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A FIXED POINT PROBLEM VIA SIMULATION FUNCTIONS IN INCOMPLETE METRIC SPACES WITH ITS APPLICATION

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Abstract. In this paper, firstly, we review the notion of the SO-complete metric spaces. This notion let us to consider some fixed point theorems for single-valued mappings in incomplete metric spaces. Secondly, as motivated by the recent work of A.H. Ansari et al. [J. Fixed Point Theory Appl. (**2017**), 1145–1163], we obtain that an existence and uniqueness result for the following problem: finding $x \in X$ such that $x = Tx$, $Ax \mathrm{R}_1 Bx$ and $Cx \mathrm{R}_2 Dx$, where (X, d) is an incomplete metric space equipped with the two binary relations R_1 and R_2, $A, B, C, D : X \to X$ are discontinuous mappings and $T : X \to X$ satisfies in a new contractive condition. This result is a real generalization of main theorem of A.H. Ansari's. Finally, we provide some examples for our results and as an application, we find that the solutions of a differential equation.

Keywords: Fixed point, Constraint inequalities, $\perp$-$\mathcal{Z}$-contraction, SO-complete metric space, Fractional differential equation.

AMS Subject Classification: 37C25, 47A45

1. Introduction

The Banach contraction Theorem is the basis of the theory of metric fixed points which is used in many practical problems [1, 3, 11, 13, 15, 16]. In recent decades, theorem conditions dropped by a large number of researchers(see [4, 9, 12, 14]). Among them, in 2015, F. Khojasteh et al. introduced in [12] the concept of simulation function in order to express different contractive in a simple, unified way. Thus, it is possible to tread some fixed point theorems from a unique, common point of view. However, in [14], the authors slightly modified the definition of simulation function and enlarged the family of all simulation functions as follows.

Definition 1.1. [14] Let $\zeta : [0, \infty) \times [0, \infty) \longrightarrow \mathbb{R}$ be a mapping. Then ζ is called a simulation function if it satisfies the following conditions:

(ζ_1) $\zeta(0, 0) = 0$;

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(ζ_2) $\zeta(t,s) < s-t$ for all $t,s>0$;
(ζ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Let $\mathcal{Z}$ be the family of all simulation functions ζ in Definition 1.1.

Example 1.1. Let $\tau \in (0,\infty)$ and $\zeta : [0,\infty)\times[0,\infty) \longrightarrow \mathbb{R}$ be a function as follow:

$$\zeta(t,s) = \begin{cases} (t-2)s & 0 \le t \le s < 1 \\ (s-2)t & 0 \le s \le t < 1 \\ s-t-\tau & t,s \ge 1. \end{cases}$$

Clearly, ζ is a simulation function.

Recently, Jleli and Samet [10] provided conditions for finding $x \in X$ such that

$$\begin{cases} x = Tx, \\ Ax \preceq_1 Bx, \\ Cx \preceq_2 Dx, \end{cases} \tag{1}$$

where X is complete matric space, $T, A, B, C, D : X \to X$ and " $\preceq_1$ " and " $\preceq_2$ " are partial orders. Ansari, Kumam and Samet in [2] proved that this problem has a unique solution without continuity of C and D.

Definition 1.2. [10] Let (X,d) be a metric space. A partial order "$\preceq$ " on X is d-regular if for any two sequences $\{u_n\}$ and $\{v_n\}$ in X, we have

$$\lim_{n\to\infty} d(u_n,u) = \lim_{n\to\infty} d(v_n,v) = 0, u_n \preceq v_n \ \textit{for all } n \Longrightarrow u \preceq v, \ \ (u,v) \in X \times X.$$

Definition 1.3. [10] Let " $\preceq_1$ " and " $\preceq_2$ " be two partial orders on X and operators $T, A, B, C, D : X \to X$ be given. The operator T is called $(A, B, C, D, \preceq_1, \preceq_2)$-stable if $x \in X$, $Ax \preceq_1 Bx \implies CTx \preceq_2 DTx$.

Let Φ be the set of all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that φ is a lower semi-continuous function and $\varphi^{-1}(\{0\}) = \{0\}$.

The main theorem presented in [2] is given by the following result.

Theorem 1.1. *Let (X,d) be a complete metric space endowed with two partial orders " $\preceq_1$ " and " $\preceq_2$ ". Let operators $T, A, B, C, D : X \to X$ be given. Suppose that the following conditions are satisfied:*

(i) " $\preceq_i$ " *is d-regular, $i = 1, 2$;*
(ii) *A, B are continuous;*
(iii) *there exists $x_0 \in X$ such that $Ax_0 \preceq_1 Bx_0$;*
(iv) *T is $(A, B, C, D, \preceq_1, \preceq_2)$-stable;*
(v) *T is $(C, D, A, B, \preceq_2, \preceq_1)$-stable;*
(vi) *there exists $\varphi \in \Phi$ such that*

$$Ax \preceq_1 Bx, Cy \preceq_2 Dy \Longrightarrow d(Tx,Ty) \le d(x,y) - \varphi(d(x,y)).$$

Then the sequence $\{T^n x_0\}$ converges to some $x^* \in X$, which is a solution to (1). *Moreover, x^* is the unique solution to* (1).

