TWMS J. App. Eng. Math. V.10, No.1, p.p. 118-127

FIXED POINT RESULTS FROM SOFT METRIC SPACES AND SOFT QUASI METRIC SPACES TO SOFT G-METRIC SPACES

NURCAN BILGILI GUNGOR[1], §

Abstract. In this paper, soft quasi-metric spaces by means of soft elements are described. Also the presentation of soft G-metric spaces and the existing fixed point results of contractive mappings defined on this kind of spaces are examined. Especially, it is shown that the most gotten fixed point theorems on this kind of spaces can be obtained directly from fixed point theorems on soft metric or soft quasi-metric spaces.

Keywords: fixed point, soft metric space, soft G-metric space, soft quasi metric space

AMS Subject Classification: [2010] 47H10,54H25

1. Introduction and Preliminaries

Problems in many fields include data that contain uncertainties. Uncertainties may be dealt with using a wide range of existing theories such as theory of probability, fuzzy set theory [20], intuitionistic fuzzy sets [2], vague sets [7], theory of interval mathematics [8], rough settheory [17], etc. All of these theories have their own difficulties which are pointed out in [13]. To overcome these difficulties, Molodtsov [13] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties. In ([13],[14]), Molodtsov pointed out several directions for the applications of soft sets, such as smoothness of functions, game theory, operations research, Riemann-integration, Perron integration, probability, theory of measurement and so on. At present, works on soft set theory and its applications are progressing rapidly. The help of rough mathematics of Pawlak [17], Maji et al. [12] defined a parameter reduction on soft sets, and presented an application of soft sets in a decision making problem. Ali et al. [1] founded new algebraic operations on soft sets. Shabir and Naz [18] presented soft topological spaces and searched their fundamental properties. Zorlutuna et al.[21] also investigated those spaces. Das and Samanta [5] presented the notions of soft real set and soft real number and gave their properties. Lately, soft set theory have a significant potential in various areas, such as see ([3],[4],[12]).

On the other hand, the wide application potential of fixed point theory is the main motivation of research activities in this field. The theoretical studies are advancing in two main directions. One of them is related with the attempts to generalize the contractive

[1] Department of Mathematics, Faculty of Science and Arts, Amasya University, 05000, Amasya, Turkey.
e-mail: bilgilinurcan@gmail.com; ORCID: https://orcid.org/0000-0001-5069-5881.

§ Manuscript received: April 20, 2018; accepted: January 22, 2019.

conditions on the maps and thus, weaken them. The other one is related with the attempts to generalize the space on which these contractions are defined. There is also a rapidly growing interest in studies combining the two directions.

In 2005, Mustafa and Sims introduced a new class of generalized metric spaces (see [15],[16]), which are called G-metric spaces, as generalization of a metric space (X, d). Subsequently, many fixed point results on such spaces appeared.

In 2012, Jleli and Samet [11] established the concept of quasi metric spaces and they showed that the most obtained fixed point theorems on G-metric spaces can be deduced immediately from fixed point theorems on metric or quasi metric spaces.

Recently, Wardowski [19] presented a concept of soft mapping and its fixed points. Further, Das and Samanta [6] established soft metric spaces and gave Banach fixed point theorem on this spaces.

In this paper, the notion of soft quasi-metric space according to soft element is introduced and some of its properties are defined. Also connections among soft metric, soft G-metric and soft quasi-metric are given. And then, it is shown that the most gotten fixed point theorems on soft G-metric spaces can be obtained directly from fixed point theorems on soft metrics or soft quasi-metric spaces. Thus, it is seen that the main results of very recent papers of Guler, Yildirim and Ozbakir [9] and Guler and Yildirim [10] are consequences of the main result of this paper.

Throughout this paper, the notations that used in [9] and [10] are followed. For the sake of completeness, some basic definitions, notations and results are given in the following.

Definition 1. *([6]) A mapping $d : SE(\widetilde{X}) \times SE(\widetilde{X}) \to \mathbb{R}(E)^*$ is said to be a soft metric on $\widetilde{X}$ if d satisfies the following conditions:*
(M1) $d(\widetilde{x}, \widetilde{y}) \widetilde{\geq} \bar{0}$, *for all* $\widetilde{x}, \widetilde{y} \widetilde{\in} \widetilde{X}$.
(M2) $d(\widetilde{x}, \widetilde{y}) = \bar{0}$ *if and only if* $\widetilde{x} = \widetilde{y}$,
(M3) $d(\widetilde{x}, \widetilde{y}) = d(\widetilde{y}, \widetilde{x})$, *for all* $\widetilde{x}, \widetilde{y} \widehat{\in} X$,
(M4) $d(\widetilde{x}, \widetilde{y}) \widetilde{\leq} d(\widetilde{x}, \widetilde{z}) + d(\widetilde{z}, \widetilde{y})$, *for all* $\widetilde{x}, \widetilde{y}, \widetilde{z} \widehat{\in} X$.
The soft set $\widetilde{X}$ with a soft metric d on $\widetilde{X}$ is said to be a soft metric space and is denoted by $(\widetilde{X}, d)$.

Definition 2. *([6]) Let $(\widetilde{x_n})$ be a sequence of soft elements in $(\widetilde{X}, d)$. The sequences $(\widetilde{x_n})$ is said to be convergent in $(\widetilde{X}, d)$, if there is a soft element $\widetilde{x} \widetilde{\in} \widetilde{X}$ such that $d(\widetilde{x_n}, \widetilde{x}) \to \bar{0}$ as $n \to \infty$.*
A sequence $(\widetilde{x_n})$ of soft elements in $(\widetilde{X}, d)$ is said to be Cauchy sequence in $\widetilde{X}$, if for every $\widetilde{\epsilon} \widetilde{\geq} \bar{0}$, there is a natural number m such that $d(\widetilde{x_i}, \widetilde{x_j}) \widetilde{\leq} \widetilde{\epsilon}$, whenever $i, j \geq m$.

