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ON THE STUDY OF FUZZY HILBERT SPACES BY FUZZY ISOMETRIC ISOMORPHISMS

B. DARABY[1], A. ROSTAMI[1], A. RAHIMI[1], §

Abstract. In this paper, by defining a fuzzy inner product on the dual of real fuzzy Hilbert space, we show that the dual of fuzzy Hilbert space is also fuzzy Hilbert space. Then the concepts of fuzzy linear operator and fuzzy isomentric are defined. It is shown that each real fuzzy Hilbert space is isometricaly isomorphic to the dual of the fuzzy Hilbert space.

Keywords: Fuzzy Hilbert space; Fuzzy isometric; Fuzzy isomorphism; Fuzzy dual space.

AMS Subject Classification: 46S40, 47S40, 26E50

1. Introduction

The idea of the fuzzy norm on a linear space was introduced at the first by Katsaras [10]. Later, many other mathematicians like Felbin [8], Cheng and Mordeson [5], Bag and Samanta [2, 3], and so on introduced the notion of fuzzy normed spaces in different approachs. A large number of papers have been published in fuzzy normed linear spaces. But studies on fuzzy inner product spaces are relatively new and a few researchs have been done in fuzzy inner product space $\mathbb{R}$. Biswas [4] and A. M. El-Abyad, H. M. Hamouly in [1] gave a definition of fuzzy inner product space as associated fuzzy norm function. Recently B. Daraby and et al. in [6] showed that the classical Hilbert space is a Felbin-fuzzy Hilbert space and thus the results obtained in classical Hilbert spaces can be established in Felbin-fuzzy Hilbert spaces in general. Moreover by an example, they showed that each Felbin-fuzzy Hilbert space is not necessarily classical Hilbert space. Also, they presented some properties of Felbin-type fuzzy inner product space and fuzzy bounded linear operators on the same space with some operator norms in [7]. This paper contains three sections. In Section 1 we present some definitions and basic theorems that will be used in the next sections. In Section 2, by defining a fuzzy inner product μ^* on H^* (the set of all strongly fuzzy bounded linear functionals over H), we show that μ^* is a fuzzy inner product on H^* and H^* is a fuzzy Hilbert space. In Section 3, the concept of fuzzy isomentric and

[1] Department of Mathematics, University of Maragheh, P. O. Box 55136-553, Maragheh, Iran.
e-mail: bdaraby@maragheh.ac.ir; ORCID: https://orcid.org/0000-0001-6872-8661.
e-mail: rostamiali1358@gmail.com; ORCID: https://orcid.org/0000-0001-5399-1061.
e-mail: rahimi@maragheh.ac.ir; ORCID: https://orcid.org/0000-0003-2095-6811.

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fuzzy linear operator is defined, and it is showed that each real Hilbert space (H, μ) is a fuzzy isometicaly isomorphic to (H^*, μ^*).

2. Preliminaries

In this section, some definitions and preliminary results are given which are used in this paper.

Definition 2.1 ([3]). *Let U be a linear space over $\mathbb{F}$ (field of Real/Complex numbers). A fuzzy subsets N of $U \times \mathbb{R}$ ($\mathbb{R}$ is the set of all real numbers) is called a fuzzy norm on U iff $\forall x, u \in U$ and $c \in \mathbb{F}$*

N1. $\forall t \in \mathbb{R}$ *with* $t \leq 0$, $N(x,t) = 0$;

N2. $\forall t \in \mathbb{R}, t > 0, N(x,t) = 1$ *iff* $x = \underline{0}$;

N3. $\forall t \in \mathbb{R}, c > 0, N(cx,t) = N(x, \frac{t}{|c|})$ *if* $c \neq 0$;

N4. $\forall t, s \in \mathbb{R}, x, u \in U \quad N(x+u, s+t) \geq \min\{N(x,s), N(u,t)\}$;

N5. $N(x,.)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $N(x,t) \to 1$ *as* $t \to \infty$.

The pair (U, N) will be referred to as a fuzzy normed linear space (FNLS in short).

Theorem 2.1 ([3]). *Let (U, N) be a fuzzy normed linear space. Assume further that,*

N6. $\forall t > 0, N(x,t) > 0 \Rightarrow x = \underline{0}$.

Define $\|x\|_\alpha = \inf\{t > 0 : N(x,t) \geq \alpha\}$, $\alpha \in (0,1)$, then $\{\| \; \|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms on U and are called α-norms on U corresponding to the fuzzy norm N on U.

