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GLOBAL ATTRACTOR OF THE HYPERBOLIC RELAXATION OF THE SWIFT-HOHENBERG EQUATION IN $\mathbb{R}^n$

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ABSTRACT. In this paper, the hyperbolic modification of the Swift-Hohenberg equation in $\mathbb{R}^n$ is dealt with. The existence of the global attractor in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ is proved. Also, the smoothness and the finite dimensionality of the global attractor are established.

Keywords: Modified Swift-Hohenberg equation, global attractor, pattern formation.

AMS Subject Classification: 35G25, 35L75.

1. Introduction

The Swift-Hohenberg equation is a parabolic nonlinear equation which was firstly introduced in [20] as a mathematical model. This model explains the thermal instability in a fluid layer near the onset of the Rayleigh-Benard convection. The Swift-Hohenberg equation is connected with the following gradient system:

$$u_t = -\frac{\delta \mathcal{L}}{\delta u} \tag{1.1}$$

where

$$\mathcal{L}(u) = \int \left(-\frac{\mu - 1}{2} u^2 + \frac{1}{4} u^4 - |\nabla u|^2 + \frac{1}{2} |\Delta u|^2 \right) dx.$$

Here, $\mathcal{L}(u)$ is the Lyapunov functional depending on u and $\frac{\delta \mathcal{L}}{\delta u}$ is the Fréchet derivative. The parabolic equation derived from (1.1), which models the spatiotemporal evolution in the order parameter $u(t)$, is as follows:

$$u_t = \mu u - (1 + \Delta)^2 - u^3. \tag{1.2}$$

In the Rayleigh-Benard convection, the order parameter $u(t)$ can be described as the temperature field which occurs as a result of heating a thin fluid layer, trapped between two horizontal plates, from the bottom. The constant μ is the parameter which controls the distance from the threshold temperature. Namely, in the case of $\mu < 0$, the heating is not enough to cause convection, whereas, in the case of $\mu > 0$, convection takes place. Also, it is important that during the Rayleigh-Benard experiment similar patterns like stripes, spots

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and spirals are formed in the fluid. As a result of this phenomena, the Swift-Hohenberg equation is one of the major models to understand the pattern formation. Therefore, the Swift-Hohenberg equations have a wide range of application areas changing from convective hydrodynamics [20], lasers [13], optical parametric oscillators [6] to instabilities in cellular flow [18], Turing pattern [7], [9].

The classical Swift-Hohenberg equation has a good agreement with the systems possessing slow dynamics. Such slow dynamics can be found in the systems close to thermodynamic equilibrium. Also, one can see slow dynamics in systems far beyond the equilibrium if the characteristic time scale of the considered process is long compared to the transient period. On the other hand, rapidly changing dynamics can be observed in far-from-equilibrium systems as well. To develop a deep understanding of the systems with fast dynamics, one should consider the transient process which can possibly characterize the relaxation to steady state. Hence, to introduce an accurate model which can describe all the dynamics of the convection from short time periods to longer periods, in [4] Galenko et al. proposed the following equation:

$$u_t = -\int_{-\infty}^{t} M(t-\tau)\frac{\partial \mathcal{L}(\tau)}{\partial u}d\tau. \tag{1.3}$$

Here $M(t-\tau)$ is the memory function which describes how the current dynamics of the system depend on their past. Moreover, choosing the memory function as the Maxwell memory function

$$M(t-\tau) = \frac{1}{\varepsilon}e^{-\frac{(t-\tau)}{\varepsilon}}$$

with $\varepsilon > 0$ as the relaxation time, from the equation (1.3) it follows that

$$\varepsilon u_{tt} + u_t = \mu u - (1+\Delta)^2 - u^3. \tag{1.4}$$

It is also crucial to understand the long-time behavior of the Swift-Hohenberg equation from the mathematical viewpoint, such as the existence of the global attractor gives information about the pattern's stability properties. Therefore, well-posedness and the long-time dynamics of the parabolic Swift-Hohenberg equation (1.2) was studied in [14]-[16]. Moreover, a modified version of (1.2) which is obtained by adding the term $b|\nabla u|^2$ to the equation (1.2) was studied in [17], [19] in terms of global attractors. On the other hand, to the best of our knowledge, [8] has been the only result obtained related to hyperbolic version (1.4). In that paper, well-posedness and the existence of the global attractor for the hyperbolic Swift-Hohenberg equation in a three-dimensional bounded domain have been proved.

In this paper, the asymptotic behavior of the hyperbolic modification of the Swift-Hohenberg equation will be investigated. The principal novelty of the paper is that the hyperbolic Swift-Hohenberg equation in an unbounded domain will be considered. In many science and engineering problems, the size of the corresponding physical domain is much larger than the pattern's wavelength. Therefore, it is more convenient to study such systems on an unbounded domain. However, this leads to several mathematical difficulties due to lack of the Sobolev compact embedding theorems in unbounded domains. By using compensated compactness technique proposed in [10], these obstacles are handled. The rest of the paper is arranged as follows: in the second section, the problem is introduced and the main results are given. In the third section, the existence of the global attractor in the phase space $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ is proved. In the fourth section, it is shown that the global attractor is bounded in $H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$. Finally, in the last section, it is established that the fractal dimension of this global attractor is finite.

