TWMS J. App. Eng. Math. V.10, N.2, 2020, pp. 360-369

SUBCLASSES OF MULTIVALENT FUNCTIONS OF COMPLEX ORDER ASSOCIATED WITH SIGMOID FUNCTION AND BERNOULLI LEMNISCATE

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Abstract. In this present work, subclasses of multivalent functions of complex order associated with simple logistic sigmoid activation function and Bernoulli Lemniscate were investigated. Early few coefficient bounds, relevant connection to Fekete-Szego inequalities and second Hankel determinant for the two classes $\mathcal{M}_L(p, \lambda, b, \Phi_{m,n})$ and $\mathcal{G}_L(p, \lambda, b, \Phi_{m,n})$ were obtained. Our results are new in this direction and give birth to many corollaries.

Keywords: Analytic function, Univalent function, Subordination, Coefficient bounds, Fekete-Szego Inequality, Hankel determinant, Sigmoid function, Bernoulli Lemniscate.

AMS Subject Classification: 30C45, 30C50

1. Introduction

Special functions deal with an information process that is inspired by the way nervous system such as brain processes information. It comprises of large number of highly interconnected processing elements (neurones) working together to solve a specific problem. The functions are outshinning by other fields like real analysis, algebra, topology, functional analysis, differential equations and so on because it mimicks the way human brain works. They can be programmed to solve a specific problem and it can also be trained by examples.

Special functions can be categorized into three namely, threshold function, ramp function and the logistic sigmoid function. The most important one among all is the logistic sigmoid function because of its gradient descendent learning algorithm. It can be evaluated in different ways, most especially by truncated series expansion. The logistic sigmoid function of the form

$$h(z) = \frac{1}{1 + e^{-z}} \tag{1}$$

is differentiable and has the following properties:

(i) it outputs real numbers between 0 and 1.

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§ Manuscript received: August 8, 2018; accepted: February 10, 2019.

(ii) it maps a very large input domain to a small range of outputs.
(iii) it never loses information because it is a one-to-one function.
(iv) it increases monotonically.
With all the aforementioned properties above, it is clear that logistic sigmoid function is very useful in geometric functions theory (see details in [3], [5], [8] and [9]).

Let A represent the class of analytic functions f defined by the unit disc $U = \{z : |z| < 1\}$ and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{2}$$

and normalized by $f(0) = f'(0) - 1 = 0$. Also, let S be the well-known subclass of A consisting of functions which are univalent. Recall that, S^* and K are the two usual classes of starlike and convex functions which their geometric conditions satisfies $Re\left(\frac{zf'(z)}{f(z)}\right) > 0$ and $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$.
Two analytic functions are said to be subordinate to each other written as $f \prec g$, if there exists a Schwartz function $\omega(z)$ which is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$, for all $z \in U$, such that $f(z) = g(\omega(z))$, and $f(U) \subset g(U)$.
Sokol and Thomas [11] introduced and studied the class S_L^* in the unit disc U, normalized by $f(0) = f' - 1 = 0$ and satisfying the condition

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z} =: q(z), \quad z \in U, \tag{3}$$

where the branch of the square root is choosen to be $q(0) = 1$.
It also noted that the set $q(U)$ lies in the region bounded by the right loop of the lemniscate of Bernoulli $\gamma_1 : (x^2 + y^2)^2 - 2(x^2 - y^2) = 0$.
Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{n=2}^{\infty} a_n z^{n+p} \tag{4}$$

which are analytic and $p-$valent in the unit disc $U = \{z : |z| < 1\}$.
Let E be the class of bounded functions

$$\omega(z) = \sum_{n=1}^{\infty} c_n z^n \tag{5}$$

which are analytic in the unit disc and satisfying the conditions $\omega(0) = 0$ and $|\omega(z)| < 1$ in U.
A function $f \in A_p$ is said to be in the class $f \in S^*_{b,p}(\varphi)$ if

$$1 + \frac{1}{b}\left[\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right] \prec \varphi(z) \qquad (p \in N, \quad z \in U), \tag{6}$$

and a function $f \in A_p$ is said to be in the class $f \in C_{b,p}(\varphi)$ if

$$1 - \frac{1}{b} + \frac{1}{bp}\left[1 + \frac{zf''(z)}{f'(z)}\right] \prec \varphi(z) \qquad (p \in N, \quad z \in U). \tag{7}$$

