}\Big) g\Big(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac{1}{p}}\Big)\mathrm{d}t \\
&\le \int_0^1 [h(1-t)f(a)+h(t)f(b)-ct(1-t)\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2] \\
&\quad \times[h(1-t)g(a)+h(t)g(b)-ct(1-t)\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2]\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= f(a)g(a)\int_0^1[h(1-t)]^2\mathrm{d}t + f(b)g(b)\int_0^1[h(t)]^2\mathrm{d}t \\
&\quad +[f(a)g(b)+f(b)g(a)]\int_0^1 h(t)h(1-t)\mathrm{d}t \\
&\quad -c\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2[f(a)+g(a)]\int_0^1 t(1-t)h(1-t)\mathrm{d}t \\
&\quad -c\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2[f(b)+g(b)]\int_0^1 t(1-t)h(t)\mathrm{d}t \\
&\quad +c^2\Big(\frac{a^p-b^p}{a^pb^p}\Big)^4\int_0^1 t^2(1-t)^2\mathrm{d}t \\
&= [f(a)g(a)+f(b)g(b)]\int_0^1[h(t)]^2\mathrm{d}t + [f(a)g(b)+f(b)g(a)]\int_0^1 h(t)h(1-t)\mathrm{d}t \\
&\quad -c\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2[f(a)+f(b)+g(a)+g(b)]\int_0^1 t(1-t)h(t)\mathrm{d}t + \frac{c^2}{30}\Big(\frac{a^p-b^p}{a^pb^p}\Big)^4 \\
&= M(a,b)\int_0^1[h(t)]^2\mathrm{d}t + N(a,b)\int_0^1 h(t)h(1-t)\mathrm{d}t \\
&\quad -c\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2 S(a,b)\int_0^1 t(1-t)h(t)\mathrm{d}t + \frac{c^2}{30}\Big(\frac{a^p-b^p}{a^pb^p}\Big)^4,
\end{aligned}
$$

which is the required result. □

4. Integral Inequalities

In this section, we are going to obtain midpoint, Simpson's and Trapezoidal like integral inequalities via relative strongly p-harmonic convex function. We need the following Lemma in order to prove our main results.

Lemma 4.1. [12]. *Let $f : I = [a,b] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a,b]$ and $\lambda \in [0,1]$, then*

$$
\begin{aligned}
&(1-\lambda)f\Big(\Big[\frac{2a^pb^p}{a^p+b^p}\Big]^{\frac{1}{p}}\Big) + \lambda\Big(\frac{f(a)+f(b)}{2}\Big) - \frac{p(a^pb^p)}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1+p}}\mathrm{d}x \\
&= \frac{(b^p-a^p)}{2p(a^pb^p)}\Big[\int_0^{\frac{1}{2}}(2t-\lambda)\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac{1}{p}} f'\Big(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac{1}{p}}\Big)\mathrm{d}t \\
&\quad + \int_{\frac{1}{2}}^1(2t-2+\lambda)\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac{1}{p}} f'\Big(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac{1}{p}}\Big)\mathrm{d}t\Big].
\end{aligned}
$$

Theorem 4.1. *Let $f : I = [a,b] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a,b]$ and $|f'|^q$ is strongly (p,h)-harmonic convex function on I, $q \geq 1$*

and $\lambda \in [0,1]$, *then*

$$\left|(1-\lambda)f\left(\frac{2a^pb^p}{a^p+b^p}\right)^{\frac{1}{p}}+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\right|$$
$$\leq \frac{b^p-a^p}{2p(a^pb^p)}\Bigg[(\mu_1(p,a,b))^{1-\frac{1}{q}}[\mu_2(p,a,b)|f'(a)|^q+\mu_3(p,a,b)|f'(b)|^q-c\mu_4(p,a,b)\left(\frac{a^p-b^p}{a^pb^p}\right)^2]^{\frac{1}{q}}$$
$$+(\mu_5(p,a,b))^{1-\frac{1}{q}}[\mu_6(p,a,b)|f'(a)|^q+\mu_7(p,a,b)|f'(b)|^q-c\mu_8(p,a,b)\left(\frac{a^p-b^p}{a^pb^p}\right)^2]^{\frac{1}{q}}\Bigg],$$