2. Statement of the problem and the main results

This paper is devoted to the following initial value problem

$$u_{tt} + \Delta^2 u + 2\Delta u + u_t + f(u) = h(x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^n, \tag{2.1}$$

$$u(0,x) = u_0(x), \;\; u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n. \tag{2.2}$$

where $h \in L^2(\mathbb{R}^n)$ and the function $f(\cdot)$ satisfies the following conditions:

$$f \in C^2(\mathbb{R}), |f''(s)| \le C\left(1 + |s|^{p-2}\right), \;\; p > 2, \;\; (n-4)p \le n, \tag{2.3}$$

$$f(s)s \ge (1+\delta)\, s^2, \text{ for some } \delta > 0 \text{ and every } s \in \mathbb{R}. \tag{2.4}$$

Firstly, the following well-posedness result is obtained.

Theorem 2.1. *Assume that the conditions (2.3)-(2.4) hold. Then for every $T > 0$ and $(u_0, u_1) \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, the problem (2.1)-(2.2) has a unique weak solution $u \in C\left([0,T]; H^2(\mathbb{R}^n)\right) \cap C^1\left([0,T]; L^2(\mathbb{R}^n)\right)$ such that*

$$\|(u(t), u_t(t))\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \le c\left(\|(u_0,u_1)\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}\right) \tag{2.5}$$

where $c : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function. Moreover, if $(u_0,u_1) \in H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$, then $u \in C\left([0,T]; H^4(\mathbb{R}^n)\right) \cap C^1\left([0,T]; H^2(\mathbb{R}^n)\right)$.

In addition, if $v, w \in C\left([0,T]; H^2(\mathbb{R}^n)\right) \cap C^1\left([0,T]; L^2(\mathbb{R}^n)\right)$ are the weak solutions to (2.1)-(2.2) with initial data $(v_0,v_1) \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and $(w_0,w_1) \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, then

$$\|v(t) - w(t)\|_{H^2(\mathbb{R}^n)} + \|v_t(t) - w_t(t)\|_{L^2(\mathbb{R}^n)}$$
$$\le \widetilde{c}(T,r)\left(\|v_0 - w_0\|_{H^2(\mathbb{R}^n)} + \|v_1 - w_1\|_{L^2(\mathbb{R}^n)}\right), \;\; \forall t \in [0,T], \tag{2.6}$$

where $\widetilde{c} : R_+ \times R_+ \to R_+$ is a nondecreasing function with respect to each variable and $r = \max\left\{\|(v_0,v_1)\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}, \|(w_0,w_1)\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}\right\}$.

Proof. Let us denote $A(w_1,w_2) = \left(w_2, -\Delta^2 w_1 - 2\Delta w_1 - w_2 - f'(0)\, w_1\right)$ and $\Phi(w_1,w_2) = \left(0, h - f(w_1) + f'(0)\, w_1\right)$. Then the problem (2.1)-(2.2) can be reduced to the following initial value problem

$$\begin{cases} \frac{d}{dt}(u(t), u_t(t)) = A(u(t), u_t(t)) + \Phi(u(t), u_t(t)), & \forall t > 0, \\ (u(0), u_t(0)) = (u_0, u_1), \end{cases} \tag{2.7}$$

in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. One can easily see that A is a maximal dissipative operator in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ by defining appropriate equivalent norm in $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Therefore, it generates a linear continuous semigroup $\left\{e^{tA}\right\}_{t\ge0}$ in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ and $H^4(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. Moreover, one can readily show that the nonlinear operator $\Phi(w_1,w_2)$ is Lipschitz continuous on bounded subsets of $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. Hence, by applying semigroup theory (see [2, p. 56-58]), for every $(u_0,u_1) \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$,the problem (2.7) has a local weak solution $(u,u_t) \in C\left([0,T_{\max}); H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\right)$. Furthermore, if $(u_0,u_1) \in H^4(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$, then (u,u_t) belonging to the class $C\left([0,T_{\max}); H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)\right)$ is a strong solution of the problem (2.7), and consequently, it is a strong solution of (2.1)-(2.2).

Now, let u be a local strong solution of (2.1)-(2.2). Then multiplying (2.1) with u_t and integrating over $(s,t)\times\mathbb{R}^n$, it is obtained that

$$L(u(t)) - \int_{\mathbb{R}^n} h(x)\,u(t,x)\,dx + \int_s^t \int_{\mathbb{R}^n} |u_t(\tau,x)|^2\,dx d\tau$$

$$= L(u(s)) - \int_{\mathbb{R}^n} h(x)\,u(s,x)\,dx, \quad t\geq s\geq 0, \tag{2.8}$$

where $L(u(t)) = \frac{1}{2}\|u_t\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{2}\|\Delta u\|^2_{L^2(\mathbb{R}^n)} - \|\nabla u\|^2_{L^2(\mathbb{R}^n)} + \int_{\mathbb{R}^n} F(u(t,x))\,dx$ and $F(z) = \int_0^z f(s)\,ds$, for all $z\in\mathbb{R}$. By using interpolation and the Young inequality,

$$\|\nabla u\|^2_{L^2(\mathbb{R}^n)} \leq \|\Delta u\|_{L^2(\mathbb{R}^n)}\|u\|_{L^2(\mathbb{R}^n)} \leq \varepsilon\|\Delta u\|^2_{L^2(\mathbb{R}^n)} + \frac{1}{4\varepsilon}\|u\|^2_{L^2(\mathbb{R}^n)} \tag{2.9}$$

for every $\varepsilon>0$. Picking $\varepsilon = \frac{1}{2+\delta}$ in (2.9), it is found that

$$\|\nabla u\|^2_{L^2(\mathbb{R}^n)} \leq \frac{1}{2+\delta}\|\Delta u\|^2_{L^2(\mathbb{R}^n)} + \frac{2+\delta}{4}\|u\|^2_{L^2(\mathbb{R}^n)}. \tag{2.10}$$