Definition 3. *([6]) A soft metric space $(\widetilde{X}, d)$ is said to be complete if every Cauchy sequence in $\widetilde{X}$ converges to some soft element of $\widetilde{X}$.*

Definition 4. *([9]) Let X be a nonempty set and E be the nonempty set of parameters. A mapping $\widetilde{G} : SE(\widetilde{X}) \times SE(\widetilde{X}) \times SE(\widetilde{X}) \to \mathbb{R}(E)^*$ is said to be a soft generalized metric or soft G-metric on $\widetilde{X}$, if $\widetilde{G}$ satisfies the following conditions:*
$(\widetilde{G}_1)$ $\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z}) = \bar{0}$, *if* $\widetilde{x} = \widetilde{y} = \widetilde{z}$,
$(\widetilde{G}_2)$ $\widetilde{G}(\widetilde{x}, \widetilde{x}, \widetilde{y}) \widetilde{>} \bar{0}$, *for all* $\widetilde{x}, \widetilde{y} \in SE(\widetilde{X})$ *with* $\widetilde{x} \neq \widetilde{y}$
$(\widetilde{G}_3)$ $\widetilde{G}(\widetilde{x}, \widetilde{x}, \widetilde{y}) \widetilde{\leq} \widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z})$, *for all* $\widetilde{x}, \widetilde{y}, \widetilde{z} \in SE(\widetilde{X})$ *with* $\widetilde{y} \neq \widetilde{z}$,
$(\widetilde{G}_4)$ $\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z}) = \widetilde{G}(\widetilde{x}, \widetilde{z}, \widetilde{y}) = \widetilde{G}(\widetilde{y}, \widetilde{z}, \widetilde{x}) = ...$

$(\widetilde{G}_5)$ $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{z})\widetilde{\leq}\widetilde{G}(\widetilde{x},\widetilde{a},\widetilde{a})+\widetilde{G}(\widetilde{a},\widetilde{y},\widetilde{z})$, *for all* $x,y,z,a\in SE(\widetilde{X})$.
The soft set $\widetilde{X}$ *with a soft G-metric* $\widetilde{G}$ *on* $\widetilde{X}$ *is said to be a soft G-metric space and is denoted by* $(\widetilde{X},\widetilde{G},E)$.

Proposition 5. *([9]) For any soft metric* d *on* $\widetilde{X}$*, we can construct a soft G-metric by the following mappings* $\widetilde{G}_s$ *and* $\widetilde{G}_m$*:*
(1) $\widetilde{G}_s(d)(\widetilde{x},\widetilde{y},\widetilde{z})=\frac{1}{3}(d(\widetilde{x},\widetilde{y})+d(\widetilde{y},\widetilde{z})+d(\widetilde{x},\widetilde{z}))$,
(2) $\widetilde{G}_m(d)(\widetilde{x},\widetilde{y},\widetilde{z})=\max\{d(\widetilde{x},\widetilde{y})+d(\widetilde{y},\widetilde{z})+d(\widetilde{x},\widetilde{z})\}$,

Proposition 6. *([9]) For any soft G-metric* $\widetilde{G}$ *on* $\widetilde{X}$*, we can construct a soft metric* $d_{\widetilde{G}}$ *on* $\widetilde{X}$ *defined by*

$$d_{\widetilde{G}}(\widetilde{x},\widetilde{y})=\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{y})+\widetilde{G}(\widetilde{x},\widetilde{x},\widetilde{y}). \tag{1.1}$$

Definition 7. *([9])* $(\widetilde{X},\widetilde{G},E)$ *be a soft G-metric space and* $(\widetilde{x_n})$ *be a sequence of soft elements in* $\widetilde{X}$*. The sequence* $(\widetilde{x_n})$ *is said to be soft G-convergent at* $\widetilde{x}$ *in* $\widetilde{X}$*, if for every* $\widetilde{\epsilon}\widetilde{\geq}\bar{0}$*, chosen arbitrarily, there exists a natural number* $N=N(\widetilde{\epsilon})$ *such that* $\bar{0}\widetilde{\leq}\widetilde{G}(\widetilde{x_n},\widetilde{x_n},\widetilde{x})\widetilde{<}\widetilde{\epsilon}$*, whenever* $n\geq N$*,i,e.,* $n\geq N\Rightarrow(\widetilde{x_n})\in B_{\widetilde{G}}(\widetilde{x},\widetilde{\epsilon})$.
We denote this by $(\widetilde{x_n})\to\widetilde{x}$ *as* $n\to\infty$ *or by* $\lim_{n\to\infty}(\widetilde{x_n})=\widetilde{x}$.

Proposition 8. *([9]) Let* $(\widetilde{X},\widetilde{G},E)$ *be a soft G-metric space, for a sequence* $(\widetilde{x_n})$ *in* $\widetilde{X}$ *and soft element* $\widetilde{x}$*, then the followings are equivalent:*
(1) $(\widetilde{x_n})$ *is soft G-convergent to* $\widetilde{x}$,
(2) $d_{\widetilde{G}}(\widetilde{x_n},\widetilde{x})\to\bar{0}$ *as* $n\to\infty$,
(3) $\widetilde{G}(\widetilde{x_n},\widetilde{x_n},\widetilde{x})\to\bar{0}$ *as* $n\to\infty$,
(4) $\widetilde{G}(\widetilde{x_n},\widetilde{x},\widetilde{x})\to\bar{0}$ *as* $n\to\infty$,
(5) $\widetilde{G}(\widetilde{x_n},\widetilde{x_m},\widetilde{x})\to\bar{0}$ *as* $n,m\to\infty$.

Definition 9. *([9]) A soft G-metric space* $(\widetilde{X},\widetilde{G},E)$ *is symmetric if*
$(\widetilde{G}_6)$ $\widetilde{G}(\widetilde{x},\widetilde{y},\widetilde{y})=\widetilde{G}(\widetilde{x},\widetilde{x},\widetilde{y})$ *for all* $x,y\in SE(\widetilde{X})$.

Definition 10. *([10]) Let* $(\widetilde{X},\widetilde{G},E)$ *be a soft G-metric space and* $(\widetilde{x_n})$ *be a sequence of soft elements in* $\widetilde{X}$.
The sequence $(\widetilde{x_n})$ *is said to be soft G-Cauchy, if for every* $\widetilde{\epsilon}\widetilde{\geq}\bar{0}$*, chosen arbitrarily, there exists a natural number* k *such that* $\widetilde{G}(\widetilde{x_n},\widetilde{x_m},\widetilde{x_l})\widetilde{<}\widetilde{\epsilon}$*, whenever* $n,m,l\geq k$.
A soft G-metric space $(\widetilde{X},\widetilde{G},E)$ *is said to be soft G-complete, if every soft G-Cauchy sequence in* $(\widetilde{X},\widetilde{G},E)$ *is soft G-convergent in* $(\widetilde{X},\widetilde{G},E)$.