In this paper, we explain sufficient conditions for the existence and uniqueness of a fixed point of T satisfying the two constraint inequalities: $Ax \ \mathrm{R}_1 \ Bx$ and $Cx \ \mathrm{R}_2 \ Dx$, where $T : X \to X$ defined on an incomplete metric space equipped with two binary relations(not necessarily two partial orders) "R_1" and "R_2" and $A, B, C, D : X \to X$ are non necessary continuous self-operators. That is, this problem contains: finding $x \in X$ such that

$$\begin{cases} x = Tx, \\ Ax \ \mathrm{R}_1 \ Bx, \\ Cx \ \mathrm{R}_2 \ Dx. \end{cases} \tag{2}$$

Also, we introduct the notation of $\perp$-$\mathcal{Z}$-contraction and give a real generalization of Banach fixed point theorem in incomplete metric spaces. In the rest of this section, we recall the notation of orthogonal set that first obtained in [8]. This notion let us to consider some fixed point theorems for single-valued mappings in incomplete metric spaces. For the depth of the subject, one can see [3, 5, 6, 7].

Definition 1.4. [5, 8] Let $X \neq \emptyset$, and $\perp \subseteq X \times X$ be a binary relation. If there exists x_0 such that $(\forall y, y \perp x_0)$ or $(\forall y, x_0 \perp y)$, then " $\perp$ " is called an orthogonality relation and pair $(X, \perp)$ an orthogonal set(briefly, O-set). We say that x_0 is an orthogonal element and elements $x, y \in X$ are $\perp$-comparable either $x \perp y$ or $y \perp x$. Let "d" be a metric on X, $(X, \perp, d)$ is called an orthogonal metric space.

Definition 1.5. [7] *Let $(X, \perp)$ be an O-set. A sequence $\{x_n\}$ is called a strongly orthogonal sequence(briefly, SO-sequence) if $(\forall n, k;\ x_n \perp x_{n+k})$ or $(\forall n, k;\ x_{n+k} \perp x_n)$.*

Definition 1.6. [7] *Let $(X, \perp, d)$ be an orthogonal metric space. X is called:*

1. *strongly orthogonal complete(briefly, SO-complete) if every Cauchy SO-sequence is convergent.*
2. *$\perp$-regular if for each SO-sequence $\{x_n\}$ with $x_n \to x$ for some $x \in X$, we conclude that $(\forall n;\ x_n \perp x)$ or $(\forall n;\ x \perp x_n)$.*

Definition 1.7. [7] *Let $(X, \perp, d)$ be an orthogonal metric space. A mapping $f : X \to X$ is strongly orthogonal continuous(briefly, SO-continuous) in $a \in X$ if for each SO-sequence $\{a_n\}$ in X, $a_n \to a$, then $f(a_n) \to f(a)$. Also, f is SO-continuous on X if f is SO-continuous in each $a \in X$.*

Definition 1.8. [5, 8] Let $(X, \perp)$ be an O-set. A mapping $T : X \to X$ is said to be $\perp$-preserving if $x \perp y$ implies $T(x) \perp T(y)$.

Definition 1.9. Let $(X, \perp, d)$ be an orthogonal metric space and $\zeta \in \mathcal{Z}$. Then a mapping $T : X \longrightarrow X$ is called a $\perp$-$\mathcal{Z}$-contraction with respect to ζ if the following condition is satisfied: $\zeta(d(Tx, Ty), d(x, y)) \geq 0$ for all $x, y \in X$ with $x \perp y$.

Example 1.2. Let $X = \mathbb{R}$, $d(x, y) = |x - y|$, for all $x, y \in X$. Define $x \perp y$ iff $xy \leq 0$ for all $x, y \in X$. Let $T : X \to X$ be a mapping defined by

$$T(x) = \begin{cases} \frac{x}{2} & x \geq 0 \\ -\frac{x}{2} & x < 0. \end{cases}$$

Then T is a $\perp$-$\mathcal{Z}$-contraction with respect to $\zeta(t, s) = \frac{1}{2}s - t$.

Definition 1.10. Let (X, d) be a metric space. An arbitrary binary relation(not necessarily partial order) "R" on X is dR-regular if for any two sequences $\{u_n\}$ and $\{v_n\}$ in X, we have

$$\lim_{n \to \infty} d(u_n, u) = \lim_{n \to \infty} d(v_n, v) = 0,\ u_n \ R \ v_n \ \textit{for all } n \Longrightarrow u \ R \ v,\ \ (u, v) \in X \times X.$$

Example 1.3. Let $X = \mathbb{R}$, $d(x,y) = |x-y|$, $x, y \in \mathbb{R}$. Define $x\,R\,y \Leftrightarrow y \leq 3x$ on X. Clearly, the binary relation(not partial order) "R" is a dR-regular.

Definition 1.11. Let "R_1" and "R_2" be two arbitrary binary relations on X and operators $T, A, B, C, D : X \to X$ be given. The operator T is called (A, B, C, D, R_1, R_2)-stable if $x \in X$, $Ax\ R_1\ Bx \implies CTx\ R_2\ DTx$.

Example 1.4. Let $X = \{(0,0), (1,1), (3,1), (3,2)\}$ and two binary relations(not necessarily partial orders) on X be defined by

$$(x,y)\ R_1\ (z,w)\ \Leftrightarrow\ yz > 1 \quad and \quad (x,y)\ R_2\ (z,w)\ \Leftrightarrow\ xw > 1.$$

Consider the operators $T, A, B, C, D : X \longrightarrow X$ as follow:

$$T(x,y) = (3,2),\ \ A(x,y) = (3,1),\ \ C(x,y) = (1,1),$$

$$B(0,0) = (1,1),\ \ B(1,1) = (3,1),\ \ B(3,1) = (3,2),\ \ B(3,2) = (3,1)$$

$$D(0,0) = (0,0),\ \ D(1,1) = (3,1),\ \ D(3,1) = (1,1),\ \ D(3,2) = (3,2).$$

If $A(x,y)\ R_1\ B(x,y)$, then $(x,y) \in \{(3,1), (3,2), (1,1)\}$, which yields $T(x,y) = (3,2)$. Therefore $CT(x,y) = C(3,2) = (1,1)\ R_2\ (3,2) = D(3,2) = DT(x,y)$.
Thus T is (A, B, C, D, R_1, R_2)-stable.