The above two classes were studied and investigated by [1]. Varying the parameters involved in (6) and (7), we have some other subclasses of analytic functions studied by many authors. For instance, taking $b = 1$ in (6) and (7), we obtain the classes of functions studied by [2]. Also, setting $p = b = 1$ gives the classes of functions introduced by [4]. The two classes become the well-known starlike and convex functions when $\varphi = \frac{1+z}{1-z}$. For $p = 1$ and $\varphi = \frac{1+z}{1-z}$ reduces to the classes of functions investigated [6] and [14]. Recently, [12] investigated the two classes of functions $\mathcal{M}_{p,\lambda}(b, \Phi_{m,n})$ and $\mathcal{G}_{p,\lambda}(b, \Phi_{m,n})$ which their geometric conditions satisfies

$$p + \frac{1}{b}\left[\frac{zf'(z) + \lambda z^2 f''(z)}{\lambda z f'(z) + (1-\lambda)f(z)} - p\right] > 0$$

and

$$p + \frac{1}{b}\left[\frac{zf'(z)}{f(z)} + \lambda \frac{z^2 f''(z)}{f(z)} - p\right] > 0,$$

for $0 \leq \lambda < 1$ and $\Phi_{m,n}$ is a modified sigmoid function and the interesting result were obtained.
In 1933, Fekete and Szego gave the sharp bound for the function $|a_3 - \mu a_2^2|$ for the class S of univalent functions when μ is real. The determination of the sharp bounds for the functional $|a_3 - \mu a_2^2|$ is known as the relevant connection to the Fekete-Szego inequality and this has been studied by many researchers for different subclasses of univalent functions. Noonman and Thomas [7] stated the q^{th} Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdot & \cdot & \cdot & a_{n+q-1} \\ a_{n+1} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n+q-1} & \cdot & & \cdot & \cdot & a_{n+2q-2} \end{vmatrix}.$$

It can be observed from the determinant that Fekete and Szego functional is $H_2(1)$ and the above determinant has been considered my many researchers in different perspectives. For $q = 2$ and $n = 2$, we have

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}.$$

which is the second Hankel determinant.
Motivation by earlier work done by [3], [10], [12] and [13], a new subclasses were introduced for the $p-$valent analytic functions of complex order related to Bernoulli Lemniscate and simple logistic sigmoid activation function. The coefficient bounds, Fekete-Szego inequality and second Hankel determinant for the two classes defined were obtained.The results are new and generates many corollaries.
For the purpose of our discussion, we shall give the following Lemmas and definitions;

Lemma 1.1. *[11] If a function $p \in P$ is given by $p(z) = 1 + p_1 z + p_2 z^2 + ...(z \in U)$, then $|p_k| \leq 2, n \in N$ where P is the family of all functions analytic in U for which $p(0) = 1$ and $Re(p(z)) > 0 (z \in U)$.*

Lemma 1.2 (Fadipe-Joseph et al.[3]). *Let h be a sigmoid function and*

$$\Phi(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right)^m \tag{8}$$

then $\Phi(z) \in P$, $|z| < 1$ where $\Phi(z)$ is a modified sigmoid function.

Lemma 1.3 (Fadipe-Joseph et al.[3]). *Let*

$$\Phi_{m,n}(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n\right) \tag{9}$$

then $|\Phi_{m,n}(z)| < 2$.

Lemma 1.4 (Fadipe-Joseph et al.[3]). *If* $\Phi(z) \in P$ *and it is starlike, then* f *is a normalized univalent function of the form (1.2).*

Setting $m = 1$, Fadipe-Joseph et al.[3] remarked that

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

where $c_n = \frac{(-1)^{n+1}}{2n!}$. As such, $|c_n| \leq 2$, $n = 1, 2, 3, ...$ and the result is sharp for each n.