With the help of (2.4), there holds the following estimate

$$F(s) \geq \frac{1+\delta}{2}s^2, \tag{2.11}$$

for some $\delta>0$. Then, considering (2.10) and (2.11) in the definition of $L(u(t))$, it is found that

$$\|(u(t),u_t(t))\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \leq c_1 L(u(t)). \tag{2.12}$$

On the other hand, recalling the conditions (2.3) and $h\in L^2(\mathbb{R}^n)$, equality (2.8) gives that

$$L(u(t)) \leq c\left(\|(u_0,u_1)\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}\right).$$

Taking into account (2.12) with the last estimate, (2.5) is obtained for strong solutions. Next, assume that $v,w\in C\left([0,T];H^2(\mathbb{R}^n)\right)\cap C^1\left([0,T];L^2(\mathbb{R}^n)\right)$ are the strong solutions to (2.1)-(2.2) with initial data $(v_0,v_1)\in H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$ and $(w_0,w_1)\in H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. Then, subtracting the aforementioned equations corresponding to v and w, multiplying the obtained equation by (v_t-w_t) and integrating over $(0,t)\times\mathbb{R}^n$, (2.6) is obtained for strong solutions. Finally, by using the standard density argument, the results are extended to weak solutions. Hence, the proof of the well-posedness theorem is completed. □

Thus, according to Theorem 2.1, by the formula $(u(t),u_t(t)) = S(t)(u_0,u_1)$, problem (2.1)-(2.2) generates a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$, where u is the weak solution of (2.1)-(2.2), determined by Theorem 2.1, with initial data (u_0,u_1).

Now, the main result can be stated.

Theorem 2.2. *Under the conditions (2.3)-(2.4), the semigroup $\{S(t)\}_{t\geq 0}$ generated by the problem (2.1)-(2.2) possesses a global attractor $\mathcal{A}$ in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. Also, the attractor is bounded in $H^4(\mathbb{R}^n)\times H^2(\mathbb{R}^n)$ and has a finite fractal dimension.*

3. Existence of the Global Attractor

Lemma 3.1. *Assume that the conditions (2.3)-(2.4) are satisfied and the sequence* $\{v_m\}_{m=1}^{\infty}$ *is weakly star convergent in* $L^{\infty}\left(0,T;H^2\left(\mathbb{R}^n\right)\right)\cap W^{1,\infty}\left(0,T;L^2\left(\mathbb{R}^n\right)\right)$. *Then, there holds the following inequality*

$$\limsup_{m\to\infty}\limsup_{l\to\infty}\int\limits_{\mathbb{R}^n}\left(\widehat{f}(v_l\left(t,x\right))-\widehat{f}(v_m\left(t,x\right))\right)\left(v_m\left(t,x\right)-v\left(t,x\right)\right)dx\leq 0 \text{ for all } t\in[0,T]$$

where $\widehat{f}\left(s\right)=f\left(s\right)-\left(1+\delta\right)s$.

Proof. To start with, it is inferred that

$$\int\limits_{\mathbb{R}^n}\left(\widehat{f}\left(v_l\right)\left(t,x\right)-\widehat{f}\left(v_m\left(t,x\right)\right)\right)\left(v_m\left(t,x\right)-v_l\left(t,x\right)\right)dx$$

$$=\int\limits_{\mathbb{R}^n}\left(\widehat{f}\left(v_l\left(t,x\right)\right)v_m\left(t,x\right)+\widehat{f}\left(v_m\left(t,x\right)\right)v_l-\widehat{f}\left(v_m\left(t,x\right)\right)v_m\left(t,x\right)-\widehat{f}\left(v_l\left(t,x\right)\right)v_l\left(t,x\right)\right)dx. \tag{3.1}$$

Then, recalling the conditions of the lemma, for any $r>0$ and $t\in[0,T]$, there exist $v\left(t\right)\in L^2\left(B(0,r)\right)$ such that

$$v_m\left(t\right)\to v\left(t\right) \text{ strongly in } L^2\left(B(0,r)\right),$$

where $B\left(0,r\right)=\{x:x\in\mathbb{R}^n,|x|<r\}$. Hence, there exists a subsequence of $\{v_m\left(t\right)\}_{m=1}^{\infty}$ which is convergent to $v\left(t\right)$ almost everywhere in $B(0,r)$ for all $r>0$. Without loss of generality, denote this subsequence again by $\{v_m\left(t\right)\}_{m=1}^{\infty}$. Next, since $f\in C^1\left(\mathbb{R}\right)$, it is obtained that

$$\widehat{f}(v_m(t,x))v_m(t,x)\to\widehat{f}(v(t,x))v(t,x) \text{ a.e. in } B(0,r) \text{ for all } r>0.$$

Moreover, by (2.3), the sequence $\left\{\widehat{f}(v_m\left(t\right))v_m\left(t\right)\right\}_{k=1}^{\infty}$ is bounded in $L^{\frac{2p}{p+1}}(\mathbb{R}^n)$. Hence, since $r>0$ is arbitrary, it is inferred that

$$\widehat{f}(v_m\left(t\right))v_m\left(t\right)\to\widehat{f}(v\left(t\right))v\left(t\right) \text{ weakly in } L^{\frac{2p}{p+1}}\left(\mathbb{R}^n\right). \tag{3.2}$$