Proposition 11. *([10]) Let* $(\widetilde{X},\widetilde{G},E)$ *be a soft G-metric space and* $(\widetilde{x_n})$ *be a sequence of soft elements in* $\widetilde{X}$*. Then the followings are equivalent:*
(1) the sequence $(\widetilde{x_n})$ *is soft G-Cauchy,*
(2) for every $\widetilde{\epsilon}\widetilde{\geq}\bar{0}$*, there exists a natural number* k *such that* $\widetilde{G}(\widetilde{x_n},\widetilde{x_m},\widetilde{x_l})\widetilde{<}\widetilde{\epsilon}$ *for any* $n,m\geq k$,
(3) $(\widetilde{x_n})$ *is a Cauchy sequence in the soft metric space* $(\widetilde{X},d_{\widetilde{G}},E)$.

Corollary 12. *([10]) Every soft G-convergent sequence in any soft G-metric space* $(\widetilde{X},\widetilde{G},E)$ *is soft G-Cauchy.*

Proposition 13. *([10]) A soft G-metric space* $(\widetilde{X},\widetilde{G},E)$ *is soft G-complete if and only if* $(\widetilde{X},d_{\widetilde{G}},E)$ *is complete soft metric space.*

It is noticed that in the symmetric case ($(\widetilde{X}, \widetilde{G}, E)$ is symmetric), many fixed point theorems on soft G-metric spaces are particular cases of existing fixed point theorems on soft metric spaces. In this paper, the non-symmetric case is handled. For this reason, soft quasi-metric space is defined and it is shown that non-symmetric soft G-metric space have a soft quasi-metric form and then many results on non-symmetric soft G-metric spaces can be reproduced from fixed point on soft quasi-metric spaces.

2. Basic Definitions and Results

Definition 14. *Let X be a nonempty set and E be the nonempty set of parameters. A mapping $\widetilde{q} : SE(\widetilde{X}) \times SE(\widetilde{X}) \to \mathbb{R}(E)^*$ is said to be soft quasi-metric on $\widetilde{X}$, if $\widetilde{q}$ satisfies the following conditions:*
($\widetilde{q}_1$) $\widetilde{q}(\widetilde{x}, \widetilde{y}) = \bar{0}$ *if and only if* $\widetilde{x} = \widetilde{y}$,
($\widetilde{q}_1$) $\widetilde{q}(\widetilde{x}, \widetilde{y}) \widetilde{\leq} \widetilde{q}(\widetilde{x}, \widetilde{z}) + \widetilde{q}(\widetilde{z}, \widetilde{y})$, *for all* $\widetilde{x}, \widetilde{y}, \widetilde{z} \in SE(\widetilde{X})$.
The soft set $\widetilde{X}$ with a soft quasi-metric $\widetilde{q}$ on $\widetilde{X}$ is said to be a soft quasi-metric space and is denoted by $(\widetilde{X}, \widetilde{q}, E)$.

Note that any soft metric space is a soft quasi-metric space, but the converse is not true in general.

Now, it is shown that soft G-metric spaces have soft quasi-metric type structure. Indeed, the following results can be obtained.

Theorem 15. *Let $(\widetilde{X}, \widetilde{G}, E)$ be a soft G-metric space. The mapping $\widetilde{q} : SE(\widetilde{X}) \times SE(\widetilde{X}) \to \mathbb{R}(E)^*$ defined by $\widetilde{q}(x, y) = \widetilde{G}(x, y, y)$ satisfies the following properties:*
($\widetilde{q}_1$) $\widetilde{q}(\widetilde{x}, \widetilde{y}) = \bar{0}$ *if and only if* $\widetilde{x} = \widetilde{y}$,
($\widetilde{q}_1$) $\widetilde{q}(\widetilde{x}, \widetilde{y}) \widetilde{\leq} \widetilde{q}(\widetilde{x}, \widetilde{z}) + \widetilde{q}(\widetilde{z}, \widetilde{y})$, *for all* $\widetilde{x}, \widetilde{y}, \widetilde{z} \in SE(\widetilde{X})$.

Proof. The proof of ($\widetilde{q}_1$) follows immediately from the properties ($\widetilde{G}_1$),($\widetilde{G}_2$),($\widetilde{G}_3$) and ($\widetilde{G}_4$) in Definition 4. Now, let $\widetilde{x}, \widetilde{y}, \widetilde{z}$ be a any points in $SE(\widetilde{X})$. Using the property ($\widetilde{G}_5$) in Definition 4,

$$\widetilde{q}(x, y) = \widetilde{G}(x, y, y) \leq \widetilde{G}(x, z, z) + \widetilde{G}(z, y, y) = \widetilde{q}(x, z) + \widetilde{q}(z, y). \tag{2.1}$$

Thus, the proof is completed. □

Now, convergence and completeness on soft quasi-metric spaces are defined in the following.

Definition 16. *Let $(\widetilde{X}, \widetilde{q}, E)$ be a soft quasi-metric space and $(\widetilde{x_n})$ be a sequence of soft elements in $\widetilde{X}$. The sequence $(\widetilde{x_n})$ is said to be soft quasi-converges to $\widetilde{x}$ in $\widetilde{X}$ if and only if*

$$\lim_{n\to\infty} \widetilde{q}(\widetilde{x_n}, \widetilde{x}) = \lim_{n\to\infty} \widetilde{q}(\widetilde{x}, \widetilde{x_n}) = \bar{0}. \tag{2.2}$$

Definition 17. *Let $(\widetilde{X}, \widetilde{q}, E)$ be a soft quasi-metric space and $(\widetilde{x_n})$ be a sequence of soft elements in $\widetilde{X}$.*
The sequence $(\widetilde{x_n})$ is said to be soft left-Cauchy, if and only if for every $\widetilde{\epsilon} \widetilde{\geq} \bar{0}$, chosen arbitrarily, there exists a natural number k such that $\widetilde{q}(\widetilde{x_n}, \widetilde{x_m}) \widetilde{<} \widetilde{\epsilon}$, whenever $n \geq m > k$.