2. Fixed point problem via simulation functions

Theorem 2.1. *Let $(X, \perp, d)$ be a SO-complete(not necessarily complete) metric space with orthogonal element x_0. Let "R_1" and "R_2" be two binary relations over X and operators $T, A, B, C, D : X \to X$ be given. Suppose that the following conditions are satisfied:*

(i) *"R_i" is dR-regular, $i = 1, 2$ and T is $\perp$-preserving;*
(ii) *A, B are SO-continuous;*
(iii) *$Ax_0\ R_1\ Bx_0$ and X is $\perp$-regular;*
(iv) *T is (A, B, C, D, R_1, R_2)-stable;*
(v) *T is (C, D, A, B, R_2, R_1)-stable;*
(vi) *there exits $\zeta \in \mathcal{Z}$ such that for each $\perp$-comparable elements $x, y \in X$*

$$(Ax\ \mathrm{R}_1\ Bx \text{ and } Cy\ \mathrm{R}_2\ Dy) \Longrightarrow \zeta(d(Tx, Ty), d(x,y)) \geq 0.$$

Then the sequence $\{T^n x_0\}$ converges to some $x^* \in X$, which is a solution to (2)*. Moreover, x^* is the unique solution of* (2)*.*

Proof. Consider the sequence $\{x_n\}$ defined by $x_n = T^n x_0$, $n = 0, 1, 2, \cdots$. From the definition of orthogonal element x_0, we have

$$(\forall n \in \mathbb{N},\ \ x_0 \perp T^n x_0 = x_n) \text{ or } (\forall n \in \mathbb{N},\ \ x_n = T^n x_0 \perp x_0).$$

Also, since T is $\perp$-preserving, we have

$$(\forall n, k \in \mathbb{N},\ x_n = T^n x_0 \perp T^{n+k} x_0 = x_{n+k}) \text{ or } (\forall n, k \in \mathbb{N},\ x_{n+k} = T^{n+k} x_0 \perp T^n x_0 = x_n).$$

Therefore $\{x_n\}$ is a SO-sequence.
On the other hand, since T is (A, B, C, D, R_1, R_2)-stable and (C, D, A, B, R_2, R_1)-stable, applying (iii), we have

$$Ax_{2n}\ R_1\ Bx_{2n} \text{ and } Cx_{2n+1}\ R_2\ Dx_{2n+1},\ \ n = 0, 1, 2, \cdots. \tag{3}$$

By settiing $a_n = d(x_n, x_{n+1})$, $n = 0, 1, 2, \cdots$, we have the following results:
(1) If there exists n_0 such that $a_{n_0} = 0$, then $Tx_{n_0} = x_{n_0}$, and the proof is finished.
(2) If for all n, $a_n \neq 0$, since $\{x_n\}$ is SO-sequence, applying (3), (vi) and symmetry for all $n \in \mathbb{N}$, we have $\zeta(d(Tx_n, Tx_{n-1}), d(x_n, x_{n-1})) \geq 0$.

Applying (ζ_2), we deduce that

$$d(x_{n+1}, x_n) < d(x_n, x_{n-1}) \quad n = 1, 2, \cdots . \tag{4}$$

Therefor there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$. Let $r > 0$. Since $x_n = T^n x_0$, applying (3), (4) and (ζ_3), we have

$$0 \leq \limsup_{n\to\infty} \zeta(d(Tx_n, Tx_{n-1}), d(x_n, x_{n-1})) = \limsup_{n\to\infty} \zeta(d(x_{n+1}, x_n)), d(x_n, x_{n-1})) < 0.$$

This is a contradiction and so $r = 0$, that is

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{5}$$

We shall prove that $\{x_n\}$ is a Cauchy SO-sequence. Suppose that $\{x_n\}$ is not a Cauchy SO-sequence. Then, there exists some $\varepsilon > 0$ and two sequences of positive integers $\{m_k\}$ and $\{n_k\}$ such that, for all positive integers k, we have $n_k > m_k > k$, $d(x_{m_k}, x_{n_k}) \geq \varepsilon$ and $d(x_{m_k}, x_{n_k-1}) < \varepsilon$. Applying triangular inequality, we have

$$\varepsilon \leq d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k}) \leq \varepsilon + d(x_{n_k-1}, x_{n_k}).$$

Letting $k \to \infty$, we obtain

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \varepsilon. \tag{6}$$

Triangle inequality, implies that $|d(x_{m_k}, x_{n_k+1}) - d(x_{m_k}, x_{n_k})| \leq d(x_{n_k}, x_{n_k+1})$. Applying (6) and (5), as $k \to \infty$, we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k+1}) = \varepsilon. \tag{7}$$

Similarly,

$$\lim_{k\to\infty} d(x_{n_k+1}, x_{m_k+1}) = \varepsilon, \tag{8}$$

and so

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k-1}) = \varepsilon. \tag{9}$$

Obviously, for all k, there exists $i(k) \in \{0, 1\}$ such that $n_k - m_k + i(k) \equiv 1(2)$. Now, applying (3), for $k \in \mathbb{N}$, we conclude that

$$Ax_{n_k} \, R_1 \, Bx_{n_k} \text{ and } Cx_{m_k-i(k)} \; R_2 \; Dx_{m_k-i(k)},$$

or

$$Ax_{m_k-i(k)} \, R_1 \, Bx_{m_k-i(k)} \text{ and } Cx_{n_k} \, R_2 \, Dx_{n_k}.$$