Definition 1.1. *For* $b \in \mathcal{C}$, *let the class* $\mathcal{M}_L(p, \lambda, b, \Phi_{m,n})$ *denote the subclass of* A_p *consisting of functions of the form (4) and satisfying the following condition*

$$p + \frac{1}{b}\left[\frac{zf'(z) + \lambda z^2 f''(z)}{\lambda z f'(z) + (1-\lambda) f(z)} - p\right] \prec p\sqrt{1+z} \tag{10}$$

where the branch of the square root is choosen to be $q(0) = 1$, $0 \leq \lambda < 1$ *and* $\Phi_{m,n}$ *is a modified sigmoid function.*

Definition 1.2. *For* $b \in \mathcal{C}$, *let the class* $\mathcal{G}_L(p, \lambda, b, \Phi_{m,n})$ *denote the subclass of* A_p *consisting of functions of the form (4) and satisfying the following condition*

$$p + \frac{1}{b}\left[\frac{zf'(z)}{f(z)} + \lambda \frac{z^2 f''(z)}{f(z)} - p\right] \prec p\sqrt{1+z} \tag{11}$$

where the branch of the square root is choosen to be $q(0) = 1$, $0 \leq \lambda < 1$ *and* $\Phi_{m,n}$ *is a modified sigmoid function.*

It also noted for two definitions that the set $q(U)$ lies in the region bounded by the right loop of the lemniscate of Bernoulli $\gamma_1 : (x^2 + y^2)^2 - 2(x^2 - y^2) = 0$.

2. Main Results

Coeffiecient Bounds.

Theorem 2.1. *If* $f \in A_p$ *of the form (4) is belonging to* $\mathcal{M}_L(p, \lambda, b, \Phi_{m,n})$, *then*

$$|a_{p+1}| \leq \frac{p|b|[1 + \lambda(p-1)]}{8[1 + \lambda p]}, \tag{12}$$

$$|a_{p+2}| \leq \frac{p|b|[1 + \lambda(p-1)]|2bp - 5|}{256[1 + \lambda(p+1)]} \tag{13}$$

and

$$|a_{p+3}| \leq \frac{p|b|[1 + \lambda(p-1)]|6p^2b^2 - 30pb + 7|}{9216[1 + \lambda(p+2)]}. \tag{14}$$

Proof. As $f(z) \in \mathcal{M}_L(p, \lambda, b, \Phi_{m,n})$, therefore

$$p + \frac{1}{b}\left[\frac{zf'(z) + \lambda z^2 f''(z)}{\lambda z f'(z) + (1-\lambda) f(z)} - p\right] = p\sqrt{1 + \omega(z)} = p\sqrt{1 + \frac{\Phi_{m,n}(z) - 1}{\Phi_{m,n}(z) + 1}} \tag{15}$$

where

$$\Phi_{m,n}(z) = 1 + \frac{z}{2} - \frac{z^3}{24} + \frac{z^5}{240} - \frac{z^6}{64} + \frac{779 z^7}{20160} - ... \tag{16}$$

Using (16) in (15) can be expanded as

$(1+\lambda p)a_{p+1}z+2(1+\lambda(p+1))a_{p+2}z^2+3(1+\lambda(p+2))a_{p+3}z^3+4(1+\lambda(p+3))a_{p+4}z^4+...$

$$=bp\left[\frac{z}{8}-\frac{5z^2}{128}+\frac{224z^3}{98304}+...\right]\left[(1+\lambda(p-1))+(1+\lambda p)a_{p+1}z+(1+\lambda(p+1))a_{p+2}z^2+(1+\lambda(p+2))a_{p+3}z^3+...\right]. \tag{17}$$

Equating the coefficient of z, z^2 and z^3 in (17). we obtain

$$a_{p+1}=\frac{pb[1+\lambda(p-1)]}{8[1+\lambda p]}, \tag{18}$$

$$a_{p+2}=\frac{pb[1+\lambda(p-1)](2bp-5)}{256[1+\lambda(p+1)]} \tag{19}$$

and

$$a_{p+3}=\frac{pb[1+\lambda(p-1)](3p^2b^2-15pb+7)}{9216[1+\lambda(p+2)]}. \tag{20}$$

Results (12), (13) and (14) can be easily obtained from (18), (19) and (20) respectively. □