Definition 18. *Let $(\widetilde{X}, \widetilde{q}, E)$ be a soft quasi-metric space and $(\widetilde{x_n})$ be a sequence of soft elements in $\widetilde{X}$.*
The sequence $(\widetilde{x_n})$ is said to be soft right-Cauchy, if and only if for every $\widetilde{\epsilon} \widetilde{\geq} \bar{0}$, chosen arbitrarily, there exists a natural number k such that $\widetilde{q}(\widetilde{x_n}, \widetilde{x_m}) \widetilde{<} \widetilde{\epsilon}$, whenever $m \geq n > k$.

Definition 19. *Let $(\widetilde{X}, \widetilde{q}, E)$ be a soft quasi-metric space and $(\widetilde{x_n})$ be a sequence of soft elements in $\widetilde{X}$.*
The sequence $(\widetilde{x_n})$ is said to be soft Cauchy, if and only if for every $\widetilde{\epsilon}\widetilde{\geq}\overline{0}$, chosen arbitrarily, there exists a natural number k such that $\widetilde{q}(\widetilde{x_n}, \widetilde{x_m})\widetilde{<}\widetilde{\epsilon}$, whenever $n, m > k$.

Apparently, a sequence $(\widetilde{x_n})$ in a soft quasi-metric space is soft Cauchy if and only if it is soft left-Cauchy and soft right-Cauchy.

Definition 20. *Let $(\widetilde{X}, \widetilde{q}, E)$ be a soft quasi-metric space. Then,*
(1) $(\widetilde{X}, \widetilde{q}, E)$ *is soft left-complete if and only if each soft left-Cauchy sequence in $\widetilde{X}$ is convergent,*
(2) $(\widetilde{X}, \widetilde{q}, E)$ *is soft right-complete if and only if each soft right-Cauchy sequence in $\widetilde{X}$ is convergent,*
(3) $(\widetilde{X}, \widetilde{q}, E)$ *is soft complete if and only if each soft Cauchy sequence in $\widetilde{X}$ is convergent.*

The following results is an immediate consequences of the above definitions and results.

Theorem 21. *Let $(\widetilde{X}, \widetilde{G}, E)$ be a soft G-metric space, $\widetilde{d} : SE(\widetilde{X}) \times SE(\widetilde{X}) \to \mathbb{R}(E)^*$ be the function defined by $\widetilde{d}(x, y) = \widetilde{G}(x, y, y)$ and $(\widetilde{x_n})$ be a sequence of soft elements in $\widetilde{X}$. Then,*

(1) $(\widetilde{X}, \widetilde{d}, E)$ *is a quasi-metric space,*
(2) $(\widetilde{x_n}) \subset \widetilde{X}$ *is soft G-convergent $\widetilde{x} \in \widetilde{X}$ if and only if $(\widetilde{x_n}) \subset \widetilde{X}$ is soft quasi-convergent to $\widetilde{x}$ in $(\widetilde{X}, \widetilde{d}, E)$,*
(3) $(\widetilde{x_n}) \subset \widetilde{X}$ *is soft G-Cauchy if and only if $(\widetilde{x_n}) \subset \widetilde{X}$ is soft Cauchy in $(\widetilde{X}, \widetilde{d}, E)$,*
(4) $(\widetilde{X}, \widetilde{G}, E)$ *is soft G-complete if and only if $(\widetilde{X}, \widetilde{d}, E)$ is soft complete.*

Every soft quasi-metric induces a soft metric, that is, if $(\widetilde{X}, \widetilde{d}, E)$ is soft quasi-metric space, then the function $\widetilde{\delta} : SE(\widetilde{X}) \times SE(\widetilde{X}) \to \mathbb{R}(E)^*$ defined by

$$\widetilde{\delta}(x, y) = \max\{\widetilde{d}(x, y), \widetilde{d}(y, x)\} \tag{2.3}$$

is a soft metric on $\widetilde{X}$.

The following results are obtained from the above definitions and results.

Theorem 22. *Let $(\widetilde{X}, \widetilde{G}, E)$ be a soft G-metric space, $\widetilde{\delta} : SE(\widetilde{X}) \times SE(\widetilde{X}) \to \mathbb{R}(E)^*$ be the function defined by $\widetilde{\delta}(x, y) = \max\{\widetilde{G}(x, y, y), \widetilde{G}(y, x, x)\}$. Then,*

(1) $(\widetilde{X}, \delta)$ *is a soft metric space,*
(2) $(\widetilde{x_n}) \subset \widetilde{X}$ *is soft G-convergent $\widetilde{x} \in \widetilde{X}$ if and only if $(\widetilde{x_n}) \subset \widetilde{X}$ is soft convergent to $\widetilde{x}$ in $(\widetilde{X}, \widetilde{\delta})$,*
(3) $(\widetilde{x_n}) \subset \widetilde{X}$ *is soft G-Cauchy if and only if $(\widetilde{x_n}) \subset \widetilde{X}$ is soft Cauchy in $(\widetilde{X}, \widetilde{\delta})$,*
(4) $(\widetilde{X}, \widetilde{G}, E)$ *is soft G-complete if and only if $(\widetilde{X}, \widetilde{\delta}, E)$ is soft complete.*

3. Remarks on Fixed Point Results on G-Metric Spaces

3.1. From Soft Metric to Soft G-Metric: The Linear Case.

In this section, it is shown that in the case of linear contractive conditions, the existing fixed point results on soft G-metric spaces are immediate consequences of existing fixed point theorems on soft metric spaces. In the following, a fixed point theorem on complete soft metric space is given and proved. Then, as a model example, it is shown that Theorem 3.5 in [10] is immediate consequence of this fixed point theorem on complete soft metric space.

