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PERIODIC AND SEMI-PERIODIC EIGENVALUES OF HILL'S EQUATION WITH SYMMETRIC DOUBLE WELL POTENTIAL

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Abstract. In this paper, some estimates are derived explicitly for periodic and semi-periodic eigenvalues of Hill' s equation with symmetric double well potentials. Also, lengths of the instability intervals are obtained and bounds for the gaps of Dirichlet and Neumann eigenvalues are given by using an auxiliary eigenvalue problem.

Keywords: Hill's equation, symmetric double well potential, periodic and semi-periodic eigenvalues, instability intervals.

AMS Subject Classification: 34B30, 34L15

1. Introduction

Consider the following differential equation

$$y''(t) + [\lambda - q(t)]\, y(t) = 0, \quad t \in [0, a] \tag{1}$$

where λ is a real parameter and $q(t)$ is a real-valued, continuous and periodic function with period a. This equation is named as Hill's equation. There are some eigenvalue problems related to this equation. A periodic problem of (1) is defined with boundary conditions $y(0) = y(a),\ y'(0) = y'(a)$. This problem has a countable infinity of eigenvalues denoted by $\{\lambda_n\}$. The other problem is a semi-periodic problem which is given with (1) and boundary conditions $y(0) = -y(a),\ y'(0) = -y'(a)$. These eigenvalues are denoted by $\{\mu_n\}$. It is known that the two sets of eigenvalues satisfy the relation [2]

$$-\infty < \lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \mu_2 \leq \mu_3 < ....$$

The equation (1) is also identified with the Dirichlet boundary conditions $y(0) = y(a) = 0$ and the Neumann boundary conditions $y'(0) = y'(a) = 0$. The eigenvalues of Dirichlet and Neumann problems are shown by Λ_n and ν_n, respectively. For all of these eigenvalues, it is indicated that Eastham [2] gained for $n = 0, 1, 2, ...$

$$\mu_{2n} \leq \Lambda_{2n} \leq \mu_{2n+1}, \qquad \lambda_{2n+1} \leq \Lambda_{2n+1} \leq \lambda_{2n+2}, \tag{2}$$

and

$$\mu_{2n} \leq \nu_{2n+1} \leq \mu_{2n+1}, \qquad \lambda_{2n+1} \leq \nu_{2n+2} \leq \lambda_{2n+2}. \tag{3}$$

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Also, the instability intervals of (1) are described as $(-\infty, \lambda_0)$, (μ_{2n}, μ_{2n+1}), $(\lambda_{2n+1}, \lambda_{2n+2})$ and called the $zero-th$, $(2n+1)-th$ and $(2n+2)-th$ instability interval, respectively. The length of the $n-th$ instability interval of (1), whether it is absent or not, will be shown by l_n. Note that the absence of an instability interval means that there is a value of λ for which all solutions of (1) have either period a, or semi-period a. Instability intervals for Hill's equation with various types of restrictions on potential have been investigated by many authors over the years [2,3,6]. [5,9] are especially referred that $q(t)$ is a symmetric double well potential in these references. Some results about the first instability interval were obtained in [5] and the eigenvalue gap for Schrodinger operators on an interval with Dirichlet and Neumann boundary conditions was considered in [9].

In this paper, estimates are obtained about the periodic eigenvalues, the semi-periodic eigenvalues and the instability intervals of (1) with $q(t)$ being of a symmetric double well potential with mean value zero. A symmetric double well potential on $[0, a]$ means a continuous function $q(t)$ on $[0, a]$ which is symmetric on $[0, a]$ as well as on $\left[0, \frac{a}{2}\right]$ and non-increasing on $\left[0, \frac{a}{4}\right]$, that is, $q(t) = q(a-t) = q\left(\frac{a}{2} - t\right)$, mathematically. The main analysis for this study is based on [1], which involves $\Lambda_n(\tau)$ the eigenvalues of (1) on the interval $[\tau, \tau + a]$ where $0 \leq \tau < a$ with Dirichlet boundary conditions

$$y(\tau) = y(\tau + a) = 0. \tag{4}$$

This problem will be referred as "auxiliary eigenvalue problem". Here, note that this problem is equivalent to the following problem [1]:

$$y''(t) + [\lambda - q(t+\tau)]\, y(t) = 0, \tag{5}$$

$$y(0) = y(a) = 0. \tag{6}$$

Also, it is indicated that $q'(t)$ exists since a monotone function on an interval I is differentiable almost everywhere on I [4].