For $\lambda=0$ in Theorem 2.1, we obtain

Corollary 2.1. *If $f\in A_p$ of the form (4) is belonging to $\mathcal{M}_L(p,\lambda,b,\Phi_{m,n})$, then*

$$|a_{p+1}|\le\frac{p|b|}{8}, \tag{21}$$

$$|a_{p+2}|\le\frac{p|b||2bp-5|}{256} \tag{22}$$

and

$$|a_{p+3}|\le\frac{p|b||3p^2b^2-15pb+7|}{9216}. \tag{23}$$

Setting $\lambda=1$ in Theorem 2.1, we have

Corollary 2.2. *If $f\in A_p$ of the form (4) is belonging to $\mathcal{M}_L(p,1,b,\Phi_{m,n})$, then*

$$|a_{p+1}|\le\frac{p^2|b|}{8[1+p]}, \tag{24}$$

$$|a_{p+2}|\le\frac{p^2|b||2bp-5|}{256[2+p]} \tag{25}$$

and

$$|a_{p+3}|\le\frac{p^2|b||3p^2b^2-15pb+7|}{9216[3+p]}. \tag{26}$$

Putting $p=1$ in Theorem 2.1 gives

Corollary 2.3. *If $f\in A$ of the form (2) is belonging to $\mathcal{M}_L(1,\lambda,b,\Phi_{m,n})$, then*

$$|a_2|\le\frac{|b|}{8[1+\lambda]}, \tag{27}$$

$$|a_3|\le\frac{|b||2b-5|}{256[1+2\lambda]} \tag{28}$$

and

$$|a_4| \leq \frac{|b||3b^2 - 15b + 7|}{9216[1 + 3\lambda]}. \tag{29}$$

Theorem 2.2. *If $f \in A_p$ of the form (4) is belonging to $\mathcal{G}_L(p, \lambda, b, \Phi_{m,n})$, then*

$$|a_{p+1}| \leq \frac{p|b|}{8[1 + \lambda p(p+1)]}, \tag{30}$$

$$|a_{p+2}| \leq \frac{p|b||2bp - 5[1 + \lambda p(p+1)]|}{128[1 + \lambda p(p+1)][2 + \lambda(p+1)(p+2)]} \tag{31}$$

and

$$|a_{p+3}| \leq \frac{p|b|}{1024[3 + \lambda(p+2)(p+3)]} \left| \frac{bp[2bp - 5[1 + \lambda p(p+1)] - 5bp[2 + \lambda(p+1)(p+2)]}{[1 + \lambda p(p+1)][2 + \lambda(p+1)(p+2)} + \frac{7}{3} \right|. \tag{32}$$

Proof. Since $f \in \mathcal{G}_L(p, \lambda, b, \Phi_{m,n})$, therefore

$$p + \frac{1}{b}\left[\frac{zf'(z)}{f(z)} + \lambda \frac{z^2 f''(z)}{f(z)} - p\right] = p\sqrt{1 + \omega(z)} = p\sqrt{1 + \frac{\Phi_{m,n}(z) - 1}{\Phi_{m,n}(z) + 1}}. \tag{33}$$

Using (16) in (33) yields

$$\lambda p(p-1) + (1 + \lambda p(p+1))a_{p+1}z + (2 + \lambda(p+1)(p+2))a_{p+2}z^2 + (3 + \lambda(p+2)(p+3))a_{p+3}z^3 + (4 + \lambda(p+3)(p+4))a_{p+4}z^4 + ...$$

$$= bp\left[\frac{z}{8} - \frac{5z^2}{128} + \frac{224z^3}{98304} + ...\right]\left[1 + a_{p+1}z + a_{p+2}z^2 + a_{p+3}z^3 + ...\right] \tag{34}$$

Equating the coefficients of z, z^2 and z^3 in (34), we obtain

$$a_{p+1} = \frac{pb}{8[1 + \lambda p(p+1)]}, \tag{35}$$

$$a_{p+2} = \frac{pb(2bp - 5[1 + \lambda p(p+1)])}{128[1 + \lambda p(p+1)][2 + \lambda(p+1)(p+2)]} \tag{36}$$

and

$$a_{p+3} = \frac{pb}{1024[3 + \lambda(p+2)(p+3)]}\left[\frac{bp[2bp - 5[1 + \lambda p(p+1)] - 5bp[2 + \lambda(p+1)(p+2)]}{[1 + \lambda p(p+1)][2 + \lambda(p+1)(p+2)]} + \frac{7}{3}\right]. \tag{37}$$

Results (35), (36) and (37) can be easily obtained from (30), (31) and (32) respectively □