Theorem 23. *Let $(\widetilde{X}, d)$ be a complete soft metric space and $T : (\widetilde{X}, d) \to (\widetilde{X}, d)$ be a mapping satisfying for all $\widetilde{x}, \widetilde{y}, \widetilde{z} \in SE(\widetilde{X})$,*

$$d(T\widetilde{x}, T\widetilde{y}) \widetilde{\leq} \bar{k} \max\{d(\widetilde{x}, T\widetilde{x}), d(\widetilde{y}, T\widetilde{y}), d(\widetilde{x}, \widetilde{y})\} \text{ where } \bar{0} \widetilde{<} \bar{k} \widetilde{<} \bar{1}. \tag{3.1}$$

Then, T has a unique fixed point.

Proof. Let $\widetilde{x_0} \in SE(\widetilde{X})$ be an arbitrary soft element and define the sequence $(\widetilde{x_n})$ by $\widetilde{x_n} = T^n(\widetilde{x_0})$. From (3.1),

$$d(\widetilde{x_n}, \widetilde{x_{n+1}}) \widetilde{\leq} \bar{k} \max\{d(\widetilde{x_{n-1}}, \widetilde{x_n}), d(\widetilde{x_n}, \widetilde{x_{n+1}}), d(\widetilde{x_{n-1}}, \widetilde{x_n})\}. \tag{3.2}$$

Assume that $\max\{d(\widetilde{x_{n-1}}, \widetilde{x_n}), d(\widetilde{x_n}, \widetilde{x_{n+1}})\} = d(\widetilde{x_n}, \widetilde{x_{n+1}})$.
In this case $d(\widetilde{x_n}, \widetilde{x_{n+1}}) \widetilde{\leq} \bar{k} d(\widetilde{x_n}, \widetilde{x_{n+1}})$, and it is a contradiction. So,

$$d(\widetilde{x_n}, \widetilde{x_{n+1}}) \widetilde{\leq} \bar{k} d(\widetilde{x_{n-1}}, \widetilde{x_n}). \tag{3.3}$$

Thus, from the triangular inequality and (3.3), for all $m, n \in \mathbb{N}$ such that $n < m$,

$$\begin{array}{rl} d(\widetilde{x_n}, \widetilde{x_m}) \widetilde{\leq} & d(\widetilde{x_n}, \widetilde{x_{n+1}}) + d(\widetilde{x_{n+1}}, \widetilde{x_{n+2}} + ... + d(\widetilde{x_{m-1}}, \widetilde{x_m})) \\ \widetilde{\leq} & ((\bar{k})^n + (\bar{k})^{n+1} + (\bar{k})^{n+2} + ... + (\bar{k})^{m-1}) d(\widetilde{x_0}, \widetilde{x_1}) \\ \widetilde{\leq} & \frac{(\bar{k})^n}{\bar{1} - \bar{k}} d(\widetilde{x_0}, \widetilde{x_1}) \\ . & \end{array} \tag{3.4}$$

Thus, $d(\widetilde{x_n}, \widetilde{x_m}) \to 0$ as $m, n \to \infty$.
Since $(\widetilde{X}, d)$ is complete soft metric space, there exists $u \in SE(\widetilde{X})$ such that $(\widetilde{x_n})$ soft converges to u.
Assume that $Tu \neq u$,i.e., $T(\widetilde{u}(\lambda_0)) \neq \widetilde{u}(\lambda_0)$ for some $\lambda_0 \in E$. then by (3.1),

$$d(\widetilde{x_n}, \widetilde{Tu}) \widetilde{\leq} \bar{k} \max\{d(\widetilde{x_{n-1}}, \widetilde{x_n}), d(\widetilde{u}, \widetilde{Tu}), d(\widetilde{x_{n-1}}, \widetilde{u})\}. \tag{3.5}$$

By taking the limit as $n \to \infty$,

$$d(\widetilde{u}, \widetilde{Tu}) \widetilde{\leq} \bar{k} d(\widetilde{u}, \widetilde{Tu}) \tag{3.6}$$

since $(\widetilde{x_n})$ soft converges to u. But, this is a contradiction, so $Tu = u$. For the uniqueness, suppose that there exists a soft element $\widetilde{v}$ such that $\widetilde{u} \neq \widetilde{u}$ and $T\widetilde{v} = \widetilde{v}$. Then by (3.1),

$$\begin{array}{rl} d(\widetilde{u}, \widetilde{v}) = d(T\widetilde{u}, T\widetilde{v}) \widetilde{\leq} & \bar{k} \max\{d(\widetilde{u}, T\widetilde{u}), d(\widetilde{v}, T\widetilde{v}), d(\widetilde{u}, \widetilde{v})\} \\ = & \bar{k} d(\widetilde{u}, \widetilde{v}). \end{array} \tag{3.7}$$

Thus, it is apparent that $\widetilde{u} = \widetilde{v}$. □

Theorem 24. *(Theorem 3.5 in [10]) Let $(\widetilde{X}, \widetilde{G}, E)$ be a soft G-complete metric space and $T : (\widetilde{X}, \widetilde{G}, E) \to (\widetilde{X}, \widetilde{G}, E)$ be a mapping that satisfies the following condition for all $\widetilde{x}, \widetilde{y}, \widetilde{z} \in SE(\widetilde{X})$,*

$$\widetilde{G}(T\widetilde{x}, T\widetilde{y}, T\widetilde{z}) \widetilde{\leq} \bar{a} \widetilde{G}(\widetilde{x}, T\widetilde{x}, T\widetilde{x}) + \bar{b} \widetilde{G}(\widetilde{y}, T\widetilde{y}, T\widetilde{y}) + \bar{c} \widetilde{G}(\widetilde{z}, T\widetilde{z}, T\widetilde{z}) + \bar{d} \widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{z}) \tag{3.8}$$

where $\bar{0} \widetilde{\leq} \bar{a} + \bar{b} + \bar{c} + \bar{d} \widetilde{<} \bar{1}$. Then T has a unique fixed point.