Now, consider the following asymptotic approximation previously obtained for the auxiliary eigenvalues ([6,7,8]) which will be used to prove our results. It was shown in ([8], p. 1275, for $N = 2$) as $n \to \infty$

$$\begin{aligned}
\Lambda_n^{1/2}(\tau) &= \frac{(n+1)\pi}{a} + \frac{a}{4(n+1)^2\pi^2} \\
&\quad \times \left[\cos\left(\frac{2(n+1)\pi}{a}\tau\right) \int_\tau^{\tau+a} q'(t) \sin\left(\frac{2(n+1)\pi}{a}t\right) dt \right. \\
&\quad \left. - \sin\left(\frac{2(n+1)\pi}{a}\tau\right) \int_\tau^{\tau+a} q'(t) \cos\left(\frac{2(n+1)\pi}{a}t\right) dt \right] \\
&\quad - \frac{a^2}{8(n+1)^3\pi^3} \int_0^a q^2(t)\, dt + o\left(n^{-3}\right).
\end{aligned} \tag{7}$$

The theorem below which involves the auxiliary eigenvalues $\Lambda_n(\tau)$ plays an important role to obtain periodic and semi-periodic eigenvalues [2]:

Theorem 1.1. *The ranges of* $\Lambda_{2n}(\tau)$ *and* $\Lambda_{2n+1}(\tau)$ *as functions of* τ *are* $[\mu_{2n}, \mu_{2n+1}]$ *and* $[\lambda_{2n+1}, \lambda_{2n+2}]$, *respectively.*

By this theorem and the fact that $\Lambda_n(\tau)$ is a continuous function of τ, it is observed that

$$\begin{aligned}
\max_\tau \Lambda_{2n}(\tau) &= \mu_{2n+1}, \ \min_\tau \Lambda_{2n}(\tau) = \mu_{2n}, \\
\max_\tau \Lambda_{2n+1}(\tau) &= \lambda_{2n+2}, \ \min_\tau \Lambda_{2n+1}(\tau) = \lambda_{2n+1}.
\end{aligned} \tag{8}$$

2. Main Results

Firstly, the following lemma will be given before proving results:

Lemma 2.1. *If $q(t)$ is a symmetric double well potential, then*
(i)

$$\int_{\tau}^{\tau+a} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt = \begin{cases} 4\int_0^{a/4} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

(ii)

$$\int_{\tau}^{\tau+a} q'(t)\cos\left(\frac{2(n+1)\pi}{a}t\right)dt = 0,$$

(iii)

$$\int_0^a q^2(t)\,dt = aq^2(a) + 2a\int_0^{a/4} q(t)q'(t)\,dt - 8\int_0^{a/4} tq(t)q'(t)\,dt.$$

Proof. (i) Since $q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)$ is a periodic function with period a

$$\begin{aligned}\int_{\tau}^{\tau+a} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt &= \int_0^{a} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt \\ &= \int_0^{a/2} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt \\ &\quad + \int_{a/2}^{a} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt \\ &= \int_0^{a/2} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt \\ &\quad - \int_{a/2}^{a} q'(a-t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt \\ &= 2\int_0^{a/2} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt.\end{aligned}$$

The last equality holds because $q(t)$ is a symmetric function and $q'(t)$ exists. Also, by using $q(t) = q\left(\frac{a}{2} - t\right)$

$$\begin{aligned}\int_{\tau}^{\tau+a} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt &= 2\int_0^{a/4} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt \\ &\quad +2\int_{a/4}^{a/2} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt \\ &= 2\int_0^{a/4} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt \\ &\quad -2\int_{a/4}^{a/2} q'\left(\frac{a}{2}-t\right)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt \\ &= 2\int_0^{a/4} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt \\ &\quad +2\cos(n+1)\pi\int_0^{a/4} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt.\end{aligned}$$

This proves the theorem.

(*ii*) It can be proved similarly.

(*iii*) Using integration by parts and $q(t) = q(a-t)$, it is obtained that

$$\begin{aligned}
\int_0^a q^2(t)\,dt &= tq^2(t)\big|_0^a - 2\int_0^a tq(t)\,q'(t)\,dt \\
&= aq^2(a) - 2\left\{\int_0^{a/2} tq(t)\,q'(t)\,dt + \int_{a/2}^a tq(t)\,q'(t)\,dt\right\} \\
&= aq^2(a) - 2\left\{\int_0^{a/2} tq(t)\,q'(t)\,dt - \int_{a/2}^a tq(a-t)\,q'(a-t)\,dt\right\} \\
&= aq^2(a) - 2\left\{\int_0^{a/2} tq(t)\,q'(t)\,dt + \int_{a/2}^0 (a-t)\,q(t)\,q'(t)\,dt\right\} \\
&= aq^2(a) + 2a\int_0^{a/2} q(t)\,q'(t)\,dt - 4\int_0^{a/2} tq(t)\,q'(t)\,dt.
\end{aligned}$$

