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# KORTEWEG-DE VRIES-BURGERS (KDVB) EQUATION WITH ATANGANA-BALEANU DERIVATIVE WITH FRACTIONAL ORDER

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ABSTRACT. In this work, we examine the Korteweg–de Vries–Burgers equation with two perturbation's levels to the concepts of fractional differentiation with no singularity. The Korteweg–de Vries–Burgers equation was constructed using the new fractional differentiation based on the generalized Mittag-Leffler function due to the non-locality. It is presented the existence of a positive set of the solutions for the KDVB equation. The uniqueness of the positive solutions was presented in detail.

Keywords: Korteweg–de Vries–Burgers equation, fractional differentiation, existence and uniqueness.

AMS Subject Classification: 26A33.

## 1. Introduction

Most real-world problems occurring in many engineering disciplines and natural sciences are defined by nonlinear ordinary or partial differential equations. The exact solution of such equations is quite difficult to find. The solution of complex equations with the derivative definition defined by Newton has become very difficult to solve. Therefore, using the fractional derivative operators defined in recent years such as Caputo, Riemann-Lioville, Caputo-Fabrizio and Atangana-Baleanu [1, 2, 3, 8], the results are very close to the real solution. On the other hand, the memory effect and the management of the time process are easier to detect. The KDVB equation is used to identify and analyze some physical theories related to wave dynamics and liquids. For example, Johnson [7] used it to investigate the propagation of waves in an elastic tube filled with a viscous liquid. Moreover, Grad and Hu [5] used it to analyze the propagation of shallow water and Hu [6] undular holes. Extending the equations, systems or models solved with classical derivatives to new models with time-fractional derivatives and analyzing them with different techniques has gained importance. Dokuyucu and et al. [4] studied the existence and uniqueness solution with the fractional cancer treatment model. In this article, we have integrated the KDVB equation into the Atangana-Baleanu fractional derivative with non-singular and non-local. Then, it is investigated existence and uniqueness solutions.

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## 2. Atangana-Baleanu Derivative

In this section we give the basic definitions of the new derivatives with fractional order by Atangana and Baleanu.

**Definition 2.1.** *The Caputo derivative of fractional derivative is defined as* [9]*:*

$$ {}^{C}_{a}D^{\nu}_{t}f(t)) = \frac{1}{\Gamma(n-\nu)}\int_a^t \frac{f^{(n)}(r)}{(t-r)^{\nu+1-n}}dr, \quad n-1<\nu<n\in\mathbb{N}. \tag{1} $$

**Definition 2.2.** *The Riemann-Liouville fractional integral is defined as* [9]*:*

$$ J^{\nu}f(t) = \frac{1}{\Gamma(\nu)}\int_a^t f(r)(t-r)^{\nu-1}dr. \tag{2} $$

**Definition 2.3.** *The Riemann-Liouville fractional derivative is defined as* [9]*:*

$$ {}^{R}_{a}D^{\nu}_{t}f(t) = \frac{1}{\Gamma(n-\nu)}\frac{d^n}{dt^n}\int_a^t \frac{f(r)}{(t-r)^{\nu+1-n}}dr, \quad n-1<\nu<n\in\mathbb{N}. \tag{3} $$

**Definition 2.4.** *The Sobolev space of order 1 in* $(a,b)$ *is defined as* [9]*:*

$$ H^1(a,b) = \{u\in L^2(a,b) : u'\in L^2(a,b)\}. $$

**Definition 2.5.** *Let a function* $u\in H^1(a,b)$ *and* $\nu\in(0,1)$ *. The AB fractional derivative in Caputo sense of order* $\nu$ *of* $u$ *with a based point* $a$ *is defined as* [1]*:*

$$ {}^{ABC}_{\phantom{AB}a}D^{\nu}_{t}u(t)) = \frac{B(\nu)}{1-\nu}\int_a^t u'(s)E_{\nu}\big[-\frac{\nu}{1-\nu}(t-s)^{\nu}\big]ds, \tag{4} $$

*where* $B(\nu)$ *has the same properties as in Caputo and Fabrizio case, and is defined as*

$$ B(\nu) = 1-\nu+\frac{\nu}{\Gamma(\nu)}, $$

$E_{\nu,\beta}(\lambda^{\nu})$ *is the Mittag-Leffler function, defined in terms of a series as the following entire function*

$$ E_{\nu,\beta}(z) = \sum_{k=0}^{\infty}\frac{(\lambda^{\nu})^k}{\Gamma(\nu k+\beta)}, \qquad \nu>0, \quad \lambda<\infty \quad and \quad \beta>0, \quad \lambda=-\nu(1-\nu)^{-1}. \tag{5} $$