Now, it can be showed that the above result is an immediate consequence of Theorem 23. Indeed, taking $\widetilde{z} = \widetilde{y}$ in (3.8),

$$\widetilde{G}(T\widetilde{x}, T\widetilde{y}, T\widetilde{y}) \widetilde{\leq} \bar{k} \max\{\widetilde{G}(\widetilde{x}, T\widetilde{x}, T\widetilde{x}), \widetilde{G}(\widetilde{y}, T\widetilde{y}, T\widetilde{y}), \widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{y})\}, \tag{3.9}$$

for all $\widetilde{x}, \widetilde{y} \in SE(\widetilde{X})$. Also from (3.8),

$$\widetilde{G}(T\widetilde{y}, T\widetilde{x}, T\widetilde{x}) \widetilde{\leq} \bar{k} \max\{\widetilde{G}(\widetilde{y}, T\widetilde{y}, T\widetilde{y}), \widetilde{G}(\widetilde{x}, T\widetilde{x}, T\widetilde{x}), \widetilde{G}(\widetilde{y}, \widetilde{x}, \widetilde{x})\}, \tag{3.10}$$

for all $\widetilde{x}, \widetilde{y} \in SE(\widetilde{X})$. Let the soft metric space $\widetilde{\delta} : SE(\widetilde{X}) \times SE(\widetilde{X}) \to \mathbb{R}(E)^*$ be the function defined by $\widetilde{\delta}(x,y) = \max\{\widetilde{G}(x,y,y), \widetilde{G}(y,x,x)\}$. It follows from (3.9) and (3.10) that

$$\widetilde{\delta}(Tx,Ty)\widetilde{\leq}\bar{k}\max\{\widetilde{\delta}(x,Tx), \widetilde{\delta}(y,Ty), \widetilde{\delta}(x,y)\}. \tag{3.11}$$

Thus, the mapping T satisfies the conditions of Theorem 23, then T has a fixed point.

3.2. **From Soft Quasi-Metric to Soft G-Metric: The Nonlinear Case.**
Sometimes, when the contractive condition is nonlinear type, the above strategy can not be used. Whereas, it is shown that fixed point results on soft G-metric spaces can be concluded from fixed point results on soft quasi-metric spaces. As a model example, a weakly contractive condition is observed. At first, the following fixed point theorem on soft quasi-metric spaces can be given.

Theorem 25. *Let $(\widetilde{X}, \widetilde{q}, E)$ be a soft complete quasi-metric space and $T : (\widetilde{X}, \widetilde{q}, E) \to (\widetilde{X}, \widetilde{q}, E)$ be a mapping satisfying for all $\widetilde{x}, \widetilde{y} \in SE(\widetilde{X})$,*

$$\widetilde{q}(T\widetilde{x}, T\widetilde{y})\widetilde{\leq}\widetilde{q}(\widetilde{x}, \widetilde{y}) - \phi(\widetilde{q}(\widetilde{x}, \widetilde{y})), \tag{3.12}$$

where $\phi : \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ is continuous with $\phi^{-1}(\{\bar{0}\}) = \{\bar{0}\}$. Then T has a unique fixed point.

Proof. Let $\widetilde{x_0} \in SE(\widetilde{X})$ be an arbitrary soft element and define the sequence $(\widetilde{x_n})$ by $\widetilde{x_n} = T^n(\widetilde{x_0})$. From (3.12),

$$\widetilde{q}(\widetilde{x_n}, \widetilde{x_{n+1}})\widetilde{\leq}\widetilde{q}(\widetilde{x_{n-1}}, \widetilde{x_n}) - \phi(\widetilde{q}(\widetilde{x_{n-1}}, \widetilde{x_n})), \text{ for all } n \geq 1. \tag{3.13}$$

This means that $\widetilde{q}(\widetilde{x_n}, \widetilde{x_{n+1}})$ is a decreasing sequence of nonnegative soft real numbers. Then there consist $\bar{r} \in \mathbb{R}(E)^*$ such that $\widetilde{q}(\widetilde{x_n}, \widetilde{x_{n+1}}) \to \bar{r}$ as $n \to \infty$. Letting $n \to \infty$ in (3.13), $\phi(\bar{r}) = \bar{0}$, that is, $\bar{r} = \bar{0}$. Hence,

$$\lim_{n\to\infty} \widetilde{q}(\widetilde{x_n}, \widetilde{x_{n+1}}) = \bar{0}. \tag{3.14}$$

Using the same metod,

$$\lim_{n\to\infty} \widetilde{q}(\widetilde{x_{n+1}}, \widetilde{x_n}) = \bar{0}. \tag{3.15}$$

Presently, it can be shown that $(\widetilde{x_n})$ is a soft Cauchy sequence in the soft quasi-metric space $(\widetilde{X}, \widetilde{q}, E)$, that is, $(\widetilde{x_n})$ is soft left-Cauchy and soft right-Cauchy. Assume that $(\widetilde{x_n})$ is not a soft left-Cauchy sequence. Then there exists $\bar{\varepsilon}\widetilde{>}\bar{0}$ for which subsequences $(\widetilde{x_{n(k)}})$ and $(\widetilde{x_{m_(k)}})$ of $(\widetilde{x_n})$ can be found with $n(k) > m(k) > k$ such that

$$\widetilde{q(x_{n(k)}}, \widetilde{x_{m(k)}})\widetilde{\geq}\bar{\varepsilon}, \tag{3.16}$$

for all k. Also, corresponding to $m(k)$, $n(k)$ can be selected sucht that it is the smallest integer with $n(k) > m(k)$ satisfying the above inequality. So

$$\widetilde{q}(\widetilde{x_{n(k)-1}}, \widetilde{x_{m(k)}})\widetilde{<}\bar{\varepsilon}, \tag{3.17}$$

for all k. On the other hand,

$$\begin{aligned} \bar{\varepsilon}\widetilde{\leq}\widetilde{q}(\widetilde{x_{n(k)}}, \widetilde{x_{m(k)}})\widetilde{\leq} &\quad \widetilde{q}(\widetilde{x_{n(k)}}, \widetilde{x_{n(k)-1}}) + \widetilde{q}(\widetilde{x_{n(k)-1}}, \widetilde{x_{m(k)}}) \\ \widetilde{<} &\quad \widetilde{q}(\widetilde{x_{n(k)}}, \widetilde{x_{n(k)-1}}) + \bar{\varepsilon}. \end{aligned} \tag{3.18}$$

Taking the limit $k \to \infty$ and using (3.15),

$$\lim_{k\to\infty} \widetilde{q}(\widetilde{x_{n(k)}}, \widetilde{x_{m(k)}})\widetilde{=}\bar{\varepsilon}. \tag{3.19}$$