Definition 2.2 ([3]). *Let (U, N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in U. Then $\{x_n\}$ is said to be convergent if $\exists x \in U$ such that*

$$\lim_{n \to \infty} N(x_n - x, t) = 1, \quad \forall t > 0.$$

Then x is called the limit of the sequence $\{x_n\}$.

Definition 2.3 ([3]). *A sequence $\{x_n\}$ in U is said to be a Cauchy sequence if*

$$\lim_{n \to \infty} N(x_n - x_{n+p}, t) = 1, \quad \forall t > 0, p = 1, 2, \ldots$$

Definition 2.4 ([3]). *Let (U, N) be a fuzzy normed linear space and $\alpha \in (0,1)$. A sequence $\{x_n\}$ in U is said to be α-convergent in U if $\exists x \in U$ such that*

$$\lim_{n \to \infty} N(x_n - x, t) > \alpha, \quad \forall t > 0$$

and x is called limit of $\{x_n\}$.

Proposition 2.1 ([3]). *Let (U, N) be a fuzzy normed linear space satisfying (N6). If $\{x_n\}$ be an α-convergent sequence in (U, N). Then $\|x_n - x\|_\alpha \to 0$ as $n \to \infty$ but converse is not necessarily true ($\| \; \|_\alpha$ denotes the α-norm of N).*

Definition 2.5 ([13]). *Let V be a linear space over the field $\mathbb{C}$ of complex numbers. Let $\mu : V \times V \times \mathbb{C} \to [0,1]$ be a mapping such that the following holds:*

(FIP-1) *For* $s, t \in \mathbb{C}$, $\mu(x+y, z, |t|+|s|) \geq \min\{\mu(x,z,|t|), \mu(y,z,|s|)\}$,

(FIP-2) *For* $s, t \in \mathbb{C}$, $\mu(x, y, |s|\,|t|) \geq \min\left\{\mu(x,x,|s|^2), \mu(y,y,|s|^2)\right\}$,

(FIP-3) *For* $t \in \mathbb{C}$, $\mu(x,y,t) = \mu(y,x,\overline{t})$,

(FIP-4) $\mu(\alpha x, y, t) = \mu\left(x, y, \frac{t}{|\alpha|}\right)$,

(FIP-5) $\mu(x,x,t) = 0, \forall x \in \mathbb{C} \backslash \mathbb{R}^+$, (FIP-6) $\mu(x, x, t) = 1, \forall t > 0$ *iff* $x = \underline{0}$.

(FIP-7) $\mu(x, x, \bullet) : \mathbb{R} \to [0, 1]$ *is a monotonic non-decreasing function of* $\mathbb{R}$ *and* $\lim_{t\to\infty} \mu(x, x, t) = 1$.

We call μ *to be the fuzzy inner product (FIP in short) function on* V *and* (V, μ) *is called a fuzzy inner product space (FIP space). In the sequel consider the following two conditions:*

(FIP-8) $\mu(x, x, t^2) > 0, \forall t > 0 \Rightarrow x = \underline{0}$,

(FIP-9) *For all* $x, y \in V$ *and* $p, q \in \mathbb{R}$,

$$\mu(x + y, x + y, 2q^2) \wedge \mu(x - y, x - y, 2p^2) \geq \min\{\mu(x, x, p^2), \mu(y, y, q^2)\}.$$

Remark 2.1 ([13]). *Let* V *be a linear space over* $\mathbb{C}$ *and* μ *be a FIP on* V. *Then we have*

$$N(x,t) = \begin{cases} \mu(x, x, t^2), t \in \mathbb{R} & \text{if } t > 0; \\ 0 & \text{if } t \leq 0. \end{cases}$$

is a fuzzy norm induced by the FIP and if (FIP-8) and (FIP-9) hold, then

$$\|x\|_\alpha = \wedge\left\{t > 0 : \mu(x, x, t^2) \geq \alpha\right\}, \quad \forall \alpha \in (0, 1)$$

is an ordinary norm α*-norm on* V *called* α*-norm on* V *generated from* μ.

