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ON A CLASS OF SAKAGUCHI FUNCTIONS RELATED TO BERNOULLI LEMNISCATE AND MODIFIED SIGMOID FUNCTION

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Abstract. The aim of this investigation is to provide further results on a class of Sakaguchi functions related to Bernoulli Lemniscate and modified sigmoid function for the class $\mathcal{S}_L(s,t,\Phi_{m,n})$. Early few coefficient bounds of $f(z)$ together with sharp inequalities for the Fekete-Szegö were obtained.

Keywords: analytic functions, starlike functions, subordination, Sakaguchi functions, Bernoulli Lemniscate, modified sigmoid function, coefficient bounds.

AMS Subject Classification: 30C45, 30C50

1. Introduction

Let A denotes a class of functions analytic in the open unit disk $U = \{z : |z| < 1\}$ and normalized by the condition $f(0) = f'(0) - 1 = 0$. Recall that, $S \subset A$ is a univalent function. Also, known that S^* and K are the starlike and convex functions which their geometrical condition satisfies $Re\frac{zf'(z)}{f(z)} > 0$ and $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$.
We say that f is subordinate to g in U and write $f(z) \prec g(z), z \in U$, if there exists a Schwarz function $w(z)$ (analytic in U with $w(0) = 0$, and $|w(z)| \leq |z|, z \in U$) such that $f(z) = g(w(z))(z \in U)$. Furthermore, if the function g is univalent in U, then we have the following equivalence:

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad and \quad f(U) \subset g(U)$$

Therefore, the class S^* may be defined as

$$S^* = \{f \in A : \frac{zf'(z)}{f(z)} \prec \frac{1+z}{1-z}\}. \quad (1.1)$$

Sokol and Stankiewicz [17] defined a class $S^*_L \subset S^*$ as follows:

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Definition 1.1. [18] *Let S_L^* denote the class of function f, analytic in the unit disc D, normalized by $f(0) = f'(0) - 1 = 0$ and satisfying the condition*

$$\frac{zf'(z)}{f(z)} \prec \sqrt{1+z} = q(z), z \in U \quad (1.2)$$

where the branch of the square root is chosen to be $q(0) = 1$.

It is noted that, the set $q(D)$ lies in the region bounded by the right loop of the lemniscate of Bernoulli $\gamma_1 = (x^2 + y^2)^2 - 2(x^2 - y^2) = 0$. He also used the class S_L^* to obtain some coefficient inequalities in Sokol [18].
Recently, Olatunji [8] introduced and studied a new class $<{}_{\lambda}^{\beta}(s, t, \phi)$ which geometric condition satisfy

$$Re\left(\frac{(s-t)z(f'(z)^{\lambda})}{f(sz) - f(tz)}\right) > \beta \quad (1.3)$$

where $s, t \in C, s \neq t, \lambda \geq 1 \in R$, $0 \leq \beta < 1$ and he obtained the early few coefficient bounds. By specializing the parameters involved in (1.3), we obtained various subclasses of analytic functions studied by many researchers. Just to mention a few, Owa et al. [12], Sakaguchi [13], Frasin [4] and so on. Researchers like Cho et al. [2], Olatunji et al. [9], Owa et al. [13], Sakaguchi [14], Sharma and Raina [15] are not also left out in the studies of Sakaguchi functions.

Sigmoid function is a special function that plays a vital role in geometric function theory. It increases the size of hypothesis space that the network can represent. It has a wide application in engineering, architecture, electronic noses, modeling and so on. It deals with an information process that is inspired by the biological nervous system such as brain to process information. This special function composed large number of highly interconnected processing element (neurones) working in unison to solve specific tasks. The function can be evaluated by truncated expansion series.

The sigmoid function

$$g(z) = \frac{1}{1+e^{-z}} \quad z \geq 0 \quad (1.4)$$

is a bounded differentiable function and has the following properties:

(1) it outputs real numbers between 0 and 1;
(2) it maps a very large output domain to a small range of inputs;
(3) it never looses information because it is an injective function; and
(4) It increases monotonically.

The aforementioned properties are very useful in geometric functions theory. We refer the reader to see Fadipe et al. [3], Murugusundaramoorthy and Janani [6], Oladipo [7], Olatunji [8], Olatunji et al. [10], Olatunji and Dansu [11].