where

$$\mu_1(p,a,b) = \int_0^{\frac{1}{2}}|2t-\lambda|\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1+\frac{1}{p}}\mathrm{d}t,$$

$$\mu_2(p,a,b) = \int_0^{\frac{1}{2}}h(1-t)|2t-\lambda|\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1+\frac{1}{p}}\mathrm{d}t, \tag{11}$$

$$\mu_3(p,a,b) = \int_0^{\frac{1}{2}}h(t)|2t-\lambda|\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1+\frac{1}{p}}\mathrm{d}t, \tag{12}$$

$$\mu_4(p,a,b) = \int_{\frac{1}{2}}^1 t(1-t)|2t-\lambda|\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1+\frac{1}{p}}\mathrm{d}t, \tag{13}$$

$$\mu_5(p,a,b) = \int_0^{\frac{1}{2}}|2t-2+\lambda|\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1+\frac{1}{p}}\mathrm{d}t,$$

$$\mu_6(p,a,b) = \int_0^{\frac{1}{2}}h(1-t)|2t-2+\lambda|\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1+\frac{1}{p}}\mathrm{d}t, \tag{14}$$

$$\mu_7(p,a,b) = \int_0^{\frac{1}{2}}h(t)|2t-2+\lambda|\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1+\frac{1}{p}}\mathrm{d}t, \tag{15}$$

$$\mu_8(p,a,b) = \int_{\frac{1}{2}}^1 t(1-t)|2t-2+\lambda|\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{1+\frac{1}{p}}\mathrm{d}t, \tag{16}$$

Proof. Using Lemma 4.1, power mean inequality and strongly (p,h)-harmonic convexity of $|f|^q$, we have

$$\left|(1-\lambda)f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\right|$$
$$\leq \frac{b^p-a^p}{2p(a^pb^p)}\left[\int_0^{\frac{1}{2}}\left|(2t-\lambda)\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}\right|\left|f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\right|\mathrm{d}t\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left|(2t-2+\lambda)\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}\right|\left|f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\right|\mathrm{d}t\right]$$

$$
\begin{aligned}
\leq\ & \frac{b^p-a^p}{2p(a^pb^p)}\Bigg[\Bigg(\int_0^{\frac12}\big|(2t-\lambda)\big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac1p}\mathrm{d}t\Bigg)^{1-\frac1q}\\
& \Bigg(\int_0^{\frac12}\big|(2t-\lambda)\big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac1p}\Big|f'\Big(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac1p}\Big)\Big|^q\mathrm{d}t\Bigg)^{\frac1q}\\
& +\Bigg(\int_{\frac12}^1\big|(2t-2+\lambda)\big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac1p}\mathrm{d}t\Bigg)^{1-\frac1q}\\
& \Bigg(\int_{\frac12}^1\big|(2t-2+\lambda)\big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac1p}\Big|f'\Big(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac1p}\Big)\Big|^q\mathrm{d}t\Bigg)^{\frac1q}\Bigg]\\
\leq\ & \frac{b^p-a^p}{2p(a^pb^p)}\Bigg[\Bigg(\int_0^{\frac12}\big|(2t-\lambda)\big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac1p}\mathrm{d}t\Bigg)^{1-\frac1q}\\
& \Bigg(\int_0^{\frac12}\big|(2t-\lambda)\big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac1p}\big[h(1-t)|f'(a)|^q+h(t)|f'(b)|^q\\
& -ct(1-t)\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2\big]\mathrm{d}t\Bigg)^{\frac1q}+\Bigg(\int_{\frac12}^1\big|(2t-2+\lambda)\big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac1p}\mathrm{d}t\Bigg)^{1-\frac1q}\\
& \Bigg(\int_{\frac12}^1\big|(2t-2+\lambda)\big|\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{1+\frac1p}\big[h(1-t)|f'(a)|^q+h(t)|f'(b)|^q\\
& -ct(1-t)\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2\big]\mathrm{d}t\Bigg)^{\frac1q}\Bigg]\\
=\ & \frac{b^p-a^p}{2p(a^pb^p)}\Bigg[(\mu_1(p,a,b))^{1-\frac1q}[\mu_2(p,a,b)|f'(a)|^q+\mu_3(p,a,b)|f'(b)|^q\\
& -c\mu_4(p,a,b)\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2]^{\frac1q}+(\mu_5(p,a,b))^{1-\frac1q}[\mu_6(p,a,b)|f'(a)|^q\\
& +\mu_7(p,a,b)|f'(b)|^q-c\mu_8(p,a,b)\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2]^{\frac1q}\Bigg],
\end{aligned}
$$