For $\lambda = 0$ in Theorem 2.2, we have

Corollary 2.4. *If $f \in A_p$ of the form (4) is belonging to $\mathcal{G}_L(p, 0, b, \Phi_{m,n})$, then*

$$|a_{p+1}| \leq \frac{p|b|}{8}, \tag{38}$$

$$|a_{p+2}| \leq \frac{p|b||2bp - 5|}{256} \tag{39}$$

and

$$|a_{p+3}| \leq \frac{p|b|}{3072}\left|\frac{bp[2bp - 5 - 10bp}{2} + \frac{7}{3}\right|. \tag{40}$$

Putting $\lambda = 1$ in Theorem 2.2, we have

Corollary 2.5. *If $f \in A_p$ of the form (4) is belonging to $\mathcal{G}_L(p, 1, b, \Phi_{m,n})$, then*

$$|a_{p+1}| \leq \frac{p|b|}{8[1 + p(p + 1)]}, \tag{41}$$

$$|a_{p+2}| \leq \frac{p|b||2bp - 5[1 + p(p + 1)]|}{128[1 + p(p + 1)][2 + (p + 1)(p + 2)]} \tag{42}$$

and

$$|a_{p+3}| \leq \frac{p|b|}{1024[3 + (p + 2)(p + 3)]} \left| \frac{bp[2bp - 5[1 + p(p + 1)] - 5bp[2 + (p + 1)(p + 2)]}{[1 + p(p + 1)][2 + (p + 1)(p + 2)} + \frac{7}{3} \right|. \tag{43}$$

Setting $p = 1$ in Theorem 2.2, gives

Corollary 2.6. *If $f \in A$ of the form (2) is belonging to $\mathcal{G}_L(1, \lambda, b, \Phi_{m,n})$, then*

$$|a_2| \leq \frac{|b|}{8[1 + 2\lambda]}, \tag{44}$$

$$|a_3| \leq \frac{|b||2b - 5[1 + 2\lambda]|}{128[1 + 2\lambda][2 + 6\lambda]} \tag{45}$$

and

$$|a_4| \leq \frac{|b|}{1024[3 + 12\lambda]} \left| \frac{b[2b - 5[1 + 2\lambda] - 5bp[2 + 6\lambda]}{[1 + 2\lambda][2 + 6\lambda} + \frac{7}{3} \right|. \tag{46}$$

Fekete-Szego Inequality.

Theorem 2.3. *If $f \in A_p$ of the form (4) is belonging to $\mathcal{M}_L(p, \lambda, b, \Phi_{m,n})$, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p|b|[1 + \lambda(p - 1)]}{64[1 + \lambda p]^2} \left| \frac{(2bp - 5)(1 + \lambda p)^2}{4[1 + \lambda p(p + 1)]} - \mu pb[1 + \lambda(p - 1)] \right|. \tag{47}$$

Proof. From (18) and (19), we obtain

$$a_{p+2} - \mu a_{p+1}^2 = \frac{pb[1 + \lambda(p - 1)]}{64[1 + \lambda p]^2} \left[\frac{(2bp - 5)(1 + \lambda p)^2}{4[1 + \lambda p(p + 1)]} - \mu pb[1 + \lambda(p - 1)] \right]. \tag{48}$$

Hence, (48) gives the desired results.

□

Taking $\lambda = 0$ in Theorem 2.3, implies

Corollary 2.7. *If $f \in A_p$ of the form (4) is belonging to $\mathcal{M}_L(p, 0, b, \Phi_{m,n})$, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p|b|}{64} \left| \frac{(2bp - 5)}{4} - \mu pb \right|.$$

Setting $\lambda = 1$ in Theorem 2.3, we have

Corollary 2.8. *If $f \in A_p$ of the form (4) is belonging to $\mathcal{M}_L(p, 1, b, \Phi_{m,n})$, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p^2|b|}{64[1 + p]^2} \left| \frac{(2bp - 5)(1 + p)^2}{4[1 + p(p + 1)]} - \mu p^2 b \right|.$$

Putting $p = 1$ in Theorem 2.3, then

Corollary 2.9. *If $f \in A$ of the form (2) is belonging to $\mathcal{M}_L(1, \lambda, b, \Phi_{m,n})$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|}{64[1+\lambda]^2}\left|\frac{(2b-5)(1+\lambda)^2}{4[1+2\lambda]} - \mu b\right|. \tag{49}$$