Now, by writing $q(t) = q\left(\frac{a}{2} - t\right)$ in the last equation, it is calculated that

$$\begin{aligned}
\int_0^a q^2(t)\,dt &= aq^2(a) + 2a\left\{\int_0^{a/4} q(t)\,q'(t)\,dt + \int_{a/4}^{a/2} q(t)\,q'(t)\,dt\right\} \\
&\quad -4\left\{\int_0^{a/4} tq(t)\,q'(t)\,dt + \int_{a/4}^{a/2} tq(t)\,q'(t)\,dt\right\} \\
&= aq^2(a) + 2a\left\{\int_0^{a/4} q(t)\,q'(t)\,dt - \int_{a/4}^{a/2} q\left(\frac{a}{2}-t\right)q'\left(\frac{a}{2}-t\right)dt\right\} \\
&\quad -4\left\{\int_0^{a/4} tq(t)\,q'(t)\,dt - \int_{a/4}^{a/2} tq\left(\frac{a}{2}-t\right)q'\left(\frac{a}{2}-t\right)dt\right\} \\
&= aq^2(a) - 4\left\{\int_0^{a/4} tq(t)\,q'(t)\,dt + \int_{a/4}^0 \frac{a}{2}q(t)\,q'(t)\,dt + \int_0^{a/4} tq(t)\,q'(t)\,dt\right\}.
\end{aligned}$$

This proves the theorem. □

Theorem 2.1. *The periodic and semi-periodic eigenvalues of (1) satisfy, as $n \to \infty$*

$$\begin{aligned}
\begin{matrix}\lambda_{2n+1}\\ \lambda_{2n+2}\end{matrix} &= \frac{(n+1)^2\pi^2}{a} \mp \frac{2}{(n+1)\pi}\left|\int_0^{a/4} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt\right| \\
&\quad -\frac{a}{4(n+1)^2\pi^2}\left[aq^2(a) + 2a\int_0^{a/4} q(t)q'(t)\,dt - 8\int_0^{a/4} tq(t)q'(t)\,dt\right] \\
&\quad +o\left(n^{-2}\right),
\end{aligned}$$

and

$$\begin{aligned} \begin{matrix}\mu_{2n}\\ \mu_{2n+1}\end{matrix} &= \frac{(n+1)^2\pi^2}{a^2} \\ &-\frac{a}{4(n+1)^2\pi^2}\left[aq^2(a)+2a\int_0^{a/4}q(t)q'(t)\,dt-8\int_0^{a/4}tq(t)q'(t)\,dt\right] \\ &+o\left(n^{-2}\right). \end{aligned}$$

Proof. From 7 and Lemma 2.1, it is observed that if n is odd

$$\begin{aligned} \Lambda_n^{1/2}(\tau) &= \frac{(n+1)\pi}{a} \\ &+\frac{a}{4(n+1)^2\pi^2}\cos\left(\frac{2(n+1)\pi}{a}\tau\right)4\int_0^{a/4}q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt \\ &-\frac{a^2}{8(n+1)^3\pi^3}\left[aq^2(a)+2a\int_0^{a/4}q(t)q'(t)\,dt-8\int_0^{a/4}tq(t)q'(t)\,dt\right] \\ &+o\left(n^{-3}\right), \end{aligned}$$

if n is even

$$\begin{aligned} \Lambda_n^{1/2}(\tau) &= \frac{(n+1)\pi}{a} \\ &-\frac{a^2}{8(n+1)^3\pi^3}\left[aq^2(a)+2a\int_0^{a/4}q(t)q'(t)\,dt-8\int_0^{a/4}tq(t)q'(t)\,dt\right] \\ &+o\left(n^{-3}\right). \end{aligned}$$

Thus, it is easily found that if n is odd

$$\begin{aligned} \Lambda_n(\tau) &= \frac{(n+1)^2\pi^2}{a^2} \\ &+\frac{2}{(n+1)\pi}\cos\left(\frac{2(n+1)\pi}{a}\tau\right)\int_0^{a/4}q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt \\ &-\frac{a}{4(n+1)^2\pi^2}\left[aq^2(a)+2a\int_0^{a/4}q(t)q'(t)\,dt-8\int_0^{a/4}tq(t)q'(t)\,dt\right] \\ &+o\left(n^{-2}\right), \end{aligned}$$

if n is even

$$\begin{aligned} \Lambda_n(\tau) &= \frac{(n+1)^2\pi^2}{a^2} \\ &-\frac{a}{4(n+1)^2\pi^2}\left[aq^2(a)+2a\int_0^{a/4}q(t)q'(t)\,dt-8\int_0^{a/4}tq(t)q'(t)\,dt\right] \\ &+o\left(n^{-2}\right). \end{aligned}$$