which is the required result. □

Theorem 4.2. *Let $f: I=[a,b]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on the interior I^o of I. If $f'\in L[a,b]$ and $|f'|^q$ is strongly (p,h)-harmonic convex function on I, $r,q>1$, $\frac1r+\frac1q=1$ and $\lambda\in[0,1]$, then*

$$
\begin{aligned}
& \Bigg|(1-\lambda)f\Big(\frac{2a^pb^p}{a^p+b^p}\Big)^{\frac1p}+\lambda\Big(\frac{f(a)+f(b)}{2}\Big)-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\Bigg|\\
\leq\ & \frac{(b^p-a^p)}{2a^pb^p}\Bigg[(\mu_9(r,p;a,b))^{\frac1r}\Bigg(\frac{f(a)+f\big(\big[\frac{2a^pb^p}{a^p+b^p}\big]^{\frac1p}\big)}{2}\int_0^1h(t)\mathrm{d}t-\frac{c}{48}\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2\Bigg)^{\frac1q}\\
& +(\mu_{10}(r,p;b,a))^{\frac1r}\Bigg(\frac{f\big(\big[\frac{2a^pb^p}{a^p+b^p}\big]^{\frac1p}\big)+f(b)}{2}\int_0^1h(t)\mathrm{d}t-\frac{c}{48}\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2\Bigg)^{\frac1q}\Bigg],
\end{aligned}
$$

where

$$
\mu_9(r,p;a,b)=\int_0^{\frac12}|2t-\lambda|^r\Big(\frac{a^pb^p}{ta^p+(1-t)b^p}\Big)^{r+\frac rp}\mathrm{d}t, \tag{17}
$$

$$\mu_{10}(r,p;b,a) = \int_{\frac{1}{2}}^{1} |2t-2+\lambda|^{r}\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{r+\frac{r}{p}} \mathrm{d}t. \tag{18}$$

Proof. Using Lemma 4.1 and the Holder's integral inequality, we have

$$\begin{aligned}
&\left|(1-\lambda)f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1+p}}\mathrm{d}x\right| \\
\leq\ & \frac{b^p-a^p}{2p(a^pb^p)}\Bigg[\int_0^{\frac{1}{2}}\left|(2t-\lambda)\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}\right|\left|f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\right|\mathrm{d}t \\
&+\int_{\frac{1}{2}}^{1}\left|(2t-2+\lambda)\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}\right|\left|f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\right|\mathrm{d}t\Bigg] \\
\leq\ & \frac{b^p-a^p}{2p(a^pb^p)}\Bigg[\left(\int_0^{\frac{1}{2}}\left|(2t-\lambda)\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}\right|^{r}\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{2}}\left|f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\right|^{q}\mathrm{d}t\right)^{\frac{1}{q}} \\
&+\left(\int_{\frac{1}{2}}^{1}\left|(2t-2+\lambda)\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}\right|^{r}\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_{\frac{1}{2}}^{1}\left|f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\right|^{q}\mathrm{d}t\right)^{\frac{1}{q}}\Bigg] \\
=\ & \frac{b^p-a^p}{2p(a^pb^p)}\Bigg[\left(\int_0^{\frac{1}{2}}|(2t-\lambda)|^{r}\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{r+\frac{r}{p}}\mathrm{d}t\right)^{\frac{1}{r}}\left(\frac{pa^pb^p}{b^p-a^p}\int_a^{[\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}}\frac{|f'(x)|^q}{x^{1+p}}\mathrm{d}x\right)^{\frac{1}{q}} \\
&+\left(\int_{\frac{1}{2}}^{1}|(2t-2+\lambda)|^{r}\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{r+\frac{r}{p}}\mathrm{d}t\right)^{\frac{1}{r}}\left(\frac{pa^pb^p}{b^p-a^p}\int_{[\frac{2a^pb^p}{a^p+b^p}]^{\frac{1}{p}}}^{b}\frac{|f'(x)|^q}{x^{1+p}}\mathrm{d}t\right)^{\frac{1}{q}}\Bigg]
\end{aligned}$$