Proposition 2.2 ([13]). *(Parallelogram Law) Let* μ *be a fuzzy inner product on* V *satisfying (FIP-8) and (FIP-9). Let* $\alpha \in (0, 1)$ *and* $\|\ \|_\alpha$ *be* α*-norm on* V *generated from the FIP on* V. *Then*

$$\|x - y\|_\alpha^2 + \|x + y\|_\alpha^2 = 2\left(\|x\|_\alpha^2 + \|y\|_\alpha^2\right). \quad x, y \in V$$

Then using Polarization identity we can get ordinary inner product, called the α*-inner product, as follows*

$$\langle x, y\rangle_\alpha = X_\alpha + iY_\alpha, \forall \alpha \in (0, 1)$$

where

$$X_\alpha = \frac{1}{4}\left(\|x - y\|_\alpha^2 - \|x + y\|_\alpha^2\right)$$

and

$$Y_\alpha = \frac{1}{4}\left(\|x + iy\|_\alpha^2 - \|x - iy\|_\alpha^2\right).$$

Definition 2.6 ([13]). *Let* (V, μ) *be a fuzzy inner product space satisfying (FIP-8).* V *is said to be level complete (l-complete) if for any* $\alpha \in (0, 1)$, *every Cauchy sequence converges in* V *w.r.t. the* α*-norm,* $\|\ \|_\alpha$ *generated by the fuzzy norm* N *which is induced by fuzzy inner product* μ.

Definition 2.7 ([13]). *Let* (V, μ) *be a FIP space.* V *is said to be a fuzzy Hilbert space, if it is level complete.*

Theorem 2.2 ([13]). *(Riesz Theorem) Let* (H, μ) *be a fuzzy Hilbert space satisfying (FIP-8) and (FIP-9) and* $f \in H^*$. *Then for each* $\alpha \in (0, 1)$, $\exists y_\alpha \in H$ *such that* $f(x) = \langle x, y_\alpha\rangle_\alpha$ *where* y_α *depends on* f *and*
$\|f\|_\alpha^* \geq \|y_\alpha\|_\alpha$ *when* $\alpha \geq \frac{1}{2}$ *and* $\|f\|_{1-\alpha}^* \leq \|y_\alpha\|_1 - \alpha$ *when* $\alpha < \frac{1}{2}$.

Definition 2.8 ([3]). *Let* $T : (U, N_1) \longrightarrow (V, N_2)$ *be a linear operator where* (U, N_1) *and* (V, N_2) *are fuzzy normed linear spaces.The operator* T *is said to be strongly fuzzy bounded on* U *iff* $\exists$ *a positive real number* M *such that*

$$N_2(T(x), s) \geq N_1(x, \frac{s}{M}). \quad \forall x \in U, \forall s \in \mathbb{R}$$

Definition 2.9 ([3]). *Let (U, N_1) and (V, N_2) be fuzzy normed linear spaces and $T : U \to V$ be a linear operator. T is said to be uniformly bounded if $\exists M > 0$ such that*

$$\|T(x)\|_\alpha^2 \leq \|x\|_\alpha^1 \qquad \forall \alpha \in (0,1)$$

where $\| \ \|_\alpha^1$ and $\| \ \|_\alpha^2$ are norms of N_1 and N_2, respectively.

Theorem 2.3 ([3]). *Let (U, N_1) and (V, N_2) be fuzzy normed linear spaces satisfying $(N6)$ and $(N7)$. Let $T : (U, N_1) \to (V, N_2)$ be a linear operator. Then T is strongly fuzzy bounded iff it is uniformly bounded with respect to norms of N_1 and N_2.*

Remark 2.2 ([3]). *The set of all strongly fuzzy bounded linear operators from a fuzzy normed linear space (U, N_1) to (V, N_2) is denoted by $F(U, V)$.*

Remark 2.3 ([3]). *Let V be a linear space where $V = \mathbb{R}$ or $\mathbb{C}$. We define a function $N_2 : V \times \mathbb{R} \to [0, 1]$ by*

$$N_2(x,t) = \begin{cases} 0 & \text{if } t \leq |x| \\ 1 & \text{if } t > |x|. \end{cases} \qquad (A)$$

Then it can be easily verified that N_2 is a fuzzy norm on V and thus (V, N_2) is a fuzzy normed linear space.

Definition 2.10 ([3]). *A strongly fuzzy bounded linear operator defined from (U, N_1) to (V, N_2) where (V, N_2) is defined by (A) is called strongly fuzzy bounded linear functional. We denote by U^* the set of all strongly fuzzy bounded linear functionals over (U, N_1).*

3. Main results

In this section, we define the concepts of fuzzy linear operator and fuzzy isomentric isomorphism. Next by defining a fuzzy inner product μ^* on H^* (the set of all strongly fuzzy bounded linear functionals over (H, μ)) we show that H and H^* is isometrically isomorphic.