Theorem 2.4. *If $f \in A_p$ of the form (4) is belonging to $\mathcal{G}_L(p, \lambda, b, \Phi_{m,n})$, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p|b|}{64[1+\lambda p(p+1)]^2}\left|\frac{[2bp-5[1+\lambda p(p+1)]][1+\lambda p(p+1)]}{2[2+\lambda(p+1)(p+2)]} - \mu pb\right|. \tag{50}$$

Proof. Using (35) and (36), we obtain

$$a_{p+2} - \mu a_{p+1}^2 = \frac{pb}{64[1+\lambda p(p+1)]^2}\left[\frac{[2bp-5[1+\lambda p(p+1)]][1+\lambda p(p+1)]}{2[2+\lambda(p+1)(p+2)]} - \mu pb\right]. \tag{51}$$

□

Taking $\lambda = 0$ in Theorem 2.4, gives

Corollary 2.10. *If $f \in A_p$ of the form (4) is belonging to $\mathcal{G}_L(p, 0, b, \Phi_{m,n})$, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p|b|}{64}\left|\frac{[2bp-5]}{4} - \mu pb\right|. \tag{52}$$

Also, setting $\lambda = 1$ in Theorem 2.4, implies

Corollary 2.11. *If $f \in A_p$ of the form (4) is belonging to $\mathcal{G}_L(p, 1, b, \Phi_{m,n})$, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \frac{p|b|}{64[1+p(p+1)]^2}\left|\frac{[2bp-5[1+p(p+1)]][1+p(p+1)]}{2[2+(p+1)(p+2)]} - \mu pb\right|. \tag{53}$$

Putting $p = 1$ in Theorem 2.4, to obtain

Corollary 2.12. *If $f \in A$ of the form (2) is belonging to $\mathcal{G}_L(1, \lambda, b, \Phi_{m,n})$, then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|}{64[1+2\lambda]^2}\left|\frac{[2bp-5[1+2\lambda]][1+2\lambda]}{2[2+6\lambda]} - \mu b\right|. \tag{54}$$

Second Hankel Determinant.

Theorem 2.5. *If $f \in A_p$ of the form (4) is belonging to $\mathcal{M}_L(p, \lambda, b, \Phi_{m,n})$, then*

$$\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \leq \frac{p^2|b|^2(1+\lambda(p-1))^2}{8192(1+\lambda(p+1))^2}\left|\frac{(3p^2b^2-15pb+7)(1+\lambda(p+1))^2}{9(1+\lambda p)(1+\lambda(p+2))} - \mu\frac{(2bp-5)^2}{8}\right| \tag{55}$$

Proof. From (18), (19) and (20), we have

$$a_{p+1}a_{p+3} - a_{p+2}^2 = \frac{p^2b^2(1+\lambda(p-1))^2}{8192(1+\lambda(p+1))^2}\left[\frac{(3p^2b^2-15pb+7)(1+\lambda(p+1))^2}{9(1+\lambda p)(1+\lambda(p+2))} - \mu\frac{(2bp-5)^2}{8}\right]. \tag{56}$$

□

Taking $\lambda = 0$ in Theorem 2.5, implies

Corollary 2.13. *If $f \in A_p$ of the form (4) is belonging to $\mathcal{M}_L(p, 0, b, \Phi_{m,n})$, then*

$$\left|a_{p+1}a_{p+3} - a_{p+2}^2\right| \leq \frac{p^2|b|^2}{8192}\left|\frac{3p^2b^2-15pb+7}{9} - \mu\frac{(2bp-5)^2}{8}\right|. \tag{57}$$

Setting $\lambda = 1$ in Theorem 2.5, gives

Corollary 2.14. *If $f \in A_p$ of the form (4) is belonging to $\mathcal{M}_L(p,1,b,\Phi_{m,n})$, then*

$$\left|a_{p+1}a_{p+3}-a_{p+2}^2\right| \le \frac{p^4|b|^2}{8192(2+p)^2}\left|\frac{(3p^2b^2-15pb+7)(2+p)^2}{9(1+p)(3+p)}-\mu\frac{(2bp-5)^2}{8}\right| \tag{58}$$