Therefore, considering (3.2) and the conditions of the lemma, passing to limit in (3.1), for all $t\in[0,T]$, it is deduced that

$$\limsup_{m\to\infty}\limsup_{l\to\infty}\int\limits_{\mathbb{R}^n}\left(\widehat{f}\left(v_l\left(t,x\right)\right)-\widehat{f}\left(v_m\left(t,x\right)\right)\right)\left(v_m\left(t,x\right)-v_l\left(t,x\right)\right)dx$$

$$=-\liminf_{m\to\infty}\liminf_{l\to\infty}\int\limits_{\mathbb{R}^n}\left(\widehat{f}\left(v_m\left(t,x\right)\right)v_m\left(t,x\right)+\widehat{f}\left(v_l\left(t,x\right)\right)v_l\left(t,x\right)-2\widehat{f}\left(v\right)v\left(t,x\right)\right)dx\leq 0.$$

□

Now, let us prove the following theorem on the asymptotic compactness of $\{S(t)\}_{t\geq 0}$ in $H^2\left(\mathbb{R}^n\right)\times L^2\left(\mathbb{R}^n\right)$, which plays a key role in the existence of the global attractor.

Theorem 3.1. *Assume that the conditions (2.3)-(2.4) hold and* B *is a bounded subset of* $H^2\left(\mathbb{R}^n\right)\times L^2\left(\mathbb{R}^n\right)$. *Then, every sequence of the form* $\{S(t_k)\varphi_k\}_{k=1}^{\infty}$, *where* $\{\varphi_k\}_{k=1}^{\infty}\subset B$, $t_k\to\infty$, *has a convergent subsequence in* $H^2\left(\mathbb{R}^n\right)\times L^2\left(\mathbb{R}^n\right)$.

Proof. Firstly, by (2.3) and (2.4), there holds

$$\sup_{t\geq 0}\sup_{\varphi\in B}\|S(t)\varphi\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}<\infty. \tag{3.3}$$

Since $\{\varphi_k\}_{k=1}^{\infty}$ is bounded in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$, by (3.3), the sequence $\{S(.)\varphi_k\}_{k=1}^{\infty}$ is bounded in $C_b\left(0,\infty;H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\right)$, where $C_b\left(0,\infty;H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\right)$ is the space of continuously bounded functions from $[0,\infty)$ to $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$. Then for any $T_0\geq 0$ there exists a subsequence $\{k_m\}_{m=1}^{\infty}$ such that $t_{k_m}\geq T_0$, and

$$\begin{cases} v_m\to v \text{ weakly star in } L^{\infty}\left(0,\infty;H^2(\mathbb{R}^n)\right), \\ v_{mt}\to v_t \text{ weakly star in } L^{\infty}\left(0,\infty;L^2(\mathbb{R}^n)\right), \end{cases} \tag{3.4}$$

for some $v\in L^{\infty}\left(0,\infty;H^2(\mathbb{R}^n)\right)\cap W^{1,\infty}\left(0,\infty;L^2(\mathbb{R}^n)\right)$, where $(v_m(t),v_{mt}(t))=S(t+t_{k_m}-T_0)\varphi_{k_m}$. By (2.1), also it is obtained that

$$v_{mtt}(t,x)-v_{ltt}(t,x)+\Delta^2(v_m(t,x)-v_l(t,x))+2\Delta(v_m(t,x)-v_l(t,x))$$
$$+(v_{mt}(t,x)-v_{lt}(t,x))+f(v_m)-f(v_l)=0. \tag{3.5}$$

It is worth mentioning that we establish the following estimates for the smooth solutions of (2.1)-(2.2) with the initial data in $H^4(\mathbb{R}^n)\times H^2(\mathbb{R}^n)$. By using the standard density arguments, these estimates can be extended to the weak solutions with the initial data in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$.
Considering (2.3)-(2.4) in (2.8), it is inferred that

$$\int_0^T\|v_{mt}(t)\|^2_{L^2(\mathbb{R}^n)}\,dt\leq c_1,\ \forall T\geq 0. \tag{3.6}$$

Multiplying (3.5) by v_m-v_l, integrating over $(0,T)\times\mathbb{R}^n$ and using (2.10), it is obtained that

$$\int_0^T\left(\|\Delta v_m(t)-\Delta v_l(t)\|^2_{L^2(\mathbb{R}^n)}+\|v_m(t)-v_l(t)\|^2_{L^2(\mathbb{R}^n)}\right)dt$$
$$\leq c_2\int_0^T\int_{\mathbb{R}^n}\left(\widehat{f}(v_m)-\widehat{f}(v_l)\right)(v_m(t,x)-v_l(t,x))\,dxdt$$
$$+c_2\int_0^T\|v_{mt}(t)-v_{lt}(t)\|^2_{L^2(\mathbb{R}^n)}\,dt-c_2\left(\int_{\mathbb{R}^n}(v_{mt}(t,x)\,v_m(t,x)-v_{lt}(t,x)\,v_l(t,x))\,dx\right)\Bigg|_0^T$$
$$-\frac{c_2}{2}\left(\int_{\mathbb{R}^n}(v_m(t,x)-v_l(t,x))^2\,dx\right)\Bigg|_0^T. \tag{3.7}$$

Taking into account (3.3), (3.6) and Lemma 3.1 in (3.7), there holds

$$\limsup_{m\to\infty}\limsup_{l\to\infty}\int_0^T\left(\|\Delta v_m(t)-\Delta v_l(t)\|^2_{L^2(\mathbb{R}^n)}+\|v_m(t)-v_l(t)\|^2_{L^2(\mathbb{R}^n)}\right)dt\leq c_3,\ \forall T\geq 0. \tag{3.8}$$

