TWMS J. App. Eng. Math. V.10, N.2, 2020, pp. 288-295

# ON SOME INTEGRAL INEQUALITIES FOR $(s,m)$-CONVEX FUNCTIONS

B. BAYRAKTAR[1], M. GÜRBÜZ[2], §

Abstract. A new identity has been handled in this paper. It allows to derive new inequalities referring to upper estimation of the Jensen functional in the class of $(s,m)$-convex functions. Also some applications for special means are given by using new inequalities.

Keywords: Convex function, $(s,m)$-convex, Hermite-Hadamard inequalitiy, Hölder inequality, power mean inequality.

AMS Subject Classification: 26A51, 26D15

## 1. Introduction

Convexity has become a very interesting topic for many researchers since antiquity because it has applications in many areas of pure and applied mathematics. Unsuspectingly, the following definition is well known in the literature.

**Definition 1.1.** *The function $f:[a,b]\to\mathbb{R}$ is said to be convex, if we have*

$$f\left(\lambda x+(1-\lambda)y\right)<\lambda f\left(x\right)+(1-\lambda)f\left(y\right) \tag{1}$$

*for all $x,y\in[a,b]$ and $\lambda\in[0,1]$.*

Also different classes of convex functions namely $m-$convex functions, $s-$convex functions, $h-$convex functions, $(\alpha,m)$-convex functions, etc., and relations between them have been established (see for example [1]-[11] and references cited therein). Many important inequalities are established for the class of convex functions, but one of the most important inequalities is Hermite-Hadamard inequality (or Hadamard's inequality). This double inequality is stated as follows in literature:

**Theorem 1.1.** *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a convex function and let $a,b\in I$ with $a<b$. Then the following double inequality holds:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f\left(x\right)dx\leq\frac{f\left(a\right)+f\left(b\right)}{2} \tag{2}$$

[1] Uludag University, Faculty of Education, Gorukle Campus, 16059, Bursa, Turkey.
e-mail: bbayraktar@uludag.edu.tr; ORCID: https://orcid.org/0000-0001-7594-8291.

[2] Agri Ibrahim Çeçen University, Faculty of Education, Dept. of Mathematics, 04100, Agri, Turkey.
e-mail: mgurbuz@agri.edu.tr; ORCID: https://orcid.org/0000-0002-7092-4298.

§ Manuscript received: December 6, 2018; accepted: October 17, 2019.

The above inequality holds in the reversed direction if $f$ is concave. G. H. Toader put forward a new definition in [10] as the following:

**Definition 1.2.** *Let $f:[0,b]\to\mathbb{R}$ with $b>0$. The function $f$ is said to be $m-$convex on $[0,b]$ if the inequality*

$$f\left(\lambda x+m\left(1-\lambda\right)y\right)\leq\lambda f\left(x\right)+m\left(1-\lambda\right)f\left(y\right)\tag{3}$$

*holds for all $x,y\in[0,b]$ and $(\lambda,m)\in[0,1]^2$.*

In [3], W. W. Breckner defined a new class of functions that are $s-$convex in the second sense:

**Definition 1.3.** *$f:[0,\infty)\to\mathbb{R}$ is said to be s-convex function in the second sense if*

$$f\left(\lambda x+\left(1-\lambda\right)y\right)\leq\lambda^s f\left(x\right)+\left(1-\lambda\right)^s f\left(y\right)\tag{4}$$

*holds for all $x,y\in[0,\infty)$, $\lambda\in[0,1]$ and for some fixed $s\in(0,1]$.*

It is clear that the ordinary convexity of functions defined on $[0,\infty)$ is obtained by choosing $s=1$ in (4).