Applying (iv), for all $k \in \mathbb{N}$, we deduce that

$$0 \leq \zeta(d(x_{n_k+1}, x_{m_k-i(k)+1}), d(x_{n_k}, x_{m_k-i(k)})). \tag{10}$$

Define $\Lambda = \{k \in \mathbb{N} : i(k) = 0\}$ and $\Delta = \{k \in \mathbb{N} : i(k) = 1\}$, and investigate two cases:

Cace1. $|\Lambda| = \infty$. Applying (10), for $k \in \Lambda$, we have

$$0 \leq \zeta(d(x_{n_k+1}, x_{m_k+1}), d(x_{n_k}, x_{m_k})).$$

Applying (6), (8) and (ζ_3), then $0 \leq \limsup_{k\to\infty} \zeta(d(x_{n_k+1}, x_{m_k+1}), d(x_{n_k}, x_{m_k})) < 0$. This is a contradiction. Hence $\varepsilon = 0$.

Cace2. $|\Lambda| < \infty$. Therefore, $|\Delta| = \infty$. Applying (10), we have

$$0 \leq \zeta(d(x_{n_k+1}, x_{m_k}), d(x_{n_k}, x_{m_k-1})).$$

Applying (7), (9) and (ζ_3), we deduce that

$$0 \leq \limsup_{k\to\infty} \zeta(d(x_{n_k+1}, x_{m_k}), d(x_{n_k}, x_{m_k-1})) < 0.$$

This is a contradiction. Thus $\varepsilon = 0$ and $\{x_n\}$ is a Cauchy SO-sequence in $(X, \perp, d)$. Since $(X, \perp, d)$ is SO-complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} d(x_n, x^*) = 0$.
Since $\{x_n\}$ is SO-sequence, we deduce that $\{x_{2n}\}$ is also SO-sequence. Applying the SO-continuity of A and B, we deduce that $\lim_{n\to\infty} d(Ax_{2n}, Ax^*) = \lim_{n\to\infty} d(Bx_{2n}, Bx^*) = 0$. Since "$R_1$" is dR-regular, (3) imply that

$$Ax^* \; R_1 \; Bx^*. \tag{11}$$

Since X is $\perp$-regular, then $x_{2n+1} \perp x^*$ or $x^* \perp x_{2n+1}$, for all $n \in \mathbb{N}$. Applying (3), (11) and (vi), we deduce that $\zeta(d(Tx^*, Tx_{2n+1}), d(x^*, x_{2n+1})) \geq 0$. If $d(Tx^*, x^*) > 0$, clearly, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we deduce that $d(Tx^*, Tx_{2n+1}) > 0$. Applying (ζ_2), for all $n \geq n_0$, we have

$$\begin{aligned} 0 \leq & \limsup_{n\to\infty} \zeta(d(Tx^*, Tx_{2n+1}), d(x^*, x_{2n+1})) \\ \leq & \limsup_{n\to\infty} [d(x^*, x_{2n+1}) - d(Tx^*, x_{2n+2})] = -d(Tx^*, x^*). \end{aligned}$$

This is a contradiction. Therefore $d(Tx^*, x^*) = 0$, that is

$$Tx^* = x^*. \tag{12}$$

Since T is (A, B, C, D, R_1, R_2)-stable, applying (11), we have $CTx^* \; R_2 \; DTx^*$. Therefore, (12) implies that

$$Cx^* \; R_2 \; Dx^*. \tag{13}$$

Applying (11), (12) and (13), we deduce that x^* is a solution of (2). We show that x^* is a unique solution. For this purpose, let $y^* \in X$ be another solution of (2), that is

$$Ty^* = y^*, \quad Ay^* \; R_1 \; By^*, \quad Cy^* \; R_2 \; Dy^*. \tag{14}$$

Since x_0 is an orthogonal element, then $x_0 \perp y^*$ or $y^* \perp x_0$. Since T is $\perp$-preserving, then

$$x_{2n} = T^{2n}x_0 \perp T^{2n}y^* = y^* \;\text{ or }\; y^* = T^{2n}y^* \perp T^{2n}x_0 = x_{2n}. \tag{15}$$

Applying (3), (14), (15) and (vi), for all $n \in \mathbb{N}$, $\zeta(d(Tx_{2n}, Ty^*), d(x_{2n}, y^*)) \geq 0$. Without loss of generality, let $d(x_n, y^*) > 0$ for all $n \in \mathbb{N}$. Therefore $d(x_{2n}, y^*) > 0$, and $d(Tx_{2n}, Ty^*) > 0$ for all $n \in \mathbb{N}$. Applying (ζ_2), we deduce that

$$0 \leq \zeta(d(x_{2n+1}, Ty^*), d(x_{2n}, y^*)) < d(x_{2n}, y^*) - d(x_{2n+1}, Ty^*).$$

Therefore $d(x_{2n+1}, Ty^*) < d(x_{2n}, y^*)$ for all $n \in \mathbb{N}$. Applying (ζ_3), we deduce that

$$0 \leq \limsup_{n\to\infty} \zeta(d(x_{2n+1}, Ty^*), d(x_{2n}, y^*)) < 0.$$

This is a contradiction, and so $d(x^*, y^*) = 0$. Therefore x^* is a unique solution of (2). □

3. Some consequences

Now, we consider some special cases, where in our result deduce several well-known fixed point theorems of the existing literature.
Setting $R_1 = R_2 = \leq$, $C = B$ and $D = A$ in Theorem 2.1, we get a generalization of Corollary 3.1 in [10].