**Definition 2.6.** *Let a function* $u\in H^1(a,b)$ *and* $\nu\in(0,1)$ *. The AB fractional derivative in Riemann-Liouville sense of order* $\nu$ *of* $u$ *with a based point* $a$ *is defined as* [1]*:*

$$ {}^{ABR}_{\phantom{AB}a}D^{\nu}_{t}u(t)) = \frac{B(\nu)}{1-\nu}\frac{d}{dt}\int_a^t u(s)E_{\nu}\big[-\frac{\nu}{1-\nu}(t-s)^{\nu}\big]ds, \tag{6} $$

*when the function* $u$ *is constant, we get zero.*

**Definition 2.7.** *The Atangana-Baleanu fractional integral of order* $\nu$ *with base point* $a$ *is defined as* [1]*:*

$$ {}^{AB}I^{\nu}_{t}u(t)) = \frac{1-\nu}{B(\nu)}u(t) + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_a^t u(s)(t-s)^{\nu-1}ds, \tag{7} $$

*when the function* $u$ *is constant, we get zero.*

## 3. KDVB Equation with ABC Derivative

The Korteweg–de Vries–Burgers (KDVB) equation is given below,

$$\rho_t(x,t) = \eta\rho_{xx}(x,t) - 2\rho(x,t)\rho_x(x,t) - \xi\rho_{xxx}(x,t). \tag{8}$$

We apply the equation (8) to the Atangana-Baleanu fractional derivative in Caputo sense,

$${}_a^{ABC}D_t^{\nu} = \eta\rho_{xx}(x,t) - 2\rho(x,t)\rho_x(x,t) - \xi\rho_{xxx}(x,t). \tag{9}$$

Using the Atangana-Baleanu integral to (9) it yields,

$$\begin{aligned}\rho(x,t) - \rho(x,0) &= \frac{1-\nu}{B(\nu)}\big(\eta\rho_{xx}(x,t) - 2\rho(x,t)\rho_x(x,t) - \xi\rho_{xxx}(x,t)\big)\\ &+ \frac{\nu}{B(\nu)\Gamma(\nu)}\int_a^t \big(\eta\rho_{xx}(x,r) - 2\rho(x,r)\rho_x(x,r) - \xi\rho_{xxx}(x,r)\big)(t-r)^{\nu-1}dr.\end{aligned} \tag{10}$$

For simplicity,

$$s(x,t,\rho(x,t)) = \eta\rho_{xx}(x,t) - 2\rho(x,t)\rho_x(x,t) - \xi\rho_{xxx}(x,t).$$

### 3.1. **Existence Solution.**

**Lemma 3.1.** *The mapping $T : H \to H$ is defined as*

$$T\rho(x,t) = \frac{1-\nu}{B(\nu)}s(x,t,\rho(x,t)) + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t s(x,r,\rho(x,r))(t-r)^{\nu-1}dr. \tag{11}$$

**Lemma 3.2.** *Let $M \subset H$ be bounded we can find $n > 0$ for KDVB equation such that*

$$||\rho(x,t_2) - \rho(x,t_1)|| \leq n||t_2 - t_1||.$$

*Then $\overline{T(M)}$ is compact.*

*Proof.* Let $N = max\{\frac{1-\nu}{B(\nu)} + s(x,t,\rho(x,t))\}$, $0 \leq \rho(x,t) \leq P$. For $\rho(x,t) \in M$, then we have the following,

$$\begin{aligned}||T\rho(x,t)|| &= \frac{1-\nu}{B(\nu)}||s(x,t,\rho(x,t))|| + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t ||s(x,r,\rho(x,r))(t-r)^{\nu-1}dr||\\ &\leq \frac{1-\nu}{B(\nu)}N + \frac{\nu}{B(\nu)\Gamma(\nu)}N\frac{t^{\nu}}{\nu}\\ &\leq \frac{1-\nu}{B(\nu)}N + \frac{\nu t^{\nu}N}{B(\nu)\Gamma(\nu+1)}.\end{aligned} \tag{12}$$

So we can say that $T$ is bounded. Now the following part, we will consider $\rho(x,t) \in M, t_1, t_2$ and $t_1 < t_2$ then for a given $\epsilon > 0$ if $|t_2 - t_1| < \delta$. Then,

$$
\begin{aligned}
||T\rho(x,t_2)-T\rho(x,t_1)|| &\leq \left|\left|\frac{1-\nu}{B(\nu)}\big(s(x,t_2,\rho(x,t_2))-s(x,t_1,\rho(x,t_1))\big)\right|\right| \\
&+\left|\left|\frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^{t_2}(t_2-r)^{\nu-1}s(x,r,\rho(x,r))dr\right|\right| \\
&-\left|\left|\frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^{t_1}(t_1-r)^{\nu-1}s(x,r,\rho(x,r))dr\right|\right| \\
&\leq \frac{1-\nu}{B(\nu)}||(s(x,t_2,\rho(x,t_2))-s(x,t_1,\rho(x,t_1))|| \\
&+\frac{\nu P}{B(\nu)\Gamma(\nu)}\left\{\int_0^{t_2}(t_2-r)^{\nu-1}dr-\int_0^{t_1}(t_1-r)^{\nu-1}dr\right\}.
\end{aligned} \tag{13}
$$