J. Park asserted a new definition given in the following and gave some properties about this class of functions in [6].

**Definition 1.4.** *For some fixed $s\in(0,1]$ and $m\in[0,1]$ a mapping $f:[0,\infty)\to\mathbb{R}$ is said to be $(s,m)$- convex in the second sense on $I$ if*

$$f\left(tx+m\left(1-t\right)y\right)\leq t^s f\left(x\right)+m\left(1-t\right)^s f\left(y\right)$$

*holds for all $x,y\in I$ and $t\in[0,1]$.*

**Proposition 1.1.** *Any positive m-convex function is an $(s,m)$- convex function since $t\leq t^s$ and $(1-t)\leq(1-t)^s$ for all $s\in(0,1]$ and $t\in[0,1]$.*

In this paper, some new upper bounds for integral variant of Jensen functional has been gathered for some special cases. With this aspect, discrete form of Jensen functional is given in the following (see [11]):

**Definition 1.5.** *Let $f:I\to\mathbb{R}$ be a function and $x_1,...,x_n\in I$ and $p_1,...,p_n\in[0,1]$ with $\sum_{i=1}^n p_i=1$. The Jensen functional is defined by*

$$\Im\left(f,x,p\right)=\sum_{i=1}^n p_i f\left(x_i\right)-f\left(\sum_{i=1}^n p_i x_i\right).$$

In this paper, a new integral identity and some new upper estimations of the Jensen functional in the class of $(s,m)$-convex functions are handled. Then some applications for special means are given.

## 2. New Results For $(s,m)$- Convex Functions

The following identity will be useful to gather new results obtained in this study.

**Lemma 2.1.** *Let $f:I=[0,b^*]\subset\mathbb{R}\to\mathbb{R}$ be a twice differentiable function on $I^\circ$ (interior of $I$) where $a,mb\in I^\circ$ and $b^*>0$. If $f''\in L[0,b^*]$, then we have*

$$\begin{aligned}&\frac{f\left(a\right)+f\left(mb\right)}{2}-f\left(\frac{a+mb}{2}\right)\\&=\frac{\left(mb-a\right)^2}{2}\left[\int_0^{1/2}tf''\left(ta+m\left(1-t\right)b\right)dt+\int_{1/2}^1\left(1-t\right)f''\left(ta+m\left(1-t\right)b\right)dt\right]\end{aligned}\tag{5}$$

*Proof.* Integrating both integrals by parts twice, we have

$$I_1 = \int_0^{1/2} t f''(ta + m(1-t)b)\,dt = \frac{1}{2(a-mb)} f'\left(\frac{a+mb}{2}\right) + \frac{f(mb) - f\left(\frac{a+mb}{2}\right)}{(mb-a)^2}$$

and

$$I_2 = \int_{1/2}^{1} (1-t) f''(ta + m(1-t)b)\,dt = -\frac{1}{2(a-mb)} f'\left(\frac{a+mb}{2}\right) + \frac{f(a) - f\left(\frac{a+mb}{2}\right)}{(mb-a)^2}.$$

By adding last two equalities we get

$$\frac{(mb-a)^2}{2}(I_1 + I_2) = \frac{f(a) + f(mb)}{2} - f\left(\frac{a+mb}{2}\right).$$

This completes the proof. □

**Corollary 2.1.** *If we choose $m = 1$ in (5) we obtain the following identity for convex functions:*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \\ &= \frac{(b-a)^2}{2}\left[\int_0^{1/2} t f''(ta+(1-t)b)\,dt + \int_{1/2}^{1}(1-t) f''(ta+(1-t)b)\,dt\right]. \end{aligned}$$

**Theorem 2.1.** *Let $f : I = [0, b^*] \to \mathbb{R}$ be a twice differentiable function on $I^\circ$ such that $f'' \in L[0, b^*]$ where $a, mb \in I^\circ$ and $b^* > 0$. If $|f''|$ is $(s,m)-$convex on $I$, for $s \in (0,1]$, $m \in [0,1]$ the following inequality holds*

$$\left|\frac{f(a)+f(mb)}{2} - f\left(\frac{a+mb}{2}\right)\right| \le \frac{(mb-a)^2}{2}\eta(|f''(a)| + m|f''(b)|)$$