Corollary 3.1. *Let $(X, \perp, d)$ be a SO-complete(not necessarily complete) metric space with orthogonal element x_0. Also, let operators $T, A, B : X \to X$ be given. Suppose that the following conditions are satisfied:*

(i) *T is $\perp$-preserving;*
(ii) *A, B are SO-continuous;*
(iii) *$Ax_0 \leq Bx_0$ and X is SO-regular;*

(iv) *for all* $x \in X$*, we have* $Ax \leq Bx \implies BTx \leq ATx$;
(v) *for all* $x \in X$*, we have* $Bx \leq Ax \implies ATx \leq BTx$;
(vi) *there exits* $\zeta \in \mathcal{Z}$ *such that for each* $\perp$*-comparable elements* $x, y \in X$

$$(Ax \leq Bx \text{ and } By \leq Ay) \Longrightarrow \zeta(d(Tx, Ty), d(x, y)) \geq 0.$$

Then the sequence $\{T^n x_0\}$ *converges to* $x^* \in X$ *satisfying* $Ax^* = Bx^*$*. Also, the point* $x^* \in X$ *is a unique solution to the problem* $\begin{cases} x = Tx, \\ Ax = Bx. \end{cases}$

Setting $A = D = I_X$ and $C = B$, we get a generalization of Corollary 3.2 in [10].

Corollary 3.2. *Let* $(X, \perp, d)$ *be a SO-complete(not necessarily complete) metric space with orthogonal element* x_0*. Also, let operators* $T, B : X \to X$ *be given. Suppose that the following conditions are satisfied:*

(i) T *is* $\perp$*-preserving;*
(ii) B *is SO-continuous;*
(iii) $x_0 \leq Bx_0$ *and* X *is SO-regular;*
(iv) *for all* $x \in X$*, we have* $x \leq Bx \implies BTx \leq Tx$;
(v) *for all* $x \in X$*, we have* $Bx \leq x \implies Tx \leq BTx$;
(vi) *there exits* $\zeta \in \mathcal{Z}$ *such that for each* $\perp$*-comparable elements* $x, y \in X$

$$(x \leq Bx \text{ and } By \leq y) \Longrightarrow \zeta(d(Tx, Ty), d(x, y)) \geq 0.$$

Then the sequence $\{T^n x_0\}$ *converges to* $x^* \in X$ *satisfying* $x^* = Tx^*$*. Also, tThe point* $x^* \in X$ *is a unique solution of the problem* $\begin{cases} x = Tx, \\ x = Bx. \end{cases}$

By setting $C = B = T$ and $A = D = I_X$, we obtain a generalization of Corollary 3.4 in [10]. Through the following we give an extension of Theorem 2.8 [12].

Corollary 3.3. *Let* $(X, \perp, d)$ *be a SO-complete metric space with orthogonal element* x_0 *and* $T : X \to X$ *be a* $\perp$*-preserving and* $\perp$*-*$\mathcal{Z}$*-contraction with respect to* ζ*. Let* X *is SO-regular. Then* T *has a unique fixed point* x^**. Also,* T *is a Picard operator, that is, for all* $x \in X$*, the sequence* $\{T^n(x)\}$ *is convergent to* x^**.*

Proof: Now we only show that T is a Picard operator. Let $x \in X$ be arbitrary. We have $[x_0 \perp x^* \text{ and } x_0 \perp x]$ or $[x^* \perp x_0 \text{ and } x \perp x_0]$.
Now, since T is $\perp$-preserving, then

$$[T^n(x_0) \perp T^n(x^*) \text{ and } T^n(x_0) \perp T^n(x)] \text{ or } [T^n(x^*) \perp T^n(x_0) \text{ and } T^n(x) \perp T^n(x_0)]$$

for all $n \in \mathbb{N}$. Hence, for all $n \in \mathbb{N}$, we get

$$\begin{aligned} 0 &\leq \zeta(d(T^n(x), T^n(x_0)), d(T^{n-1}(x), T^{n-1}(x_0))) \\ &< d(T^{n-1}(x), T^{n-1}(x_0)) - d(T^n(x), T^n(x_0)). \end{aligned}$$

Then $\{d(T^n(x), T^n(x_0))\}$ is a decreasing sequence and bounded below. Thus there exists $r \geq 0$ such that $\lim_{n\to\infty} d(T^n(x), T^n(x_0)) = r$. Let $r > 0$, applying (ζ_3), we have

$$0 \leq \limsup_{n\to\infty} \zeta(d(T^n(x), T^n(x_0)), d(T^{n-1}(x), T^{n-1}(x_0))) < 0.$$

This is a contraction. Thus $r = \lim_{n\to\infty} d(T^n(x), T^n(x_0)) = 0$.
Hence $x^* = \lim_{n\to\infty} T^n(x_0) = \lim_{n\to\infty} T^n(x)$. This completes the proof.

4. Some exampels

Example 4.1. *Let $X = (-2, 2]$. Define $x \perp y \iff 0 \leq x \leq y \leq 1$ or $x = 0$. Then $(X, \perp)$ is an O-set with orthogonal element $x_0 = 0$. Clearly, X with the Euclidean metric is not a complete metric space, but it is SO-complete. We see that X is $\perp$-regular. Now, define relation "R" as $x\ R\ y \iff x + y \in [-1, 2]$. Clearly, "$R$" is not partial order. We take $R_1 = R_2 := R$. Let $T : X \to X$ be the mapping defined by*

$$T(x) = \begin{cases} 0 & x \leq 1 \\ -x & 1 < x < 2 \\ 1 & x = 2. \end{cases}$$

If $x = 0$, then $Tx = 0$, and if $x \neq 0$, we have $0 < x \leq y \leq 1$, and so $Tx = 0$. Hence $Tx \perp Ty$ and T is $\perp$-preserving. Consider the mappings $A, B, C, D : X \to X$ defined by $Ax = x$,

$$B(x) = \begin{cases} \frac{x}{2} + 2 & x < 0 \\ 2 & x \geq 0, \end{cases} \quad C(x) = \begin{cases} 0 & x \leq 1 \\ x - 1 & x > 1, \end{cases} \quad D(x) = \begin{cases} -x & x \leq 1 \\ 2 & x > 1. \end{cases}$$