Definition 3.1. *Let U_1 and U_2 be two fuzzy normed linear spaces (over $F = \mathbb{R}$ or $\mathbb{C}$), the mapping $\Phi : U_1 \times (0,1) \to U_2$ is called a fuzzy linear opertator if $\forall x_1, x_2 \in U_1$, $\forall \alpha \in (0,1)$ and $\lambda \in \mathbb{R}$ or $\mathbb{C}$ we have*

$$\Phi(\alpha, x_1 + x_2)(y) = \Phi(\alpha, x_1)(y) + \Phi(\alpha, x_2)(y)$$
$$\Phi(\alpha, \lambda x)(y) = \lambda \Phi(\alpha, x)(y)$$

and the mapping $\Phi : U_1 \times (0,1) \to U_2$ is called anti-fuzzy linear opertator if $\forall x_1, x_2 \in U_1$, $\forall \alpha \in (0,1)$ and $\lambda \in \mathbb{R}$ or $\mathbb{C}$ we have

$$\Phi(\alpha, x_1 + x_2)(y) = \Phi(\alpha, x_1)(y) + \Phi(\alpha, x_2)(y)$$
$$\Phi(\alpha, \lambda x)(y) = \bar{\lambda} \Phi(\alpha, x)(y).$$

Definition 3.2. *Let U_1 and U_2 be two fuzzy normed linear spaces (over $F = \mathbb{R}$ or $\mathbb{C}$) and $\Phi : U_1 \times (0,1) \to U_2$ be a fuzzy linear opertator. U_1 and U_2 are isometrically isomorphic, if $\forall x \in U_1$, $\forall \alpha \in (0,1)$ we have*

$$\|x\|_\alpha = \|\Phi(x)\|_\alpha.$$

Now, to prove the main theorem, we show that the space of strongly fuzzy bounded linear functionals is a fuzzy inner product space.

Theorem 3.1. *Let (H,μ) be a fuzzy Hilbert space satisfying (FIP-8) and (FIP-9) and H^* be the set of all strongly fuzzy bounded linear functionals over H as defined in Definition 2.15, and $\langle\ ,\ \rangle_\alpha$ be its $\alpha-$inner product $\forall\alpha\in(0,1)$.*
Define the function $\mu^*: H^*\times H^*\times\mathbb{C}\to[0,1]$ as

$$\mu^*(f,g,t)=0$$

if $f=g$ and $\forall t\in\mathbb{C}-\mathbb{R}^+$ and elsewhere as

$$\mu^*(f,g,t)=\sup\{\alpha\in(0,1):|\langle x_\alpha,y_\alpha\rangle_\alpha|\le|t|\}$$

where according Riesz Theorem $\alpha\in(0,1)$, there is x_α and y_α

$$f_{x_\alpha}(x)=\langle x,x_\alpha\rangle_\alpha$$
$$g_{y_\alpha}(y)=\langle y,y_\alpha\rangle_\alpha.$$

Then μ^* is a fuzzy inner product on H^* if $\langle\ ,\ \rangle_\alpha$ is an increasing function on $\mathbb{R}$.

Proof. **(FIP1)**. For $s,t\in\mathbb{C}$ and $f,g,h\in H^*$, we have to show that

$$\mu^*(f+g,h,|t|+|s|)\ge\min\{\mu^*(f,h,|t|),\mu^*(g,h,|s|)\}$$

Let $p=\mu^*(f,h,|t|)$ and $q=\mu^*(g,h,|t|)$. Without loss of generality assume that $p\le q$. Let $0<r<p\le q$. Then $\exists\alpha,\beta>r$ such that

$$|\langle x_\alpha,z_\alpha\rangle_\alpha|<|s|$$
$$|\langle y_\beta,z_\beta\rangle_\beta|<|t|.$$

Let $\gamma=\alpha\wedge\beta>r$. Since $\langle\ ,\ \rangle_\alpha$ is increasing, then

$$|\langle x_\gamma,z_\gamma\rangle_\gamma|\le|\langle x_\alpha,z_\alpha\rangle_\alpha|<|s|$$

and

$$|\langle y_\gamma,z_\gamma\rangle_\gamma|\le|\langle y_\beta,z_\beta\rangle_\beta|<|t|.$$