The modified sigmoid function of the form

$$\Phi(z) = \frac{2}{1+e^{-z}} \quad (1.5)$$

were studied by Fadipe et al. [3] and they obtained another series of modified sigmoid function as

$$\Phi_{m,n}(z) = 1 + (\sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} (\frac{(-1)^n}{n!} z^n)^m) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 + \therefore \quad (1.6)$$

which shows that $\Phi_{m,n}(z) \in P$.

In this work, further results were obtained for the class of Sakaguchi functions related to Bernoulli Lemniscate and modified sigmoid function by employing Sokol and Thomas [16] method. The early few coefficient bounds were obtained which are also used to generate the relevant connection to Fekete-Szegö theorem for the class defined. Our results serve as a new generalization in this direction because the results obtained are not yet found in literature.

For the purpose of our results, the following lemma and definition shall be necessary.

Lemma 1.1. [1] *Let $p(z)$ be the class of functions P satisfying $Re[p(z)] > 0$, $z \in U$, with the form*

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \quad (1.7)$$

Definition 1.2. *Let $f \in \mathcal{S}_L(s, t, \Phi_{m,n})$ and given by*

$$f(z) = z + \sum_{n=1}^{\infty} a_n z^n, z \in U \quad (1.8)$$

which geometrical condition satisfy

$$\frac{(s-t)zf'(z)}{f(st) - f(tz)} \prec \sqrt{1+z} := q(z) \quad (1.9)$$

where $s, t \in C, s \neq t$ and the branch of the square root is chosen to be $q(0) = 1$.

2. Main Result

In this section, we shall discuss the main findings of this paper.

Theorem 2.1. *If $f \in \mathcal{S}_L(s, t, \Phi_{m,n})$ and is given by (1.8), then for $n \geq 2$,*

$$(n(s-t) - (s^n - t^n))^2 |a_n|^2 \leq \sum_{k=1}^{n-1} |a_k|^2 \{[k(s-t)\delta - (s^k - t^k)]^2 - [k(s-t) - (s^k - t^k)]^2\} \quad (2.1)$$

where $\delta = \sqrt{2} - 1$.

Proof. First note that from (1.9), we have

$$\frac{(s-t)zf'(z)}{f(sz) - f(tz)} \prec \sqrt{1+z} \prec \frac{1+z}{1+\delta z}$$

and so

$$\frac{(s-t)zf'(z)}{f(sz) - f(tz)} = \frac{1+w(z)}{1+\delta w(z)}$$

where $w(0) = 0$ and $|w(z)| < 1$ for $z \in U$, and

$$w(z) = \frac{\Phi_{m,n}(z) - 1}{\Phi_{m,n}(z) + 1} = \frac{1}{4}z - \frac{1}{16}z^2 - \frac{1}{192}z^3 - \frac{5}{768}z^4 - \dots = \sum_{k=1}^{\infty} c_k z^k \quad (2.2)$$

implies that $c_1 = \frac{1}{4}, c_2 = -\frac{1}{16}$ and so on.
Thus, we obtain

$$(s-t)zf'(z) - (f(sz) - f(tz)) = w(z)((f(sz) - f(tz)) - \delta(s-t)zf'(z))$$

and from (1.8) and (2.2), we have

$$\sum_{k=1}^{\infty}[k(s-t) - (s^k - t^k)]a_k z^k = w(z)\sum_{k=1}^{\infty}[(s^k - t^k) - k(s-t)\delta]a_k z^k, a_1 = 1.$$

Now write

$$\sum_{k=1}^{n}[k(s-t) - (s^k - t^k)]a_k z^k + \sum_{k=n+1}^{\infty}[k(s-t) - (s^k - t^k)]a_k z^k$$

$$= w(z)\{\sum_{k=1}^{n-1}((s^k - t^k) - k(st)\delta)a_k z^k + \sum_{k=n}^{\infty}((s^k - t^k) - k(s-t)\delta)a_k z^k\}$$

which can be written as

$$\sum_{k=1}^{n}[k(s-t)-(s^k-t^k)]a_k z^k + \sum_{k=n+1}^{\infty}[k(s-t)-(s^k-t^k)]a_k z^k - w(z)\sum_{k=n}^{\infty}[((s^k-t^k)-k(s-t)\delta)]a_k z^k$$