Also,

$$\widetilde{q}(\widetilde{x_{n(k)-1}}, \widetilde{x_{m(k)-1}}) \widetilde{\leq} \widetilde{q}(\widetilde{x_{n(k)-1}}, \widetilde{x_{n(k)}}) + \widetilde{q}(\widetilde{x_{n(k)}}, \widetilde{x_{m(k)}}) + \widetilde{q}(\widetilde{x_{m(k)}}, \widetilde{x_{m(k)-1}}) \tag{3.20}$$

and

$$\widetilde{q}(\widetilde{x_{n(k)}}, \widetilde{x_{m(k)}}) \widetilde{\leq} \widetilde{q}(\widetilde{x_{n(k)}}, \widetilde{x_{n(k)-1}}) + \widetilde{q}(\widetilde{x_{n(k)-1}}, \widetilde{x_{m(k)-1}}) + \widetilde{q}(\widetilde{x_{m(k)-1}}, \widetilde{x_{m(k)}}). \tag{3.21}$$

Taking the limit $k \to \infty$ in the above inequalities and using (3.14),(3.15) and (3.19),

$$\lim_{k\to\infty} \widetilde{q}(\widetilde{x_{n(k)-1}}, \widetilde{x_{m(k)-1}}) \widetilde{=} \bar{\varepsilon}. \tag{3.22}$$

Thus, from (3.13), for all k,

$$\widetilde{q}(\widetilde{x_{n(k)}}, \widetilde{x_{m(k)}}) \widetilde{\leq} \widetilde{q}(\widetilde{x_{n(k)-1}}, \widetilde{x_{m(k)-1}}) - \phi(\widetilde{q}(\widetilde{x_{n(k)-1}}, \widetilde{x_{m(k)-1}})). \tag{3.23}$$

Taking the limit $k \to \infty$ in the above inequality and using (3.19) and (3.22),

$$\bar{\varepsilon} \widetilde{\leq} \bar{\varepsilon} - \phi(\bar{\varepsilon}), \tag{3.24}$$

which implies that $\bar{\varepsilon} \widetilde{=} \bar{0}$. This is a contradiction with $\bar{\varepsilon} \widetilde{>} \bar{0}$. So, $(\widetilde{x_n})$ is a soft left-Cauchy sequence. Similarly, it is shown that $(\widetilde{x_n})$ is a soft right-Cauchy sequence. Hence $(\widetilde{x_n})$ is a soft Cauchy sequence in the soft complete quasi-metric space $(\widetilde{X}, \widetilde{q}, E)$. This means that there exists $\widetilde{a} \in SE(\widetilde{X})$ such that

$$\lim_{n\to\infty} \widetilde{q}(\widetilde{x_n}, \widetilde{a}) \widetilde{=} \widetilde{q}(\widetilde{a}, \widetilde{x_n}) \widetilde{=} \bar{0}. \tag{3.25}$$

On the other hand,

$$\widetilde{q}(\widetilde{x_n}, T\widetilde{a}) \widetilde{=} \widetilde{q}(T\widetilde{x_{n-1}}, T\widetilde{a}) \widetilde{\leq} \widetilde{q}(\widetilde{x_{n-1}}, \widetilde{a}) - \phi(\widetilde{q}(\widetilde{x_{n-1}}, \widetilde{a})). \tag{3.26}$$

Taking the limit $n \to \infty$ in the above inequality and using (3.25),

$$\lim_{n\to\infty} \widetilde{q}(\widetilde{x_n}, T\widetilde{a}) \widetilde{=} \bar{0}. \tag{3.27}$$

In the same way,

$$\lim_{n\to\infty} \widetilde{q}(T\widetilde{a}\widetilde{x_n}) \widetilde{=} \widetilde{q}(T\widetilde{a}, T\widetilde{x_{n-1}}) \widetilde{\leq} \widetilde{q}(\widetilde{a}, \widetilde{x_{n-1}}) - \phi(\widetilde{q}(\widetilde{a}, \widetilde{x_{n-1}}). \tag{3.28}$$

Taking the limit $n \to \infty$ in the above inequality and using (3.25),

$$\lim_{n\to\infty} \widetilde{q}(T\widetilde{a}, \widetilde{x_n}) \widetilde{=} \bar{0}. \tag{3.29}$$

Thus,

$$\lim_{n\to\infty} \widetilde{q}(\widetilde{x_n}, T\widetilde{a}) \widetilde{=} \widetilde{q}(T\widetilde{a}, \widetilde{x_n}) \widetilde{=} \bar{0}. \tag{3.30}$$

Using (3.25) and (3.30), $\widetilde{a} \widetilde{=} T\widetilde{a}$, that is, $\widetilde{a}$ is a fixed point of T.
To show the uniqueness of the fixed point, suppose that $\widetilde{b}$ is also a fixed point of T. From (3.12),

$$\widetilde{q}(\widetilde{a}, \widetilde{b}) \widetilde{=} \widetilde{q}(T\widetilde{a}, T\widetilde{b}) \widetilde{\leq} \widetilde{q}(\widetilde{a}, \widetilde{b}) - \phi(\widetilde{q}(\widetilde{a}, \widetilde{b})), \tag{3.31}$$

which implies that $\widetilde{q}(\widetilde{a}, \widetilde{b}) \widetilde{=} \bar{0}$, that is, $\widetilde{a} \widetilde{=} \widetilde{b}$. So $\widetilde{a}$ is the unique fixed point of T.

□

Now, using the Theorem 25, the following fixed point theorem on soft G-metric spaces can be obtained.