By using (3.6) and (3.8), it is obtained that

$$\limsup_{m\to\infty}\limsup_{l\to\infty}\int_0^T E\left(v_m\left(t\right)-v_l\left(t\right)\right)dt\leq c_4,\ \forall T\geq 0. \tag{3.9}$$

where $E\left(u\left(t\right)\right)=\|\Delta u(t)\|^2_{L^2(\mathbb{R}^n)}+\|u(t)\|^2_{L^2(\mathbb{R}^n)}+\|u_t\left(t\right)\|^2_{L^2(\mathbb{R}^n)}$ Now, testing (3.5) by $2t\left(v_{mt}-v_{lt}\right)+\varepsilon t\left(v_m-v_l\right)$, recalling Lemma 3.1 and applying Young inequality, for sufficiently small $\varepsilon>0$, it is inferred that

$$TE\left(v_m\left(T\right)-v_l\left(T\right)\right)\leq c_5\kappa_{m,l}\left(T\right)+c_5\int_0^T E\left(v_m\left(t\right)-v_l\left(t\right)\right)dt$$

$$+c_5\int_0^T t\left(\|v_{mt}\left(t\right)\|^2_{L^2(\mathbb{R}^n)}+\|v_{lt}\left(t\right)\|^2_{L^2(\mathbb{R}^n)}\right)E\left(v_m\left(t\right)-v_l\left(t\right)\right)dt,$$

where $\kappa_{m,l}\left(T\right)=2T\int_{\mathbb{R}^n}\left(\widehat{f}\left(v_m\left(T,x\right)\right)-\widehat{f}\left(v_l\left(T,x\right)\right)\right)\left(v_m\left(T,x\right)-v_l\left(T,x\right)\right)dx$. Hence, applying Gronwall inequality in the last estimate, there holds

$$TE\left(v_m\left(T\right)-v_l\left(T\right)\right)$$

$$\leq c_6\left(\kappa_{m,l}\left(T\right)+\int_0^T E\left(v_m\left(t\right)-v_l\left(t\right)\right)dt\right)e^{c_5\int_0^T\left(\|v_{mt}(t)\|^2_{L^2(\mathbb{R}^n)}+\|v_{lt}(t)\|^2_{L^2(\mathbb{R}^n)}\right)dt}. \tag{3.10}$$

Therefore, by using (3.6), (3.9) and Lemma 3.1 in (3.10), it is obtained that

$$\limsup_{m\to\infty}\limsup_{l\to\infty}E\left(v_m\left(T\right)-v_l\left(T\right)\right)\leq\frac{c_7}{T}.$$

Recalling the definition of the $E\left(v_m\left(T\right)-v_l\left(T\right)\right)$, from the last estimate, it follows that

$$\limsup_{m\to\infty}\limsup_{l\to\infty}\ \|v_m(T)-v_l(T)\|^2_{H^2(\mathbb{R}^n)}+\|v_{mt}\left(T\right)-v_{lt}\left(T\right)\|^2_{L^2(\mathbb{R}^n)}\leq\frac{c_8}{T},\ \ \forall T\geq 0,$$

which means

$$\limsup_{m\to\infty}\limsup_{l\to\infty}\|S(T+t_{k_m}-T_0)\varphi_{k_m}-S(t+t_{k_l}-T_0)\varphi_{k_l}\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}\leq\frac{c_9}{\sqrt{T}},\ \forall T>0.$$

Picking $T=T_0$, it is deduced that

$$\limsup_{m\to\infty}\limsup_{l\to\infty}\|S(t_{k_m})\varphi_{k_m}-S(t_{k_l})\varphi_{k_l}\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}\leq\frac{c_9}{\sqrt{T_0}},\ \forall T_0\geq 1.$$

Finally, from the previous sequential limit estimate, it is observed that

$$\liminf_{m\to\infty}\liminf_{l\to\infty}\|S(t_k)\varphi_k-S(t_m)\varphi_m\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}=0.$$

Hence, with the help of the argument at the end of the proof of [12, Lemma 3.4], the asymptotic compactness of $\{S(t)\}_{t\geq 0}$ in $H^2\left(\mathbb{R}^n\right)\times L^2\left(\mathbb{R}^n\right)$ is obtained. □

Now, the existence of the global attractor can be proved. Recalling (2.8), it is obtained that the $\left(S\left(t\right),H^2\left(\mathbb{R}^n\right)\times L^2\left(\mathbb{R}^n\right)\right)$ is a gradient system because the problem (2.1)-(2.2) possesses a strict Lyapunov function

$$\Lambda\left(u\left(t\right)\right)=L\left(u\left(t\right)\right)-\int_{\mathbb{R}^n}h\left(x\right)u\left(t,x\right)dx,\ \forall t\in\mathbb{R}.$$

Consequently, using [3, Corollary 7.5.7], Theorem 2.2 is established.