Obviously, "R" is dR-regular. Moreover, A and B are SO-continuous mappings. If for some $x \in X$, we have $Ax\ R\ Bx$, then $x \leq 0$, which yields that $Tx = 0$. Thus $CT(x)\ R\ DT(x)$. If for some $x \in X$, we have $Cx\ R\ Dx$, then $x \leq 1$, and so $Tx = 0$. Hence $AT(x)\ R\ BT(x)$. Thus T is (A, B, C, D, R_1, R_2)-stable and (C, D, A, B, R_2, R_1)-stable. For all $(x, y) \in X \times X$, we have

$$Ax\ R_1\ Bx,\ Cy\ R_2\ Dy \implies (x \leq 0 \text{ and } y \leq 1).$$

Set $\zeta(t, s) = \frac{1}{2}s - t$ for all $t, s \in [0, \infty)$. We show that condition (vi) of Theorem 2.1 is satisfied. We have $x \leq 0$ and $y \leq 1 \implies Tx = Ty = 0$. This implies that $\zeta(d(Tx, Ty), d(x, y)) = \frac{1}{2}d(x, y) - d(Tx, Ty) = \frac{1}{2}d(x, y) \geq 0$. Therefore there exists $\zeta \in \mathcal{Z}$ such that for all $x, y \in X$ with $x \perp y$ and $(Ax\ R_1\ Bx$ and $Cy\ R_2\ Dy)$, $\zeta(d(Tx, Ty), d(x, y)) \geq 0$. Applying Theorem 2.1, (2) has unique solution $x^* = 0$.

Example 4.2. *Let $X = (-1, \infty)$. Suppose that $x \perp y \iff xy = 0$. Then $(X, \perp)$ is an O-set with orthogonal element $x_0 = 0$. Clearly, X with the Euclidean metric is not a complete metric space, but it is SO-complete. We see that X is $\perp$-regular. We take $R_1 = R_2 := \leq$. Set $T : X \to X$ defined by*

$$T(x) = \begin{cases} \frac{x}{2} & x < 1 \\ \frac{3}{2x} & x \geq 1. \end{cases}$$

If $x = 0$, then $Tx = 0$, and if $x \neq 0$, we have $y = 0$. Hence $Ty = 0$, and so $Tx \perp Ty$. Then T is $\perp$-preserving. Consider the mappings $A, B, C, D : X \to X$ defined by $Ax = x + 1$,

$$B(x) = \begin{cases} 1 & x \geq 1 \\ x + 2 & x < 1, \end{cases} \quad C(x) = \begin{cases} 2 & \frac{1}{2} < x \leq 3 \\ x & o.w., \end{cases} \quad D(x) = \begin{cases} \frac{1}{2} & \frac{1}{2} < x \leq 3 \\ x^2 + 1 & o.w.. \end{cases}$$

Obviously, "R_i" is dR-regular, $i = 1, 2$. Moreover, A and B are SO-continuous mappings. If for some $x \in X$, we have $Ax \leq Bx$, then $x < 1$, which yields $Tx = \frac{x}{2}$. Thus $CT(x) \leq DT(x)$. Therefore T is (A, B, C, D, R_1, R_2)-stable. If for some $x \in X$, we have $Cx \leq Dx$, then $x \leq \frac{1}{2}$ or $x > 3$. We consider two caces:

(1) *If* $x \leq \frac{1}{2}$, *then* $Tx = \frac{x}{2}$, *and so* $AT(x) \leq BT(x)$.

(2) *If* $x > 3$, *then* $Tx = \frac{3}{2x}$, *and so* $AT(x) \leq BT(x)$.

Thus T *is* (C, D, A, B, R_2, R_1)*-stable. For all* $(x, y) \in X \times X$, *we have*

$$Ax\ R_1\ Bx,\ Cy\ R_2\ Dy \implies (x < 1 \text{ and } (y \leq \frac{1}{2} \text{ or } y > 3)).$$

Set $\zeta(t, s) = s\varphi(s) - t$ *for all* $t, s \in [0, \infty)$, *where* $\varphi : [0, \infty) \to [0, 1)$ *define by*

$$\varphi(s) = \begin{cases} \dfrac{s}{s+1} & s > 2 \\ \\ \dfrac{1}{2} & s \leq 2. \end{cases}$$

Since $\limsup_{t \to \to r^+} \varphi(t) < 1$ *for all* $r > 0$, *then* $\zeta(x, y)$ *is a simulation function. We show that condition* (vi) *of Theorem* 2.1 *is satisfied. We have two cases:*

(1) *If* $x < 1$ *and* $y \leq \frac{1}{2}$, *then* $d(x, y) \leq 2$, *and so*

$$\zeta(d(Tx, Ty), d(x, y)) = d(x, y)\varphi(d(x, y)) - d(Tx, Ty) = \frac{1}{2}d(x, y) - \frac{1}{2}d(x, y) = 0.$$

(2) *If* $x < 1$ *and* $y > 3$, *then* $d(x, y) > 2$, *and so*

$$\zeta(d(Tx, Ty), d(x, y)) = d(x, y)\varphi(d(x, y)) - d(Tx, Ty) = d(x, y)\frac{d(x, y)}{d(x, y) + 1} - d(\frac{x}{2}, \frac{3}{2y}).$$

Then there exists $\zeta \in \mathcal{Z}$ *such that for all* $x, y \in X$ *with* $x \perp y$, $Ax\ R_1\ Bx$, $Cy\ R_2\ Dy$ *and* $\zeta(d(Tx, Ty), d(x, y)) \geq 0$. *Applying Theorem 2.1, Problem* (2) *has unique solution* $x^* = 0$.