By minimizing and maximizing last two equations, it is written that if n is odd

$$\begin{aligned}\min_{\tau}\Lambda_n(\tau) &= \frac{(n+1)^2\pi^2}{a}-\frac{2}{(n+1)\pi}\left|\int_0^{a/4}q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt\right| \\ &-\frac{a}{4(n+1)^2\pi^2}\left[aq^2(a)+2a\int_0^{a/4}q(t)q'(t)\,dt-8\int_0^{a/4}tq(t)q'(t)\,dt\right] \\ &+o\left(n^{-2}\right), \qquad (9)\end{aligned}$$

$$\begin{aligned}\max_{\tau}\Lambda_n(\tau) &= \frac{(n+1)^2\pi^2}{a}+\frac{2}{(n+1)\pi}\left|\int_0^{a/4}q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt\right| \\ &-\frac{a}{4(n+1)^2\pi^2}\left[aq^2(a)+2a\int_0^{a/4}q(t)q'(t)\,dt-8\int_0^{a/4}tq(t)q'(t)\,dt\right] \\ &+o\left(n^{-2}\right), \qquad (10)\end{aligned}$$

and if n is even

$$\begin{aligned}\min_{\tau}\Lambda_n(\tau) &= \max_{\tau}\Lambda_n(\tau)=\frac{(n+1)^2\pi^2}{a^2} \\ &-\frac{a}{4(n+1)^2\pi^2}\left[aq^2(a)+2a\int_0^{a/4}q(t)q'(t)\,dt-8\int_0^{a/4}tq(t)q'(t)\,dt\right] \\ &+o\left(n^{-2}\right). \qquad (11)\end{aligned}$$

From (8)

$$\min_{\tau}\Lambda_{2n+1}(\tau)=\lambda_{2n+1},\quad \max_{\tau}\Lambda_{2n+1}(\tau)=\lambda_{2n+2},$$

so now with the values of $\min_{\tau}\Lambda_{2n+1}(\tau)$ and $\max_{\tau}\Lambda_{2n+1}(\tau)$ in (9) and (10), λ_{2n+1} and λ_{2n+2} are obtained. Also, from (8)

$$\min_{\tau}\Lambda_{2n}^{1/2}(\tau)=\mu_{2n},\quad \max_{\tau}\Lambda_{2n}^{1/2}(\tau)=\mu_{2n+1}$$

and by using this with (11), find μ_{2n} and μ_{2n+1} as required. □

Corollary 2.1. *Let $q(t)$ be a symmetric double well potential on $[0,a]$. Then, as $n\to\infty$*

$$\begin{aligned}\begin{matrix}\Lambda_{2n+1}-\Lambda_{2n}\\ \nu_{2n+2}-\nu_{2n+1}\end{matrix} &\geq \frac{(2n+1)^2\pi^2}{a^2}-\frac{1}{(n+1)\pi}\left|\int_0^{a/4}q'(t)\sin\left(\frac{4(n+1)\pi}{a}t\right)dt\right| \\ &-\frac{(4n+3)a}{16(n+1)^2(2n+1)\pi^2}\left\{aq^2(a)+2a\int_0^{a/4}q(t)q'(t)\,dt-8\int_0^{a/4}tq(t)q'(t)\,dt\right\} \\ &+o\left(n^{-2}\right)\end{aligned}$$

and

$$\begin{aligned}\begin{matrix}\Lambda_{2n+1}-\Lambda_{2n}\\ \nu_{2n+2}-\nu_{2n+1}\end{matrix} &\leq \frac{(2n+1)^2\pi^2}{a^2}-\frac{1}{(n+1)\pi}\left|\int_0^{a/4}q'(t)\sin\left(\frac{4(n+1)\pi}{a}t\right)dt\right| \\ &-\frac{(4n+3)a}{16(n+1)^2(2n+1)\pi^2}\left\{aq^2(a)+2a\int_0^{a/4}q(t)q'(t)\,dt-8\int_0^{a/4}tq(t)q'(t)\,dt\right\} \\ &+o\left(n^{-2}\right).\end{aligned}$$

Proof. By recasting (2) and (3) and using Theorem 2.1, the corollary is proved as required. □

Corollary 2.2. *l_n satisfies, as $n \to \infty$*

$$\begin{aligned} l_{2n+1} &= o\left(n^{-3}\right), \\ l_{2n+2} &= \frac{a}{4\left(n+1\right)\pi^2}\left|\int_0^{a/4} q'\left(t\right)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt\right| \\ &\quad +o\left(n^{-3}\right). \end{aligned}$$

Proof. Proof of the corollary follows from Theorem 2.1 and definition of l_n. □

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