4. Regularity of the global attractor

Theorem 4.1. *Under the assumptions of Theorem 2.2, the global attractor $\mathcal{A}$ for the problem (2.1)-(2.2) is bounded in $H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$.*

Proof. Let $(u_0, u_1) \in \mathcal{A}$. By the invariance of $\mathcal{A}$, it follows that (see [1, p. 159]) there exists an invariant trajectory $\gamma = \{(u(t), u_t(t)) : t \in \mathbb{R}\} \subset \mathcal{A}$ such that $(u(0), u_t(0)) = (u_0, u_1)$. By an invariant trajectory we mean a curve $\gamma = \{(u(t), u_t(t)) : t \in \mathbb{R}\}$ such that $S(t)(u(\tau), u_t(\tau)) = (u(t+\tau), u_t(t+\tau))$ for all $t \geq 0$ and $\tau \in \mathbb{R}$ (see [1, p. 157]). Let $(u_0, u_1) \in B$ and $S(t)(u_0, u_1) := (u(t), u_t(t))$. Defining

$$v(t,x) := \frac{u(t+\tau, x) - u(t,x)}{\tau},\ \tau > 0,$$

by (2.1), we get

$$\begin{aligned} &v_{tt}(t,x) + \Delta^2 v(t,x) + 2\Delta v(t,x) + v_t(t,x) \\ &\frac{f(u(t+\tau,x)) - f(u(t,x))}{\tau} = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n. \end{aligned} \tag{4.1}$$

Using the definition of v, from (2.7), it follows that

$$\|v(s)\|_{L^2(\mathbb{R}^n)} = \left\| \frac{u(s+\tau,x) - u(s,x)}{\tau} \right\|_{L^2(\mathbb{R}^n)} \leq \sup_{0 \leq s < \infty} \|u_t(s)\|_{L^2(\mathbb{R}^n)} < \widehat{C}. \tag{4.2}$$

Then, testing (4.1) by $2v_t + \beta v$, exploiting (2.4), (4.2) and applying Young inequality, for sufficiently small β, it is deduced that

$$\begin{aligned} &\frac{d}{dt}(\Psi(v(t))) + c_1 E(v(t)) \\ &\leq c_2 + c_2 \left(\|u_t(t+\tau)\|^2_{L^2(\mathbb{R}^n)} + \|u_t(t)\|^2_{L^2(\mathbb{R}^n)} \right) E(v(t)) \end{aligned} \tag{4.3}$$

where $c_1 > 0$ and

$$\Psi(t)) = \|v_t(t)\|^2_{L^2(\mathbb{R}^n)} + \|\Delta(v(t))\|^2_{L^2(\mathbb{R}^n)} - 2\|\nabla v(t)\|^2_{L^2(\mathbb{R}^n)} + \left(1 + \delta + \frac{\beta}{2}\right) \|v(t)\|^2_{L^2(\mathbb{R}^n)}$$

$$+ \int_{\mathbb{R}^n} \int_0^1 \widehat{f}'(\sigma u(t+\tau,x) + (1-\sigma) u(t,x)) d\sigma (v(t,x))^2 dx + \beta \int_{\mathbb{R}^n} v_t(t,x) v(t,x) dx.$$

It is easy to see that, for small enough β there exist constants $c > 0$, $\widetilde{c} > 0$ and $M > 0$ such that

$$c_3 \Psi(t)) - M \leq E(v(t)) \leq \widetilde{c}_3 \Psi(t)) + M. \tag{4.4}$$

Then, taking into account (4.4) in (4.3), there holds

$$\frac{d}{dt}(\Psi(t)) + c_4 \Psi(t) \leq c_5 + c_5 \left(\|u_t(t+\tau)\|^2_{L^2(\mathbb{R}^n)} + \|u_t(t)\|^2_{L^2(\mathbb{R}^n)} \right) \Psi(t)$$

which entails

$$\Psi(t) \leq c_6 + c_5 \int_0^t \left(\|u_t(s+\tau)\|^2_{L^2(\mathbb{R}^n)} + \|u_t(s)\|^2_{L^2(\mathbb{R}^n)} \right) \Psi(s)) ds,\ \forall t \geq 0.$$

Hence, applying Gronwall inequality in the last estimate, it is obtained that

$$\Psi(t)) \leq c_6 e^{c_7 \int_0^\infty \|u_t(s)\|^2_{L^2(\mathbb{R}^n)} ds}, \quad \forall t \geq 0, \tag{4.5}$$

On the other hand, recalling (2.8), there holds the estimate

$$\int_0^\infty \|u_t(s)\|^2_{L^2(\mathbb{R}^n)}\, ds \leq \widehat{c}.$$

Therefore, taking into account the last estimate in (4.5) and exploiting once more (4.4), it is inferred

$$E\left(v\left(t\right)\right) \leq c_8, \quad \forall t \geq 0. \tag{4.6}$$

Hence, by passing to limit as $\tau \to 0$ in (4.6) and recalling the definition of v, it is eventually obtained that

$$E\left(u\left(t\right)\right) \leq c_9, \quad \forall t \geq 0.$$

Considering the last one in (2.1), it can be seen that

$$\|u\left(t\right)\|_{H^4(\mathbb{R}^n)} \leq c_{10}, \quad \forall t \geq 0.$$

Consequently, last two estimates amount to proving that

$$\|\left(u\left(t\right), u_t\left(t\right)\right)\|_{H^4(\mathbb{R}^n)\times H^2(\mathbb{R}^n)} \leq C,$$

for some constant $C > 0$, and the proof is complete. □

5. Finite dimensionality of the global attractor

In this section, the idea of the [11] will be used to obtain the finite dimensionality.