Putting $p=1$ in Theorem 2.5, to obtain

Corollary 2.15. *If $f \in A$ of the form (2) is belonging to $\mathcal{M}_L(1,\lambda,b,\Phi_{m,n})$, then*

$$\left|a_2a_4-a_3^2\right| \le \frac{|b|^2}{8192(1+2\lambda)^2}\left|\frac{(3b^2-15b+7)(1+2\lambda)^2}{9(1+\lambda)(1+3\lambda)}-\mu\frac{(2b-5)^2}{8}\right| \tag{59}$$

Theorem 2.6. *If $f \in A_p$ of the form (4) is belonging to $\mathcal{G}_L(p,\lambda,b,\Phi_{m,n})$, then*

$$\left|a_{p+1}a_{p+3}-a_{p+2}^2\right| \le \frac{p^2|b|}{8192} \tag{60}$$

$$\left|\frac{1}{[1+\lambda p(p+1)][3+\lambda(p+2)(p+3)]}\left[\frac{bp[2bp-5(1+\lambda p(p+1))]-5bp[2+\lambda(p+1)(p+2)]}{[1+\lambda p(p+1)][2+\lambda(p+1)(p+2)]}+\frac{7}{3}\right]-\mu\frac{[2bp-5(1+\lambda p(p+1))]^2}{2[1+\lambda p(p+1)]^2[2+\lambda(p+1)(p+2)]^2}\right|. \tag{61}$$

Proof. From (35), (36) and (37), we have

$$a_{p+1}a_{p+3}-a_{p+2}^2 = \frac{p^2|b|}{8192} \tag{62}$$

$$\left[\frac{1}{[1+\lambda p(p+1)][3+\lambda(p+2)(p+3)]}\left[\frac{bp[2bp-5(1+\lambda p(p+1))]-5bp[2+\lambda(p+1)(p+2)]}{[1+\lambda p(p+1)][2+\lambda(p+1)(p+2)]}+\frac{7}{3}\right]-\mu\frac{[2bp-5(1+\lambda p(p+1))]^2}{2[1+\lambda p(p+1)]^2[2+\lambda(p+1)(p+2)]^2}\right]. \tag{63}$$

□

Taking $\lambda=0$ in Theorem 2.5, gives

Corollary 2.16. *If $f \in A_p$ of the form (4) is belonging to $\mathcal{G}_L(p,0,b,\Phi_{m,n})$, then*

$$\left|a_{p+1}a_{p+3}-a_{p+2}^2\right| \le \frac{p^2|b|}{8192}\left|\frac{1}{3}\left[\frac{bp[2bp-5]-10bp}{2}+\frac{7}{3}\right]-\mu\frac{[2bp-5]^2}{8}\right|. \tag{64}$$

Putting $\lambda=1$ in Theorem 2.5, implies

Corollary 2.17. *If $f \in A_p$ of the form (4) is belonging to $\mathcal{G}_L(p,1,b,\Phi_{m,n})$, then*

$$\left|a_{p+1}a_{p+3}-a_{p+2}^2\right| \le \frac{p^2|b|}{8192} \tag{65}$$

$$\left|\frac{1}{[1+p(p+1)][3+(p+2)(p+3)]}\left[\frac{bp[2bp-5(1+p(p+1))]-5bp[2+(p+1)(p+2)]}{[1+p(p+1)][2+(p+1)(p+2)]}+\frac{7}{3}\right]-\mu\frac{[2bp-5(1+p(p+1))]^2}{2[1+p(p+1)]^2[2+(p+1)(p+2)]^2}\right|. \tag{66}$$

Setting $p=1$ in Theorem 2.5, gives

Corollary 2.18. *If $f \in A$ of the form (2) is belonging to $\mathcal{G}_L(1,\lambda,b,\Phi_{m,n})$, then*

$$\left|a_2a_4-a_3^2\right| \le \frac{|b|}{8192}\left|\frac{1}{[1+2\lambda][3+12\lambda]}\left[\frac{b[2b-5(1+2\lambda)]-5b[2+6\lambda]}{[1+2\lambda][2+6\lambda]}+\frac{7}{3}\right]-\mu\frac{[2bp-5(1+2\lambda)]^2}{2[1+2\lambda]^2[2+6\lambda]^2}\right|. \tag{67}$$

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