Now

$$|\langle x_\gamma+y_\gamma,z_\gamma\rangle_\gamma|\le|\langle x_\gamma,z_\gamma\rangle_\gamma|+|\langle y_\gamma,z_\gamma\rangle_\gamma|<|s|+|t|$$

therefore $\mu^*(f+g,h,|t|+|s|)\ge\gamma>r$. Since $0<r<\gamma$ is arbitrary, thus

$$\mu^*(f+g,h,|t|+|s|)\ge\min\ \mu^*(f,h,t|t|),\mu^*(g,h,|s|)\}.$$

(FIP2). For $s,t\in\mathbb{C}$ and $f,g,h\in H^*$ we have to show that

$$\mu(f,g,|s||t|)\ge\min\{\mu(f,f,|s|^2),\mu(g,g,|s|^2)\}.$$

Let $p=\mu^*(f,f,|t|^2)$ and $q=\mu^*(g,g,|t|^2)$. Without loss of generality assume that $p\le q$. Let $0<r<p\le q$. Then $\exists\alpha,\beta>r$ such that

$$|\langle x_\alpha,x_\alpha\rangle_\alpha|<|s|^2$$
$$|\langle y_\beta,y_\beta\rangle_\beta|<|t|^2$$

Let $\gamma=\alpha\wedge\beta>r$ Since $\langle,t\rangle_\alpha$ is increasing. Then

$$|\langle x_\gamma,x_\gamma\rangle_\gamma|\le|\langle x_\alpha,x_\alpha\rangle_\alpha|<|s|^2$$

and

$$|\langle y_\gamma,y_\gamma\rangle_\gamma|\le|\langle y_\beta,y_\beta\rangle_\beta|<|t|^2.$$

Now

$$|\langle x_\gamma,y_\gamma\rangle_\gamma|\le\sqrt{|\langle x_\gamma,x_\gamma\rangle_\gamma|}\sqrt{|\langle y_\gamma,y_\gamma\rangle_\gamma|}<|s||t|.$$

Since $0<r<\gamma$ is arbitrary, thus

$$\mu(f,g,|s||t|)\ge\min\{\mu(f,f,|s|^2),\mu(g,g,|s|^2)\}$$

(FIP3). For $t \in \mathbb{C}$ $\mu^*(f,g,t) = \mu^*(f,g,\bar{t}) = 0$ if $f = g$ and $\forall t \in \mathbb{C} - \mathbb{R}^+$.
Now let $t \in \mathbb{C}$ and $x \neq y$. then

$$\begin{aligned}\mu^*(f,g,t) &= \sup\{\alpha \in (0,1) : |\langle y_\alpha, x_\alpha\rangle|_\alpha \leq |\bar{t}|\} \\ &= \sup\{\alpha \in (0,1) : |\langle y_\alpha, x_\alpha\rangle|_\alpha \leq |\bar{t}|\} \\ &= \mu^*(g,f,\bar{t})\end{aligned}$$

(FIP4). For $t \in \mathbb{C}$

$$\begin{aligned}\mu^*(cf,g,t) &= \sup\{\alpha \in (0,1) : |\langle cx_\alpha, y_\alpha\rangle|_\alpha \leq |t|\} \\ &= \sup\{\alpha \in (0,1) : |\langle x_\alpha, y_\alpha\rangle|_\alpha \leq \frac{|t|}{|c|}\} \\ &= \mu^*(f,g,\frac{t}{|c|}).\end{aligned}$$

(FIP5). For $t \in \mathbb{C}$: $\forall t \in \mathbb{C} - \mathbb{R}^+, \mu^*(f,g,t) = 0$ [By definition].
(FIP6).

$$\begin{aligned}\mu^*(f,f,t) = 1 \ \forall t > 0 &\Leftrightarrow \sup\{\alpha \in (0,1) : |\langle x_\alpha, x_\alpha\rangle|_\alpha \leq |t|\} = 1 \quad \forall t > 0 \\ &\Leftrightarrow \langle x_\alpha, x_\alpha\rangle_\alpha = 0 \quad \forall \alpha \in (0,1) \\ &\Leftrightarrow x_\alpha = 0 \quad \forall \alpha \in (0,1) \\ &\Leftrightarrow f_{x_\alpha}(x) = \langle x, x_\alpha\rangle_\alpha = 0 \quad \forall \alpha \in (0,1), \forall x \in H \\ &\Leftrightarrow f = 0.\end{aligned}$$