Theorem 5.1. *Under the assumptions of Theorem 2.2, the global attractor $\mathcal{A}$ for the problem (2.1)-(2.2) has a finite fractal dimension.*

Proof. Let $\{T\left(t,s\right)\}_{t\geq s}$ be the process generated by the problem

$$\begin{cases} \frac{d}{dt}\left(w\left(t\right), w_t\left(t\right)\right) = A\left(w\left(t\right), w_t\left(t\right)\right), & t \geq s, \\ \left(w\left(s\right), w_t\left(s\right)\right) = \left(w_0, w_1\right), & s \geq 0 \end{cases}$$

in $H^2\left(\mathbb{R}^n\right) \times L^2\left(\mathbb{R}^n\right)$. Then, one can readily see that, there exist $M > 1$ and $\gamma > 0$ such that

$$\|T\left(t,s\right)\|_{L\left(H^{2(1+i)}(\mathbb{R}^n)\times H^{2i}(\mathbb{R}^n)\right)} \leq Me^{-\gamma(t-s)}, \forall t \geq s \tag{5.1}$$

where $i \in [-1, 1]$ and $L\left(X\right)$ is the space of linear bounded operators in X.
Now, assume that θ_1, $\theta_2 \in \mathcal{A}$ and $\left(u\left(t\right), u_t\left(t\right)\right) = S\left(t\right)\theta_1$, $\left(v\left(t\right), v_t\left(t\right)\right) = S\left(t\right)\theta_2$. Define $z\left(t\right) := v\left(t\right) - u\left(t\right)$. Then, it is found that

$$z_{tt}(t,x)+\Delta^2 z(t,x)+2\Delta z(t,x)+z_t(t,x)+z\left(t,x\right)+\int_0^1 \widehat{f}'\left(\sigma v\left(t,x\right)+\left(1-\sigma\right)u\left(t,x\right)\right)z\left(t,x\right) = 0. \tag{5.2}$$

Hence, by the variation of parameters formula,

$$\left(z\left(t\right), z_t\left(t\right)\right) = T\left(t,0\right)\left(z\left(0\right), z_t\left(0\right)\right) + \int_0^t T\left(t,s\right)F\left(s\right)ds, \tag{5.3}$$

where $F\left(t\right) = \left(0, -\int_0^1 \widehat{f}'\left(\sigma v\left(t,x\right)+\left(1-\sigma\right)u\left(t,x\right)\right)z\left(t,x\right)\right)$. Considering (5.1) in (5.3), there holds the estimate

$$\|\left(z\left(t\right), z_t\left(t\right)\right)\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \leq Me^{-\gamma t}\|\theta_2-\theta_1\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}$$

$$+\int_0^t M_1 e^{-\gamma(t-s)} \|(z(s), z_t(s))\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} ds. \tag{5.4}$$

Applying Gronwal lemma, from (5.4) it follows that

$$\|S(t)\theta_2 - S(t)\theta_1\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \le M e^{(M_1-\gamma)t} \|\theta_2-\theta_1\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}. \tag{5.5}$$

On the other hand, since the attractor $\mathcal{A}$ is bounded in $H^4(\mathbb{R}^n)\times H^2(\mathbb{R}^n)$, in the case $n>8$, from (5.3) it follows that

$$\|(z(t), z_t(t))\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \le M e^{-\gamma t}\|\theta_2-\theta_1\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}$$

$$+\int_0^t M_2 e^{-\gamma(t-s)} \|(z(s), z_t(s))\|_{H^{\frac{2(n-8)}{n-4}}(\mathbb{R}^n)\times H^{\frac{-8}{n-4}}(\mathbb{R}^n)} ds. \tag{5.6}$$

It is easy to get the desired estimates by using embedding theorems when $n\le 8$.

Now, the last term on the right-hand side of (5.6) will be evaluated. Using (5.1) in (5.3) once more, it is obtained that

$$\int_0^t e^{-\gamma(t-s)} \|(z(s), z_t(s))\|_{H^{\frac{2(n-8)}{n-4}}(\mathbb{R}^n)\times H^{\frac{-8}{n-4}}(\mathbb{R}^n)} ds \le M t e^{-\gamma t}\|\theta_2-\theta_1\|_{H^{\frac{2(n-8)}{n-4}}(\mathbb{R}^n)\times H^{\frac{-8}{n-4}}(\mathbb{R}^n)}$$

$$+ M_3\int_0^t\int_0^s e^{-\gamma(t-\tau)}\|(z(\tau), z_t(\tau))\|_{L^2(\mathbb{R}^n)\times H^{-2}(\mathbb{R}^n)} d\tau ds. \tag{5.7}$$

Considering (5.7) in (5.6), it is deduced that

$$\|(z(t), z_t(t))\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} \le M\left(e^{-\gamma t}+te^{-\gamma t}\right)\|\theta_2-\theta_1\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}$$

$$+M_4\int_0^t\int_0^s e^{-\gamma(t-\tau)}\|(z(\tau), z_t)(\tau)\|_{L^2(\mathbb{R}^n)\times H^{-2}(\mathbb{R}^n)} d\tau ds$$

$$\le M\left(e^{-\gamma t}+te^{-\gamma t}\right)\|\theta_2-\theta_1\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)} + \frac{1}{\gamma^2}\sup_{0\le\tau\le t}\|(z(\tau), z_t)(\tau)\|_{L^2(B(0,r))\times H^{-2}(B(0,r))}$$

$$+c_1\int_0^t\int_0^s e^{-\gamma(t-\tau)}\|(z(\tau), z_t)(\tau)\|_{L^2(\mathbb{R}^n\backslash B(0,r))\times H^{-2}(\mathbb{R}^n\backslash B(0,r))} d\tau ds, \tag{5.8}$$