5. Application to solve system of fractional hybrid differential equations

Let $X = C(J, \mathbb{R})$ be the class of continuous functions $f : J \to \mathbb{R}$ that $J = (t_0, t_0 + a)$ denote a bounded interval in $\mathbb{R}$ for some $a, t_0 \in \mathbb{R}$ with $a > 0$. Consider the following system of fractional hybrid differential equations(in short FHDE) of order $0 < q < 1$

$$\begin{cases} D^q[x(t) - f(t, x(t))] = h(t, x(t)), & t \in J, \\ D^q[x(t) - g(t, x(t))] = k(t, x(t)), & t \in J, \\ x(t_0) = x_0 = 0, \end{cases} \tag{16}$$

where $f, g, h, k : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions for which:

(C_1) The functions $x \to x - f(t, x)$ and $x \to x - g(t, x)$ are increasing in $\mathbb{R}$ for all $t \in J$.

(C_2) There exist two continuous functions $s, u : J \to \mathbb{R}$ such that $|h(t, x)| \leq s(t)$ and $|k(t, x)| \leq u(t)$, $t \in J$ for all $x \in \mathbb{R}$.

(C_3) $f(t, 0) = h(t, 0) = 0$ for all $t \in J$ and $g(t_0, 0) = 0$.

(C_4) (i) For all $x \in X$, we have

$$x(t) \leq g(t, x(t)) + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} k(s, x(s)) ds, \ \forall t \in J,$$

$$\Rightarrow\ g(t, x) \leq f(t, x) \text{ and } k(t, x) \leq h(t, x), \ \forall t \in J.$$

(ii) For all $x \in X$, we have

$$g(t, x(t)) + \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} k(s, x(s)) ds \leq x(t), \ \forall t \in J,$$

$$\Rightarrow\ f(t,x) \leq g(t,x)\ \textit{and}\ h(t,x) \leq k(t,x),\ \forall t \in J.$$

(C_5) $g(t,x)$ and $k(t,x)$ are decreasing related to the second variable.

(C_6) There exist $0 < \lambda < 1$ such that for all $x \in X$

$$|f(t,x(t))| \leq \frac{\lambda}{2}\|x\| \ \text{ and } \ |h(t,x(t))| \leq \frac{\lambda\Gamma(q+1)}{2(t-t_0)^q}\|x\|.$$

Theorem 5.1. *Let the above conditions are satisfied. Then the system of fractional hybrid differential equations* (16) *has a unique solution.*

Proof. We define two operator equations $T, B : X \to X$ as follow:

$$Tx(t) = f(t,x(t)) + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}h(s,x(s))ds,$$

$$Bx(t) = g(t,x(t)) + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}k(s,x(s))ds.$$

Now, using the hypotheses (C_1) and (C_2) it can be shown that the FHDE (16) has a unique solution if and only if T and B have a unique common fixed point in X. We consider the following orthogonality relation in X:

$$x \perp y \ \Leftrightarrow\ x = 0 \ \text{ or } \ y = 0 \quad \forall x, y \in X. \tag{17}$$

Since (X,d) is a complete metric space, then $(X,\perp,d)$ is SO-complete. We take $\preceq_1 = \preceq_2 = \leq$. From definition, " $\leq$ " is dR-regular and X is $\perp$-regular. Clearly, B is SO-continuous. Now, we prove the following four steps to complete the proof.

Step 1: T is $\perp$-preserving. Let $x \perp y$ that is $x = 0$ or $y = 0$. Let $x = 0$. Applying (C_3), we have $f(t,x) = 0$ and $h(t,x) = 0$. Furthermore $Tx = 0$. Similarly, if $y = 0$, we have $Ty = 0$. Then T is $\perp$-preserving.

Step 2: Prove that $x \in X$, $x(t) \leq Bx(t)$, $\forall t \in J \implies BTx(t) \leq Tx(t)$.
Let $x \in X$ with $x(t) \leq Tx(t)$, $\forall t \in J$. Applying part (i) of (C_4), we have $g(t,x(t)) \leq f(t,x(t))$ and $k(t,x(t)) \leq h(t,x(t))$. Then for all $t \in J$,

$$\begin{aligned} x(t) \leq & g(t,x(t)) + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}k(s,x(s))ds \\ \leq & f(t,x(t)) + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}h(s,x(s))ds = Tx(t). \end{aligned}$$

Also, applying (C_5), for all $t \in J$, we have

$$\begin{aligned} BTx(t) &= g(t,Tx(t)) + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}k(s,Tx(s))ds \\ &\leq g(t,x(t)) + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}k(s,x(s))ds \\ &\leq f(t,x(t)) + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}h(s,x(s))ds = Tx(t). \end{aligned}$$

Step 3: Prove that for all $x \in X$, $Bx(t) \leq x(t) \implies Tx(t) \leq BTx(t)$, $\forall t \in J$.
Let $x \in X$ with $Bx(t) \leq x(t)$. Applying part (ii) of (C_4), we have $f(t,x(t)) \leq g(t,x(t))$

and $h(t, x(t)) \leq k(t, x(t))$. Then for all $t \in J$,

$$\begin{aligned} x(t) \geq & g(t, x(t)) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} k(s, x(s)) ds \\ \geq & f(t, x(t)) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} h(s, x(s)) ds = Tx(t). \end{aligned}$$