Theorem 26. *Let* $(\widetilde{X}, \widetilde{G}, E)$ *be a soft* G*-complete metric space and* $T : (\widetilde{X}, \widetilde{G}, E) \to (\widetilde{X}, \widetilde{G}, E)$ *be a mapping that satisfies the following condition for all* $\widetilde{x}, \widetilde{y} \in SE(\widetilde{X})$,

$$\widetilde{G}(T\widetilde{x}, T\widetilde{y}, T\widetilde{y}) \widetilde{\leq} \widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{y}) - \phi(\widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{y})), \tag{3.32}$$

where $\phi : \mathbb{R}(E)^* \to \mathbb{R}(E)^*$ *is continuous with* $\phi^{-1}(\{\bar{0}\}) = \{\bar{0}\}$. *Then,* T *has a unique fixed point.*

Proof. Consider the soft quasi-metric $\widetilde{d}(\widetilde{x}, \widetilde{y}) \widetilde{=} \widetilde{G}(\widetilde{x}, \widetilde{y}, \widetilde{y})$, for all $\widetilde{x}, \widetilde{y} \in SE(\widetilde{X})$. From (3.32),

$$\widetilde{d}(T\widetilde{x}, T\widetilde{y}) \widetilde{\leq} \widetilde{d}(\widetilde{x}, \widetilde{y}) - \phi(\widetilde{d}(\widetilde{x}, \widetilde{y})), \tag{3.33}$$

for all $\widetilde{x}, \widetilde{y} \in SE(\widetilde{X})$. Then the result follows from Theorem 25. □

4. Conclusions

It is noticed that if $(\widetilde{X}, \widetilde{G}, E)$ is symmetric, many fixed point theorems on soft G-metric spaces are particular cases of existing fixed point theorems on soft metric spaces. In this paper, the non-symmetric case is handled. For this reason, in section 2, the notion of soft quasi-metric space according to soft element is introduced and some of its properties are defined. And it is shown that if there is a soft G-metric on $SE(\widetilde{X})$, then, a soft metric and a soft quasi-metric on $SE(\widetilde{X})$ can be defined by using this soft G-metric .

In subsection 1 of section 3, it is shown that in case of the linear contractive condition, the fixed point results on soft G-metric spaces are immediate consequences of fixed point theorems on soft metric spaces. Therefore, a fixed point theorem in complete soft metric space is proved and then, as a model example, it is shown that Theorem 3.5 in [10] is an immediate consequence of this fixed point theorem in complete soft metric space.

However, when the contractive condition is nonlinear type, the strategy is given in subsection 1 of section 3, can not be used. Whereas, in subsection 2 of section 3, it is shown that fixed point results on soft G-metric spaces can be obtained from fixed point results on soft quasi-metric spaces. As a model example, a weakly contractive condition is observed and a fixed point theorem on soft quasi-metric spaces is proved. Thus, it is seen that the most gotten fixed point theorems on soft G-metric spaces can be obtained from fixed point theorems on soft metrics or soft quasi-metric spaces which were given before.

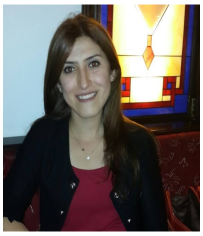
5. Acknowledgements

This research article has been supported by Amasya University Research Fund Project (FMB-BAP 17-0272).

References

[1] Ali, M.I., Feng,F., Liu,X., Min,W.K. and Shabir,M., (2003), On some new operations in soft set theory, Computer and Mathematics with Application, 57, (9), 1547- 1553.
[2] Atanassov, K.,(1986), Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 8796.
[3] Cagman, N. and Enginoglu, S.,(2010), Soft set theory and unit-int decision making, European J. Oper. Res., 207, 848-855.
[4] Cetkin, V.,Aygunoglu, A. and Aygun, H.(2016), A new approach in handling soft decision making problems, J. Nonlinear Sci. Appl., 9, 231-239.
[5] Das, S. and Samanta, S. K.,(2012), Soft real set, soft real number and their properties, J. Fuzzy Math., 20, (3), 551-576.
[6] Das, S. and Samanta, S. K, (2013), On soft metric spaces, J. Fuzzy Math., 21 (3), 707-734.
[7] Gau, W.L. and Buehrer, D.J.,(1993) Vague sets, IEEE Transactions on Systems, Man and Cybernetics, 23/2, 610614.

[8] Gorzalzany, M.B.,(1987) A method of inference in approximate reasoning based on interval-valued fuzzy sets, Fuzzy Sets and Systems, 21, 117.
[9] Guler, A. C., Yildirim, E. D. and Ozbakir, O. B.,(2016) A Fixed point theorem on soft G-metric spaces, J. Nonlinear Sci. Appl., 9, 885-894.
[10] Guler, A. C.and Yildirim, E. D.,(2016), A note on soft G-metric spaces about fixed point theorems, Annals of Fuzzy Mathematics and Informatics, Volume 12, No. 5, 691-701.
[11] Jleli, M. and Samet, B.,(2012), Remarks on G-metric spaces and fixed point theorems, Fixed Point Theory and Applications, 2012:210.
[12] Maji, P.K., Roy, A.R. and Biswas,R., (2002), An application of soft sets in a decision making problem, Computers and Mathematics with Applications, 44, 10771083.
[13] Molodtsov, D.,(1999), Soft set theory first results, Computers and Mathematics with Applications, 37, 1931.
[14] Molodtsov,D.,(2004), The Theory of Soft Sets, URSS Publishers, Moscow, (in Russian).
[15] Mustafa, Z., (2005), A new structure for generalized metric spaces-with applications to fixed point theory, PhD thesis, the University of Newcastle, Australia
[16] Mustafa, Z. and Sims, B.,(2006) A new approach to generalized metric spaces, J. Nonlinear Convex Anal., 7(2), 289-297.
[17] Pawlak, Z.,(1982), Rough sets, International Journal of Information and Computer Sciences, 11, 341-356.
[18] Shabir M. and Naz,M.,(2011), On soft topological spaces, Comput. Math. Appl., 61 (7), 1786-1799.
[19] Wardowski, D.,(2013), On A Soft Mapping And Its Fixed Points, Fixed Point Theory Appl., 11 pages.
[20] Zadeh, L.A.,(1965), Fuzzy Sets, Information and Control, 8, 338353.
[21] Zorlutuna, I., Akdag, M., Min, W. K. and Atmaca, S.,(2012) Remarks On Soft Topological Spaces, Ann. Fuzzy Math. Inform., 3 , 171-185.



Nurcan BILGILI GUNGOR graduated from Faculty of Science and Arts, Gazi University, Ankara (2007). She received her M.Sc. (2009) and Ph.D.(2014) degrees from Gazi University. She has been working as an Assistant Professor in the Faculty of Science and Arts, Amasya University. Her research area includes fixed point theory and its applications.