*where*

$$\eta = \frac{2^{s+1}-1}{(s+1)(s+2)2^{s+1}}.$$

*Proof.* Using the triangle inequality for the equality in Lemma 2.1, we can write

$$\begin{aligned} &\left|\frac{f(a)+f(mb)}{2} - f\left(\frac{a+mb}{2}\right)\right| \qquad (6) \\ &\le \frac{(mb-a)^2}{2}\left[\int_0^{1/2} t\left|f''(ta+m(1-t)b)\right|dt + \int_{1/2}^{1}(1-t)\left|f''(ta+m(1-t)b)\right|dt\right] \\ &= \frac{(mb-a)^2}{2}(|I_1| + |I_2|) \end{aligned}$$

Since $|f''|$ is $(s,m)$ −convex on $I$ we have

$$\begin{aligned}|I_1| &\le \left|f''(a)\right| \int_0^{1/2} t^{s+1}dt + m\left|f''(b)\right| \int_0^{1/2} t(1-t)^s dt \\ &= \frac{1}{(s+2)\,2^{s+2}}\left(\left|f''(a)\right| + m\frac{2^{s+2}-s-3}{s+1}\left|f''(b)\right|\right)\end{aligned}$$

and

$$\begin{aligned}|I_2| &\le \left|f''(a)\right| \int_{1/2}^{1} t^s(1-t)\,dt + m\left|f''(b)\right| \int_{1/2}^{1} t(1-t)^{s+1} dt \\ &= \frac{1}{(s+2)\,2^{s+2}}\left(\frac{2^{s+2}-s-3}{s+1}\left|f''(a)\right| + m\left|f''(b)\right|\right)\end{aligned}$$

Adding the last two inequalities we obtain

$$|I_1| + |I_2| \le \eta(\left|f''(a)\right| + m\left|f''(b)\right|) \tag{7}$$

where

$$\eta = \frac{2^{s+1}-1}{(s+1)(s+2)\,2^{s+1}}$$

Taking into account (6) and (7), the proof is completed. □

**Corollary 2.2.** *In Theorem 2.1 if we choose $s = m = 1$, we have*

$$\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \le \frac{(b-a)^2}{16}\left(\left|f''(a)\right| + \left|f''(b)\right|\right).$$

**Corollary 2.3.** *If $\|f''\|_\infty = \sup|f''(x)|$ be a real number and $s = 1, m = 1$ we have*

$$\frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \le \frac{(b-a)^2}{8}\|f''\|_\infty. \tag{8}$$

On the other hand by reorganizing the Hermite-Hadamard integral inequality we have

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \tag{9}$$

From (8) and (9), we obtained a trapezoid rule as:

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{(b-a)^2}{8}\|f''\|_\infty.$$

**Theorem 2.2.** *Let $f : I = [0, b^*] \to \mathbb{R}$ be a twice differentiable function on $I^\circ$ such that $f'' \in L[0, b^*]$ where $a, mb \in I^\circ$ and $b^* > 0$. If $|f''|^q$ is $(s,m)$-convex on $I$, for $s \in (0,1], m \in [0,1]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, the following inequality holds*

$$\begin{aligned}&\frac{f(a)+f(mb)}{2} - f\left(\frac{a+mb}{2}\right) \\ &\le \frac{(mb-a)^2}{2}\mu\nu\left\{\left[\left|f''(a)\right|^q + m\gamma\left|f''(b)\right|\right]^{1/q} + \left[\gamma\left|f''(a)\right| + m\left|f''(b)\right|\right]^{1/q}\right\}\end{aligned}$$

*where $\mu = \left[\frac{1}{(p+1)2^{p+1}}\right]^{1/p}$, $\nu = \left[\frac{1}{(s+1)2^{s+1}}\right]^{1/q}$ and $\gamma = \left(2^{s+1}-1\right)$.*