Using the the inequalities (6) and (7), we obtained the required result. □

Theorem 4.3. *Let $f : I = [a,b] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a,b]$ and $|f'|^q$ is strongly (p,h)-harmonic convex function on I, $r,q > 1$, $\frac{1}{r}+\frac{1}{q} = 1$ and $\lambda \in [0,1]$, then*

$$\begin{aligned}
&\left|(1-\lambda)f\left(\frac{2a^pb^p}{a^p+b^p}\right)^{\frac{1}{p}}+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1+p}}\mathrm{d}x\right| \\
\leq\ & \frac{b^p-a^p}{2p(a^pb^p)}\times\left(\frac{\lambda^{r+1}+(1-\lambda)^{r+1}}{2(r+1)}\right)^{\frac{1}{r}}\big[\big(\mu_{11}(q,p;a,b)|f'(a)|^q+\mu_{12}(q,p;a,b)|f'(b)|^q \\
&-c\mu_{13}(q,p;a,b)\left(\frac{a^p-b^p}{a^pb^p}\left(\right)^{\frac{1}{q}}+\big(\mu_{14}(q,p;a,b)|f'(a)|^q+\mu_{15}(q,p;a,b)|f'(b)|^q\right. \\
&-c\mu_{16}(q,p;a,b)\left(\frac{a^p-b^p}{a^pb^p}\left(\right)^{\frac{1}{q}}\big],\right.
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
\mu_{11}(q,p;a,b) &= \int_0^{\frac{1}{2}} h(1-t)\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{q}{p}+q}\mathrm{d}t, \\
\mu_{12}(q,p;b,a) &= \int_0^{\frac{1}{2}} h(t)\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{q}{p}+q}\mathrm{d}t, \\
\mu_{13}(q,p;b,a) &= \int_0^{\frac{1}{2}} t(1-t)\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{q}{p}+q}\mathrm{d}t,
\end{aligned}$$

$$
\begin{aligned}
\mu_{14}(q,p;a,b) &= \int_{\frac{1}{2}}^{1} h(1-t)\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{q}{p}+q}\mathrm{d}t,\\
\mu_{15}(q,p;b,a) &= \int_{\frac{1}{2}}^{1} h(t)\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{q}{p}+q}\mathrm{d}t\\
\mu_{16}(q,p;b,a) &= \int_{\frac{1}{2}}^{1} t(1-t)\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{q}{p}+q}\mathrm{d}t.
\end{aligned}
$$

Proof. Using Lemma 4.1 and the Holder's integral inequality, we have

$$
\begin{aligned}
&\left|(1-\lambda)f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{p(a^pb^p)}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1+p}}\mathrm{d}x\right|\\
\leq\ & \frac{b^p-a^p}{2p(a^pb^p)}\left[\int_0^{\frac{1}{2}}|(2t-\lambda)|\left|\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\right|\mathrm{d}t\right.\\
&\left.+\int_{\frac{1}{2}}^{1}|(2t-2+\lambda)|\left|\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\right|\mathrm{d}t\right]\\
\leq\ & \frac{b^p-a^p}{2p(a^pb^p)}\left[\left(\int_0^{\frac{1}{2}}\big|(2t-\lambda)\big|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{2}}\left|\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^{1}\big|(2t-2+\lambda)\big|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_{\frac{1}{2}}^{1}\left|\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{1+\frac{1}{p}}f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right]\\
=\ & \frac{b^p-a^p}{2p(a^pb^p)}\left[\left(\int_0^{\frac{1}{2}}\big|(2t-\lambda)\big|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{2}}\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{\frac{q}{p}+q}\left|f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^{1}\big|(2t-2+\lambda)\big|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_{\frac{1}{2}}^{1}\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{\frac{q}{p}+q}\left|f'\left(\left[\frac{a^pb^p}{ta^p+(1-t)b^p}\right]^{\frac{1}{p}}\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}\right]\\
\leq\ & \frac{b^p-a^p}{2p(a^pb^p)}\left[\left(\int_0^{\frac{1}{2}}\big|(2t-\lambda)\big|^r\mathrm{d}t\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{2}}\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{\frac{q}{p}+q}\big[h(1-t)|f'(a)|^q\right.\right.\\
&\left.+h(t)|f'(b)|^q-ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\big]\mathrm{d}t\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{1}\big|(2t-2+\lambda)\big|^r\mathrm{d}t\right)^{\frac{1}{r}}\\
&\left.\left(\int_{\frac{1}{2}}^{1}\left(\frac{a^pb^p}{ta^p+(1-t)b^p}\right)^{\frac{q}{p}+q}\big[h(1-t)|f'(a)|^q+h(t)|f'(b)|^q-ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\big]\mathrm{d}t\right)^{\frac{1}{q}}\right]\\
=\ & \frac{b^p-a^p}{2p(a^pb^p)}\times\left(\frac{\lambda^{r+1}+(1-\lambda)^{r+1}}{2(r+1)}\right)^{\frac{1}{r}}\big[\big(\mu_{11}(q,p;a,b)|f'(a)|^q+\mu_{12}(q,p;a,b)|f'(b)|^q\\
&-c\mu_{13}(q,p;a,b)\left(\frac{a^p-b^p}{a^pb^p}\right(\big)^{\frac{1}{q}}+\big(\mu_{14}(q,p;a,b)|f'(a)|^q+\mu_{15}(q,p;a,b)|f'(b)|^q\\
&-c\mu_{16}(q,p;a,b)\left(\frac{a^p-b^p}{a^pb^p}\right(\big)^{\frac{1}{q}}\big],
\end{aligned}
$$