Also, applying (C_5), we have

$$\begin{aligned} BTx(t) = & g(t, Tx(t)) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} k(s, Tx(s)) ds \\ \geq & g(t, x(t)) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} k(s, x(s)) ds \\ \geq & f(t, x(t)) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} h(s, x(s)) ds = Tx(t). \end{aligned}$$

Step 4: Prove that there exits $\zeta \in \mathcal{Z}$ such that for each $\perp$-comparable elements $x, y \in X$, $(Ax \ \mathrm{R}_1 \ Bx \ and \ Cy \ \mathrm{R}_2 \ Dy) \Longrightarrow \zeta(d(Tx, Ty), d(x, y)) \geq 0$.
Since $x \perp y$, therefore $x = 0$ or $y = 0$. Let $y = 0$ and so $Ty(t) = 0$. Applying (C_6), we have

$$\begin{aligned} |Tx(t) - Ty(t)| = |Tx(t)| = & |f(t, x(t)) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} h(s, x(s)) ds| \\ \leq & |f(t, x(t))| + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} |h(s, x(s))| ds \\ \leq & \frac{\lambda}{2} \|x\| + \frac{\lambda \Gamma(q+1)}{2(t-t_0)^q} \|x\| \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} ds \\ \leq & \lambda \|x\| = \lambda \|x - y\|. \end{aligned}$$

Set $\xi(t, s) = \lambda s - t$ for all $t, s \in [0, \infty)$. Therefore

$$\xi(d(Tx(t), Ty(t)), d(x(t), y(t))) = \lambda |x(t) - y(t)| - |Tx(t) - Ty(t)| \geq 0.$$

Finally, applying Corollary 3.2, T and B have a unique solution in X which is a unique solution of system of fractional hybrid differential equations (16). □

Remark 5.1. *By Corollary 3.2 in* [10] *we can not guarantee the establishment of Theorem 5.1 unless we put the following condition in place of condition* (C_6)*:*
There exist $0 < \lambda < 1$ *such that for all* $x, y \in X$

$$|f(t, x(t)) - f(t, y(t))| \leq \frac{\lambda}{2} \|x - y\| \ \ and \ |h(t, x(t)) - h(t, y(t))| \leq \frac{\lambda \Gamma(q+1)}{2(t-t_0)^q} \|x - y\|.$$

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References

[1] Ahmad, B., Nieto, J.J., (2013), Boundary value problems for a class of sequential integrodifferential equations of fractional order, J. Funct. Spaces Appl., Article ID, pp. 149-659.

[2] Ansari, A.H., Kumam, P., Samet, B., (2017), A fixed point problem with constraint inequalities via an implicit contraction, J. Fixed Point Theory and Appl., pp. 1145-1163.

[3] Ahmadi, Z., Lashkaripour, R., Baghani, H., A fixed point problem with constraint inequalities via a contraction in incomplete metric spaces, Filomat, to appear.

[4] Babu, G.V.R., Sarma, K.K.M., Krishna, P.H., (2014), Fixed points of ψ-weak Geraghty contractions in partially ordered metric spaces, J. Adv. Res. Pure Math., 6, pp. 9-23.

[5] Baghani, H., Eshaghi Gordji, M., Ramezani, M., (2016), Orthogonal sets: The axiom of choice and proof of a fixed point theorem, J. Fixed Point Theory and Appl., 18, pp. 465-477.

[6] Baghani, H., Ramezani, M, (2017), A fixed point theorem for a new class of set-valued mappings in R-complete(not necessarily complete) metric spaces, Filomat, 31, pp. 3875-3884.

[7] Baghani, H., Ramezani, M., (2017), Contractive gauge functions in strongly orthogonal metric spaces, Int. J. Nonlinear Anal., 2, 23-28.

[8] Eshaghi Gordji, M., Ramezani, M., De La Sen, M., Cho, Y.J., (2017), On orthogonal sets and Banach fixed Point theorem, Fixed Point Theory, 18, 569-578.

[9] Hussain, N., Kadelburg, Z., Radenovic, S., Al-Solamy, F.R., (2012), Comparison functions and fixed point results in partial metric spaces, Abstract and Applied Analysis Article ID., 15, 605-781.

[10] Jleli, M., Samet, B., (2016), A fixed point problem under two constraint inequalities, Fixed Point Theory and Appl., doi:10.1186/s13663-016-0504-9.

[11] Kawasaki, T., Toyoda, M., (2015), Fixed point theorem and fractional differential equations with multiple delays related with chaos neuron models, Appl. Math., 6, 2192-2198.

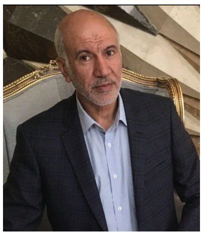
[12] Khojasteh, F., Shukla, S., Radenovi, S., (2015), A new approach to the study of fixed point theorems via simulation functions, Filomat, 29, 1189-1194.

[13] Ntouyas, S.K., Tariboon, J., (2017), Fractional boundary value problems with multiple orders of fractional with multiple orders of fractional derivatives and integrals, Electronic Journal of Diferential Equations, 100, 1-18.

[14] Roldan-Lopez-de-Hierro, AF., Karapinar, E, Roldan-Lopez-de-Hierro, C., Martinez-Moreno, J., (2015), Coincidence point theorems on metric spaces via simulation functions, J. Comput. Appl. Math., 275, 345-355.

[15] Wang, W.X., Zhang, L., Liang, Zh., (2006),Initial value problems for nonlinear impulsive integro-differential equations in Banach space, J. Math. Anal. Appl., 320, 510-527.

[16] Yang, Sh., Zhang, Sh., (2016), Impulsive boundary value problem for a fractional differential equation, Boundary Value Problems, doi:10.1186/s13661-016-0711-7.



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