*Proof.* Using triangle inequality and the well-known Hölder's integral inequality on equality (5) we have

$$\left|\frac{f(a)+f(mb)}{2}-f\left(\frac{a+mb}{2}\right)\right| \tag{10}$$
$$\leq \frac{(mb-a)^2}{2}\left[\left(\int_0^{1/2} t^p dt\right)^{1/p}\left(\int_0^{1/2}\left|f''(ta+m(1-t)b)\right|^q dt\right)^{1/q}\right.$$
$$\left.+\left(\int_{1/2}^{1}(1-t)^p dt\right)^{1/p}\left(\int_{1/2}^{1}\left|f''(ta+m(1-t)b)\right|^q dt\right)^{1/q}\right]$$
$$=\frac{(mb-a)^2}{2}\left(\frac{1}{(p+1)2^{p+1}}\right)^{1/p}\left(|I_1|^{1/q}+|I_2|^{1/q}\right)$$

Taking into account the fact that the function $|f''|^q$ is $(s,m)$-convex, we consider each of the integrals separately as:

$$|I_1|\leq\left|f''(a)\right|^q\int_0^{1/2}t^s dt+m\left|f''(b)\right|^q\int_0^{1/2}(1-t)^s dt=\frac{|f''(a)|^q+m\left(2^{s+1}-1\right)|f''(b)|^q}{(s+1)2^{s+1}}$$

and

$$|I_2|\leq\left|f''(a)\right|^q\int_{1/2}^{1}t^s dt+m\left|f''(b)\right|^q\int_{1/2}^{1}(1-t)^s dt=\frac{\left(2^{s+1}-1\right)|f''(a)|^q+m|f''(b)|^q}{(s+1)2^{s+1}}.$$

By adding the last two inequalities we get

$$|I_1|^{1/q}+|I_2|^{1/q}\leq\left(\frac{1}{(s+1)2^{s+1}}\right)^{1/q}\left\{\left[\left|f''(a)\right|^q+m(2^{s+1}-1)\left|f''(b)\right|^q\right]^{1/q}\right. \tag{11}$$
$$\left.+\left[\left(2^{s+1}-1\right)\left|f''(a)\right|^q+m\left|f''(b)\right|^q\right]^{1/q}\right\}.$$

Taking into account inequalities (10) and (11), the proof is completed. $\square$

**Corollary 2.4.** *In Theorem 2.2 if we choose $s=m=1$ and $p=q=2$ we have*

$$\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)$$
$$\leq\frac{(b-a)^2}{16\sqrt{3}}\left\{\left[\left|f''(a)\right|^2+3\left|f''(b)\right|^2\right]^{1/2}+\left[3\left|f''(a)\right|^2+\left|f''(b)\right|^2\right]^{1/2}\right\}.$$

**Theorem 2.3.** *Let $f: I=[0,b^*]\to\mathbb{R}$ be a twice differentiable function on $I^\circ$ such that $f''\in L[0,b^*]$ where $a,mb\in I^\circ$ and $b^*>0$. If $|f''|^q$ is $(s,m)$-convex on $I$, for $s\in(0,1]$, $m\in[0,1]$ and $q\geq 1$, the following inequality holds*

$$\frac{f(a)+f(mb)}{2}-f\left(\frac{a+mb}{2}\right)$$
$$\leq 2^{(3-4q)/q}(mb-a)^2\left\{\left[\eta\left|f''(a)\right|^q+m\zeta\left|f''(b)\right|^q\right]^{1/q}+\left[\zeta\left|f''(a)\right|^q+m\eta\left|f''(b)\right|^q\right]^{1/q}\right\}$$

*where $\eta=\frac{1}{2^{s+2}(s+2)}$ and $\zeta=\frac{2^{s+2}-s-3}{2^{s+2}(s+1)(s+2)}$.*