$$= w(z)\sum_{k=1}^{n-1}[(s^k - t^k) - k(s-t)\delta]a_k z^k.$$

Applying the method of Sokol and Thomas [16], we now write

$$\sum_{k=1}^{n}[k(s-t) - (s^k - t^k)]a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k = w(z)\sum_{k=1}^{n-1}[(s^k - t^k) - k(s-t)\delta]a_k z^k$$

for some $b_k, n+1 \leq k < \infty$ where b_k can be expressed in terms of the coefficients a_k and c_k as

$$b_k = (k(s-t) - (s^k - t^k))a_k - \sum_{j=1}^{k-n}[(s^k - t^k) - k(s-t)\delta]c_j a_{k-j}.$$

This gives

$$\left|\sum_{k=1}^{n}[k(s-t) - (s^k - t^k)]a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k\right|^2 = \left|w(z)\sum_{k=1}^{n-1}[s^k - t^k - k(s-t)\delta]a_k z^k\right|^2$$

$$\leq \left|\sum_{k=1}^{n-1}[(s^k - t^k) - k(s-t)\delta]a_k z^k\right|^2,$$

where

$$\sum_{k=1}^{n}[k(s-t) - (s^k - t^k)]a_k z^k + \sum_{k=n+1}^{\infty} b_k z^k := \sum_{k=1}^{\infty} d_k z^k$$

is an analytic function in the unit disc. Parseval's Theorem [5] gives

$$\int_0^{2\pi} |\sum_{k=1}^{\infty} d_k (re^{i\theta})^k|^2 d\theta = 2\pi \sum_{k=1}^{\infty} |d_k|^2 r^{2k}$$

and so integrating with respect to θ from θ to 2π any r, $0 < r < 1$, we obtain

$$\sum_{k=1}^{n}[k(s-t)-(s^k-t^k)]^2|a_k|^2r^{2k}+\sum_{k=n+1}^{\infty}|b_k|^2r^{2k}\leq\sum_{k=1}^{n-1}[(s^k-t^k)-k(s-t)\delta]^2|a_k|^2r^{2k}.$$

Therefore

$$\sum_{k=1}^{n}[k(s-t)-(s^k-t^k)]^2|a_k|^2r^{2k}\leq\sum_{k=1}^{n-1}[(s^k-t^k)-k(s-t)\delta]^2|a_k|^2r^{2k}.$$

Letting $r\to 1$ gives

$$\sum_{k=1}^{n}[k(s-t)-(s^k-t^k)]^2|a_k|^2+\sum_{k=n+1}^{\infty}|b_k|^2r^{2k}\leq\sum_{k=1}^{n-1}[k(s-t)\delta-(s^k-t^k)]^2|a_k|^2,$$

and this leads to the desired result (2.1). □

Corollary 2.1. *If $f\in\mathcal{S}_L(s,t,\Phi_{m,n})$ is given by (1.8) then for $n\geq 2$*

$$|a_n|\leq\frac{(s-t)(2-\sqrt{2})}{n(s-t)-(s^n-t^n)}.\quad (2.3)$$

Proof. From (2.1), we have

$$[n(s-t)-(s^n-t^n)]^2\leq\sum_{k=1}^{n-1}|a_k|^2[[(k(s-t)\delta)-(s^k-t^k)]^2-(k(s-t)-(s^k-t^k))^2]$$

$$=(s-t)^2(\delta-1)^2-\sum_{k=2}^{n-1}|a_k|^2[(k(s-t)-(s^k-t^k))-(k(s-t)\delta-(s^k-t^k))]$$

$$\leq(s-t)^2(\delta-1)^2=(s-t)^2(\sqrt{2}-2)^2$$

this gives (2.3).