which is the required result. □

We now discuss some applications of our results.

I. For $\lambda=0$, Theorem 4.2 reduces to:

Corollary 4.1. *Let $f : I = [a,b] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a,b]$ and $|f'|^q$ is strongly (p,h)-harmonic convex function on I, $r,q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, then*

$$\begin{aligned} &\left| f\left(\frac{2a^pb^p}{a^p+b^p}\right)^{\frac{1}{p}} - \frac{p(a^pb^p)}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1+p}}\mathrm{d}x\right| \\ \leq\ & \frac{(b^p-a^p)}{2a^pb^p}\Bigg[(\mu_9(r,p;a,b))^{\frac{1}{r}}\left(\frac{f(a)+f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)}{2}\int_0^1 h(t)\mathrm{d}t - \frac{c}{48}\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right)^{\frac{1}{q}} \\ &+(\mu_{10}(r,p;b,a))^{\frac{1}{r}}\left(\frac{f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)+f(b)}{2}\int_0^1 h(t)\mathrm{d}t - \frac{c}{48}\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right)^{\frac{1}{q}}\Bigg], \end{aligned}$$

where $\mu_9(r,p;b,a)$ and $\mu_{10}(r,p;b,a)$ are given by (17) and (18), respectively.

II. For $\lambda = 1$, Theorem 4.2 reduces to:

Corollary 4.2. *Let $f : I = [a,b] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on the interior I^o of I. If $f' \in L[a,b]$ and $|f'|^q$ is strongly (p,h)-harmonic convex function on I, $r,q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$, then*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{p(a^pb^p)}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1+p}}\mathrm{d}x\right| \\ \leq\ & \frac{(b^p-a^p)}{2a^pb^p}\Bigg[(\mu_9(r,p;a,b))^{\frac{1}{r}}\left(\frac{f(a)+f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)}{2}\int_0^1 h(t)\mathrm{d}t - \frac{c}{48}\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right)^{\frac{1}{q}} \\ &+(\mu_{10}(r,p;b,a))^{\frac{1}{r}}\left(\frac{f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)+f(b)}{2}\int_0^1 h(t)\mathrm{d}t - \frac{c}{48}\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right)^{\frac{1}{q}}\Bigg], \end{aligned}$$

where $\mu_9(r,p;b,a)$ and $\mu_{10}(r,p;b,a)$ are given by (17) and (18), respectively.

Remark 4.1. (i). *For appropriate and suitable choice of p,q,r and h, one can obtain several new and known results as special cases for various classes of convex functions and their variant forms.*
(ii). *For $c \neq 0$ and $h(t) = t$, one can obtain new results for strongly p-harmonic convex functions.*
(iii). *For $c = 0$ and $h(t) = t$, all the results in this paper reduces to* [11, 12]. *This shows that our newly introduced class is more general and unifying one.*

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