*Proof.* Using the triangle inequality for the equality (5) in Lemma 2.1, we can write

$$\left|\frac{f(a)+f(mb)}{2}-f\left(\frac{a+mb}{2}\right)\right|$$
$$\leq\frac{(mb-a)^2}{2}\left[\int_0^{1/2}t\left|f''(ta+m(1-t)b)\right|dt+\int_{1/2}^{1}(1-t)\left|f''(ta+m(1-t)b)\right|dt\right]$$
$$=\frac{(mb-a)^2}{2}(I_1+I_2). \tag{12}$$

Using the power mean inequality and $(s,m)$-convexity of $|f''|^q$ on $[a,b]$ we get

$$I_1\leq\left(\int_0^{1/2}tdt\right)^{1-(1/q)}\left[\int_0^{1/2}t\left|f''(ta+m(1-t)b)\right|^q dt\right]^{1/q}$$
$$\leq 2^{3(1-q)/q}\left[\left|f''(a)\right|^q\int_0^{1/2}t^{s+1}dt+m\left|f''(b)\right|^q\int_0^{1/2}t(1-t)^s dt\right]^{1/q}.$$

So we have

$$I_1\leq 2^{\frac{3(1-q)}{q}}\left[\eta\left|f''(a)\right|^q+\zeta m\left|f''(b)\right|^q\right]^{1/q} \tag{13}$$

where $\eta=\frac{1}{2^{s+2}(s+2)}$ and $\zeta=\frac{-s+2^{s+2}-3}{2^{s+2}(s+1)(s+2)}$. Similarly for $I_2$ we can write

$$I_2\leq 2^{\frac{3(1-q)}{q}}\left[\zeta\left|f''(a)\right|^q+\eta m\left|f''(b)\right|^q\right]^{1/q} \tag{14}$$

Substituting these inequalities for (13) and (14) into inequality (12) and rearranging we complete the proof. □

**Corollary 2.5.** *In Theorem 2.3 if we choose $s=m=1$, we have*

$$\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)$$
$$\leq\frac{2^{\frac{1-4q}{q}}(b-a)^2}{3^{1/q}}\left\{\left[0.5|f''^q+|f''^q\right]^{1/q}+\left[|f''^q+0.5|f''^q\right]^{1/q}\right\}$$

## 3. Applications To Special Means

We now consider the means for arbitrary real numbers $\alpha$ and $\beta$.
We now consider the means for arbitrary real numbers $\alpha$ and $\beta$

$$1.\mathit{Arithmetic\ mean}: A=A(\alpha,\beta)=\frac{\alpha+\beta}{2};\qquad \alpha,\beta\in R$$
$$2.\mathit{Harmonic\ \ mean}: H=H(\alpha,\beta)=\frac{2\alpha\beta}{\alpha+\beta};\qquad \alpha,\beta>0$$
$$3.\mathit{Quadratic\ \ mean}: Q=Q(\alpha,\beta)=\sqrt{\alpha^2+\beta^2};\quad \alpha,\beta\in R$$

Now, using some results we give some applications to special means of positive real numbers.

**Proposition 3.1.** *Let $a, b \in \mathbb{R}^+$, $a < b$ and $n \in Z$, $|n| \geq 2$. Then, we have*

$$A\left(a^n, b^n\right) - A^n\left(a, b\right) \leq \frac{(b-a)^2}{8} n\left(n-1\right) A\left(a^{n-2}, b^{n-2}\right).$$

*Proof.* The assertion follows from Corollary 2.2 applied to the function $f(x) = x^n$, $x \in \mathbb{R}^+$. □

**Proposition 3.2.** *Let $a, b \in \mathbb{R}^+, a < b$. Then, we have*

$$A\left(\ln a, \ln b\right) - \ln\left(A\left(a, b\right)\right) \leq \frac{(b-a)^2}{16\sqrt{3}}\left\{Q\left(\frac{1}{a^2}, \frac{\sqrt{3}}{b^2}\right) + Q\left(\frac{\sqrt{3}}{a^2}, \frac{1}{b^2}\right)\right\}.$$