(FIP7).

$$\begin{aligned}\mu^*(f,f,t) &= \sup\{\alpha \in (0,1) : |\langle x_\alpha, x_\alpha\rangle|_\alpha \leq |t|\} \\ &= \sup\{\alpha \in (0,1) : \|x_\alpha\|_\alpha^2 \leq |t|\} \\ &= \sup\{\alpha \in (0,1) : \|x_\alpha\|_\alpha \leq \sqrt{|t|}\}.\end{aligned}$$

If $t_1 > t_2$ then we have

$$\sqrt{|t|_1} > \sqrt{|t|_2}$$

It follows that $\alpha \in (0,1) : \|x_\alpha\|_\alpha \leq \sqrt{|t_1|}\} \supset \{\alpha \in (0,1) : \|x_\alpha\|_\alpha \leq \sqrt{|t_2|}$. So

$$\sup\{\alpha \in (0,1) : \|x_\alpha\|_\alpha \leq \sqrt{|t_1|}\} \geq \sup\{\alpha \in (0,1) : \|x_\alpha\|_\alpha \leq \sqrt{|t_2|}\}.$$

Hence

$$\mu^*(f,f,t_2) > \mu^*(f,f,t_2).$$

Therefore $\mu^*(f,f,.) : \mathbb{R}^+ \longrightarrow [0,1]$ is increasing and $\lim_{t\to\infty} \mu^*(f,f,t) = 1$. Thus μ^* is a fuzzy inner product on H. □

Theorem 3.2. *Let (H,μ) be a fuzzy Hilbert space satisfying (FIP-8) and (FIP-9) and H^* be dual of H. Then (H^*,μ^*) is a fuzzy Hilbert space.*

Proof. We show that the property (FIP-8) in H^* holds. For all $f \in H^*$ and $x \in H$, we have

$$\begin{aligned}
&\mu^*(f, f, t^2) > 0, \forall t > 0 \\
&\Rightarrow \sup\{\alpha \in (0,1) : |\langle x_\alpha, x_\alpha\rangle_\alpha| \leq |t|\} > 0,\ \forall t > 0 \\
&\Rightarrow \exists \alpha \in (0,1), |\langle x_\alpha, x_\alpha\rangle_\alpha| \leq |t|,\ \forall t > 0 \\
&\Rightarrow \|x_\alpha\|_\alpha^2 \leq |t|,\ \forall t > 0 \Rightarrow x_\alpha = 0 \\
&\Rightarrow f_{x_\alpha}(x) = \langle x, x_\alpha\rangle_\alpha = \langle x, 0\rangle_\alpha = 0 \\
&\Rightarrow f = 0.
\end{aligned}$$

Then the property (FIP-8) in H^* holds. Now we define a function $N : H^* \times \mathbb{C} \to [0,1]$ as

$$N(f,t) = \begin{cases} \mu^*(f, f, t^2), t \in \mathbb{R} & \text{if } \ t > 0 \\ 0 & \text{if } \ t \leq 0 \end{cases}$$

Then N is fuzzy norm induced by the FIP and since (FIP-8) is hold, so we can define

$$\|f\|_\alpha^* = \wedge\left\{t > 0 : \mu^*(f, f, t^2) \geq \alpha\right\}, \quad \forall \alpha \in (0,1),$$

That it is a $\alpha-$norm on H^*, called α-norm on H^* generated from μ^*.
We show that

$$\|f\|_\alpha^* = \|x_\alpha\|_\alpha,\ \textit{where}\ f_{x_\alpha}(x) = \langle x, x_\alpha\rangle_\alpha,\ \ \forall \alpha \in (0,1).$$

For all $\alpha \in (0,1)$ and $f \in H^*$ we have

$$\begin{aligned}
\|f\|_\alpha^* &= \wedge\left\{t > 0 : \mu^*(f, f, t^2) \geq \alpha\right\} \\
&= \wedge\{t > 0 : \sup\{\beta \in (0,1) : |\langle x_\beta, x_\beta\rangle_\beta \leq |t|\}\} \geq \alpha\} \\
&= \|x_\alpha\|_\alpha
\end{aligned} \tag{1}$$