For $n=6$ the condition gives

$$|a_6|\leq\frac{2-\sqrt{2}}{6-(s^5+s^4t+s^3t^2+s^2t^3+st^4+t^5)}.$$

□

3. Coefficient Bounds for the class $\mathcal{S}_L(s,t,\Phi_{m,n})$

Theorem 3.1. *If $f\in\mathcal{S}_L(s,t,\Phi_{m,n})$ and is given by (1.8), then*

$$|a_2|\leq\frac{1}{8(2-(s+t))}$$

$$|a_3|\leq\frac{1}{64(3-(s^2+st+t^2))}\left(\frac{5}{2}+\frac{(s+t)(s+t-2)}{(2-(s+t))^2}\right)\quad (3.1)$$

and

$$|a_4|\leq$$

$$\frac{1}{4-(s^3+s^2t+st^2+t^3)}\left|\frac{224}{98304}+\frac{(s+t)^2(s+t-2)}{512(2-(s+t))^3}+\frac{2(s^2+st+t^2)(s+t-1)-3(s+t)}{512(2-(s+t))(3-(s^2+st+t^2))}\left(\frac{5}{2}+\frac{(s+t)(s+t-2)}{(2-(s+t))^2}\right)\right|.$$

Proof. First note that from (1.9), we have

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} = \sqrt{1+w(z)}$$

where $w(0)$ and $|w(z)| < 1$ for $z \in U$. On the other hand, it is known that

$$w(z) = \frac{\Phi_{m,n}(z)-1}{\Phi_{m,n}(z)+1} = \frac{1}{4}z - \frac{1}{16}z^2 - \frac{1}{192}z^3 - \frac{5}{768}z^4 - ... \quad (3.2)$$

for some $\Phi_{m,n}(z) \in P$, which gives

$$\frac{(s-t)zf'(z)}{f(sz)-f(tz)} = \sqrt{1+\frac{\Phi(z)-1}{\Phi(z)+1}} \quad (3.3)$$

from (1.6) and (3.3), equality coefficients gives, after simplification

$$a_2 = \frac{1}{8(2-(s+t))} \quad (3.4)$$

$$a_3 = -\frac{1}{64(3-(s^2+st+t^2))}\left(\frac{5}{2}+\frac{(s+t)(s+t-2)}{(2-(s+t))^2}\right) \quad (3.5)$$

and

$$a_4 =$$

$$\frac{1}{4-(s^3+s^2t+st^2+t^3)}\left[\frac{224}{98304}+\frac{(s+t)^2(s+t-2)}{512(2-(s+t))^3}+\frac{2(s^2+st+t^2)(s+t-1)-3(s+t)}{512(2-(s+t))(3-(s^2+st+t^2))}\left(\frac{5}{2}+\frac{(s+t)(s+t-2)}{(2-(s+t))^2}\right)\right]. \quad (3.6)$$

which yields (3.1). □

Taking $s = 1$, we have

Corollary 3.1. *If $f \in \mathcal{S}_L(1, t, \Phi_{m,n})$ and is given by (1.8), then*

$$|a_2| \leq \frac{1}{8(2-(1+t))}$$

$$|a_3| \leq \frac{1}{64(3-(1+t+t^2))}\left(\frac{5}{2}+\frac{(1+t)(t-1)}{(2-(1+t))^2}\right) \quad (3.7)$$

$$|a_4| \leq \frac{1}{4-(1+t+t^2+t^3)}\left|\frac{224}{98304}+\frac{(1+t)^2(t-1)}{512(2-(1+t))^3}+\frac{2(1+t+t^2)(t)-3(1+t)}{512(2-(1+t))(3-(1+t+t^2))}\left(\frac{5}{2}+\frac{(1+t)(t-1)}{(2-(1+t))^2}\right)\right|.$$

Setting $t = -1$, we have

Corollary 3.2. *If $f \in \mathcal{S}_L(1, -1, \Phi_{m,n})$ and is given by (1.8), then*

$$|a_2| \leq \frac{1}{16}$$

$$|a_3| \leq \frac{5}{256} \quad (3.8)$$

$$|a_4| \leq \frac{1}{24576}.$$

Using corollary 3.2, we have

Theorem 3.2. *If $f \in \mathcal{S}_L(1, -1, \Phi_{m,n})$ and is given by (1.8), then*

$$|a_2a_4 - a_3^2| \leq \frac{149}{393216}. \quad (3.9)$$

Theorem 3.3. *If* $f \in S_L(1, -1, \Phi_{m,n})$ *and is given by (1.8), then*

$$|a_3^2 - \lambda a_2^2| \leq \left| \frac{25}{65536} - \frac{\lambda}{256} \right|. \quad (3.10)$$

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