*Proof.* The assertion follows from Corollary 2.4 applied to the function $f(x) = \ln x$, $x \in \mathbb{R}^+$. □

**Proposition 3.3.** *Let $a, b \in \mathbb{R}^+, a < b$. Then, we have*

$$H^{-1}\left(a, b\right) - A^{-1}\left(a, b\right) \leq \frac{2^{(2-3q)/q}\left(b-a\right)^2}{3^{1/q}}\left[A^{1/q}\left(\frac{1}{2a^{3q}}, \frac{1}{b^{3q}}\right) + A^{1/q}\left(\frac{1}{a^{3q}}, \frac{1}{2b^{3q}}\right)\right].$$

*Proof.* The assertion follows from Corollary 2.5 applied to the function $f(x) = 1/x$, $x \in \mathbb{R}^+$. □

## 4. Conclusion

New bounds for upper estimation of the Jensen functional in the class of $(s, m)$-convex functions can be gathered by revealing a new lemma or using different techniques. Also these inequalities have been gathered in the class of $(s, m)$-convex functions for some special cases of the Jensen functional. Therefore these results can be generalized for normalized Jensen functional or in the class of other classes of convex functions.

## References


[1] Dragomir, S. S., Agarwal, R. P., & Barnet, N. S. (2000) Inequalities for Beta and Gamma Functions via Some Classical and New Integral Inequalities. J. Ineq. Appl., 5, (2), 103-165.

[2] Dragomir, S. S., & Pearce, C.E.M. (2000). Selected Topics on Hermite-Hadamard Inequalities and Applications. RGMIA Monographs, Victoria University,[ONLINE. http://rgmia.org/papers/monographs/Master.pdf].

[3] Breckner, W. W. (1978). Stetigkeitsaussagen für eine klasse verallgemeinerter konvexer funktionen in topologischen linearen räumen. Publ. Inst. Math, 23 (37), 13-20.

[4] Kirmaci, U. S. (2017), On Some Hermite-Hadamard Type Inequalities for Twice Differentiable $(\alpha, m)$-convex functions and Applications. RGMIA Res. Rep. Coll., 20, Art. 51.

[5] Özdemir, M. E., Avci, M., & H. Kavurmaci. (2011). Hermite-Hadamard-type Inequalities via $(\alpha, m)$-convexity. Computer and Math. with Appl., 61, 2614-2620.

[6] Park, J. (2011). Generalization of Ostrowski-type inequalities for differentiable real $(s, m)$-convex mappings. Far East J. of Math. Sci., 49(2), 157-171.

[7] Park, J. (2011). Ostrowski type inequalities for mappings whose derivatives are $(s, m)$ $-$convex in the second sense. Far East J. of Math. Sci., 49 (2), 181-195.

[8] Sarikaya, M. Z., & Aktan, N. (2011). On the generalization of some integral inequalities and their applications. Math. and Comp. Modelling, 54, 2175-2182.

[9] Set, E., Sardari, M., Özdemir, M. E., & Rooin, J. (2012). On Generalizations of the Hadamard Inequality for $(\alpha, m)$-Convex Functions. Kyungpook Math. J., 52, 307-317

[10] Toader, G.H. (1984). Some Generalisations of the Convexity. Proc. Colloq. Approx. Optim, Cluj-Napoca, 329-338, Romania.

[11] Dragomir, S.S. (2006). Bounds for the normalized Jensen functional. Bull. Aust. Math. Soc. 74(3), 471-476.

**Bahtiyar BAYRAKTAR** for the photograph and short biography, see TWMS J.Appl. and Eng. Math.,V.6, No.2, 2016.

**Dr. Mustafa Gürbüz** received his B.S. degree, M.S. degree and Ph.D. degree from Atatürk University, Erzurum, Turkey. He is an assistant professor in the Department of Mathematics, Ağrı İbrahim Çeçen University, Ağrı, Turkey. His area of interest includes Convex Functions Theory and Functional Analysis.