($\langle,\rangle_\alpha$ is an increasing function on $\mathbb{R}$ by hypothesis) For showing the H^* is a fuzzy Hilbert space, we must show that H^* is level complete (l-complete). We show that for any $\alpha \in (0,1)$, every Cauchy sequence $\{f^n\}$ converges in H^* w.r.t. the α-norm, $\|.\|_\alpha^*$.

$$f^n_{x^n_\alpha}(x) = \langle x, x^n_\alpha\rangle_\alpha$$
$$f^n_{x^m_\alpha}(y) = \langle x, x^m_\alpha\rangle_\alpha.$$

Then from (3.1) we obtain that

$$\|f^n - f^m\|_\alpha^* = \|x^n_\alpha - x^m_\alpha\|_\alpha.$$

Since H is the Hilbert space and it is level complete (l-complete), then $\{x^n_\alpha\}$ is converges in H^* w.r.t. the α-norm and there is $x \in H$ such that

$$\|x^n_\alpha - x\|_\alpha \to 0.$$

We define the function f as follows

$$f_x(y) = \langle y, x\rangle_\alpha,$$

it is clear that $f_x \in H^*$ and

$$\|f^n - f\|_\alpha^* = \|x^n_\alpha - x\|_\alpha \to 0.$$

□

Theorem 3.3. *Let (H, μ) be a fuzzy Hilbert space and (H^*, μ^*) be a fuzzy dual of H, then*

(a) *H and H^* are fuzzy isometrically isomorphic, if (H, μ) be the fuzzy real Hilbert space.*
(b) *H and H^* are anti-fuzzy isometrically isomorphic, if (H, μ) be the fuzzy complex Hilbert space.*

Proof. For $\alpha \in (0, 1)$, we define the mapping $\Phi : H \times (0, 1) \to H^*$ by

$$\Phi(\alpha, x) = f_x^\alpha$$

where

$$f_x^\alpha(y) = \langle y, x\rangle_\alpha \quad \forall \alpha \in (0, 1).$$

According to Riesz Theorem Φ is bijective, and from Theorem 3.4, we have

$$\|x_\alpha\|_\alpha = \|\Phi(x)\|_\alpha^* = \|f_x^\alpha\|_\alpha^*.$$

It follows that Φ is a fuzzy isometric mapping.
Φ is additive:

$$\begin{aligned}\Phi(\alpha, x_1 + x_2)(y) = f_{x_1+x_2}^\alpha(y) &= \langle y, x_1 + x_2\rangle_\alpha \\ &= \langle y, x_1\rangle_\alpha + \langle y, x_2\rangle_\alpha = f_{x_1}^\alpha(y) + f_{x_2}^\alpha(y) \\ &= \Phi(\alpha, x_1)(y) + \Phi(\alpha, x_2)(y).\end{aligned}$$

if (H, μ) be the fuzzy real Hilbert space, then for each real number, we have

$$\Phi(\alpha, \lambda x)(y) = f_{\lambda x}^\alpha(y) = \langle y, \lambda x\rangle_\alpha = \lambda\langle y, x\rangle_\alpha = \lambda\Phi(\alpha, x)(y).$$

It follows that Φ is fuzzy isometric and H and H^* are fuzzy isometrically isomorphic.
if (H, μ) be the fuzzy complex Hilbert space, then $\Phi(\lambda x) = \bar{\lambda}\Phi(x)$ for all complex numbers λ, where $\bar{\lambda}$ denotes the complex conjugation of λ.

$$\Phi(\alpha, \lambda x)(y) = f_{\lambda x}^\alpha(y) = \langle y, \lambda x\rangle_\alpha = \bar{\lambda}\langle y, x\rangle_\alpha = \bar{\lambda}\Phi(\alpha, x)(y).$$

It follows that Φ is anti-fuzzy isometric and H and H^* are fuzzy anti-isometrically isomorphic. □

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Bayaz Daraby received his B.S. degree from University of Tabriz, M.S. degree from Tarbiat Modares University in Iran and Ph.D. degree from Puna University, Puna, India. He is a full prof. in the Department of Mathematics, University of Maragheh, Iran. His area of interest includes Fuzzy Analysis and Fuzzy Topology.

A. Rpstami is a Ph.D. candidate at the University of Maragheh. He is studying on Fuzzy Hilbert spaces, Fuzzy dual spaces.

Asghar Rahimi is a full professor of Mathematical Analysis from University of Maragheh, Maragheh, Iran. His main research area is on Frame Theory, Wavelet Analysis.