for all $t\ge 0$ and $r>0$. Now, the last term on the right-hand side of (5.8) will be estimated. Let $\eta\in C^\infty(\mathbb{R}^n)$, $0\le\eta(x)\le 1$, $\eta(x)=\begin{cases}0, & |x|\le 1\\ 1, & |x|\ge 2\end{cases}$ and $\eta_r(x)=\eta\left(\frac{x}{r}\right)$. Multiplying (5.2) by η_r and denoting $z_r(t)=\eta_r z(t)$, it is obtained that

$$z_{rtt}(t,x)+\Delta^2 z_r(t,x)+2\Delta z_r(t,x)+z_{rt}(t,x)+z_r(t,x)$$

$$+\int_0^1 \widehat{f'}(\sigma v(t,x)+(1-\sigma)u(t,x))\, d\sigma z_r(t,x) = f_1(t),$$

where

$$f_1(t)=\Delta^2\eta_r z+2\Delta\eta_r\Delta z+2\sum_{i=1}^n(\Delta\eta_r)_{x_i} z_{x_i}+2\sum_{i=1}^n(\eta_r)_{x_i}\Delta z_{x_i}$$

$$+4\sum_{i,j=1}^{n}(\eta_r)_{x_ix_j}z_{x_ix_j}+2\Delta\eta_r z+4\sum_{i=1}^{n}(\eta_r)_{x_i}z_{x_i}.$$

Then, by the variation of parameters formula,

$$(z_r(t),z_{rt}(t))=T(t,0)(z_r(0),z_{rt}(0))+\int_0^t T(t,\tau)G_r(\tau)\,d\tau, \tag{5.9}$$

where

$$G_r(t):=\left(0,-\int_0^1\widehat{f}'(\sigma v(t,x)+(1-\sigma)u(t,x))\,d\sigma z_r(t,x)+f_1(t)\right).$$

Hence, applying (5.1) to (5.9) and recalling (5.5), it is inferred that

$$\|(z_r(t),z_{rt}(t))\|_{L^2(\mathbb{R}^n)\times H^{-2}(\mathbb{R}^n)}\le Me^{-\gamma t}\|\theta_2-\theta_1\|_{L^2(\mathbb{R}^n)\times H^{-2}(\mathbb{R}^n)}$$

$$+\widetilde{c}_2e^{-\gamma t}\int_0^t e^{\gamma\tau}\left(\|v(\tau)\|_{H^2(\mathbb{R}^n\backslash B(0,r))}+\|u(\tau)\|_{H^2(\mathbb{R}^n\backslash B(0,r))}\right)\|z(\tau)\|_{H^2(\mathbb{R}^n)}\,d\tau$$

$$+\frac{\widetilde{c}_2}{r}e^{-\gamma t}\int_0^t e^{\gamma\tau}\|z(\tau)\|_{H^2(\mathbb{R}^n)}\,d\tau\le\widetilde{c}_3\left(e^{-\gamma t}+\Upsilon_re^{(M_1-\gamma)t}\right)\|\theta_2-\theta_1\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)},$$

for all $t\ge 0$ and $r\ge 1$, where

$$\Upsilon_r:=\sup_{t\ge 0}\left(\|u(t)\|_{H^2(\mathbb{R}^n\backslash B(0,r))}+\|v(t)\|_{H^2(\mathbb{R}^n\backslash B(0,r))}\right)+\frac{1}{r}.$$

Therefore, from the last estimate, it is obtained that

$$\int_0^t\int_0^s e^{-\gamma(t-\tau)}\|(z(\tau),z_t)(\tau)\|_{L^2(\mathbb{R}^n\backslash B(0,r))\times H^{-2}(\mathbb{R}^n\backslash B(0,r))}\,d\tau ds$$

$$\le\widetilde{c}_4\left(e^{-\gamma t}+\Upsilon_re^{(M_1-\gamma)t}\right)t^2\|\theta_2-\theta_1\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)},\qquad\forall t\ge 0\text{ and }\forall r\ge 1. \tag{5.10}$$

Hence, considering (5.10) in (5.8), it is deduced that

$$\|S(t)\theta_2-S(t)\theta_1\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}$$

$$\le\widetilde{c}_5\left(e^{-\gamma t}\left(1+t+t^2\right)+\Upsilon_re^{(M_1-\gamma)t}t^2\right)\|\theta_2-\theta_1\|_{H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)}$$

$$+\widetilde{c}_5\sup_{0\le\tau\le t}\|S(\tau)\theta_2-S(\tau)\theta_1\|_{H^2(B(0,r))\times L^2(B(0,r))},\qquad\forall t\ge 0\text{ and }\forall r\ge 1. \tag{5.11}$$

Since the global attractor $\mathcal{A}$ is compact in $H^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)$, as r goes to infinity Υ_r goes to zero uniformly with respect to the trajectories emanating from $\mathcal{A}$. Hence, (5.5) and (5.11) concludes the proof of the theorem by applying [3, Theorem 7.9.6]. □

Conclusion 5.1. *In this study, the long-time dynamics of the hyperbolic version of the well-known Swift-Hohenberg equation, which plays the central role in pattern formation, is investigated. The existence of the smooth global attractor, which gives the information about the pattern stability, is proved. Furthermore, the finite dimensionality of the global attractor is also obtained. Physically, it means that one can describe the corresponding infinite dimensional dynamical system by a finite number of degrees of freedom.*

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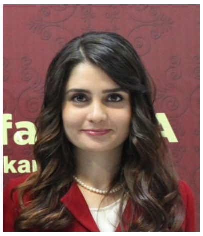

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