Now firstly we start the integral part,

$$
\begin{aligned}
\int_0^{t_2}(t_2-r)^{\nu-1}dr-\int_0^{t_1}(t_1-r)^{\nu-1}dr &= \int_0^{t_1}\big((t_1-r)^{\nu-1} \\
&-(t_2-r)^{\nu-1}\big)dr+\int_{t_1}^{t_2}(t_2-r)^{\nu-1}dr \\
&= 2\frac{(t_2-t_1)^{\nu}}{\nu}.
\end{aligned} \tag{14}
$$

Then we will investigate the following part,

$$
||(s(x,t_2,\rho(x,t_2))-s(x,t_1,\rho(x,t_1))|| = ||\eta\rho_{xx}(x,t_2)-2\rho(x,t_2)\rho_x(x,t_2)-\xi\rho_{xxx}(x,t_2)||. \tag{15}
$$

Because all the solution are bounded, let us find appropriate different positive constants, $a_1, b_1, c_1$ for all $t$. So that, if we use Lipschitz condition of the derivative, equality (16) can be reconsidered as below,

$$
\begin{aligned}
||(s(x,t_2,\rho(x,t_2))-s(x,t_1,\rho(x,t_1))|| &\leq \eta a_1^2||\rho(x,t_2)-\rho(x,t_1)|| \\
&+2b_1||\rho(x,t_2)-\rho(x,t_1)||+\xi c_1^3||\rho(x,t_2)-\rho(x,t_1)|| \\
&\leq (\eta a_1^2+2b_1+\xi c_1^3)||\rho(x,t_2)-\rho(x,t_1)|| \\
&\leq A||t_2-t_1||.
\end{aligned} \tag{16}
$$

Now putting equations (14) – (16) in equation (13), we obtain,

$$
\begin{aligned}
||T\rho(x,t_2)-T\rho(x,t_1)|| &\leq \frac{1-\nu}{B(\nu)}A||t_2-t_1|| \\
&+2\frac{\nu P}{B(\nu)\Gamma(\nu)}\frac{||t_2-t_1||^{\nu}}{\nu} \\
&\leq \frac{1-\nu}{B(\nu)}A||t_2-t_1||+\frac{2\nu P}{B(\nu)\Gamma(\nu+1)}||t_2-t_1||^{\nu},
\end{aligned} \tag{17}
$$

$$\delta = \frac{\epsilon}{\frac{1-\nu}{B(\nu)}A + \frac{2\nu P}{B(\nu)\Gamma(\nu+1)}}.$$

Such that $||T\rho(x,t_2) - T\rho(x,t_1)|| \leq \epsilon$ is satisfied. So $T(M)$ is equi-continuous and with the help of Arzela-Ascoli theorem $\overline{T(m)}$ is compact. □

3.2. **Uniqueness Solution.** In this subsection we will show the uniqueness solution. So the solution is presented as below,

$$\begin{aligned}
||T\rho(x,t) - Tq(x,t)|| \leq & \Bigg|\Bigg|\frac{1-\nu}{B(\nu)}\big((s(x,t,\rho(x,t)) - (s(x,t,q(x,t))\big) \\
& + \frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t (t-r)^{\nu-1}\big(s(x,r,\rho(x,r)) - s(x,r,q(x,r))\big)dr\Bigg|\Bigg| \\
& \leq \frac{1-\nu}{B(\nu)}||(s(x,t,\rho(x,t)) - (s(x,t,q(x,t))|| \\
& + \frac{\nu}{B(\nu)\Gamma(\nu)}\Bigg|\Bigg|\int_0^t (t-r)^{\nu-1}\big(s(x,r,\rho(x,r)) - s(x,r,q(x,r))\big)dr\Bigg|\Bigg|.
\end{aligned} \tag{18}$$

First we will solve the second part of the equation (18).

$$\begin{aligned}
\frac{\nu}{B(\nu)\Gamma(\nu)}\int_0^t & (t-r)^{\nu-1}||\big(\eta\rho_{xx}(x,t) - 2\rho(x,t)\rho_x(x,t) - \xi\rho_{xxx}(x,t) \\
& - \eta q_{xx}(x,t) - 2q(x,t)q_x(x,t) - \xi q_{xxx}(x,t)\big)|| \\
& \leq \frac{\nu}{B(\nu)\Gamma(\nu)}a^\nu||\eta\big(\rho_{xx}(x,t) - q_{xx}(x,t)\big) + 2\big(q(x,t)q_x(x,t) - \rho(x,t)\rho_x(x,t)\big) \\
& + \xi\big(q_{xxx}(x,t) - \rho_{xxx}(x,t)\big)|| \\
& \leq \frac{\nu}{B(\nu)\Gamma(\nu)}a^\nu\big[\eta||\big(\rho_{xx}(x,t) - q_{xx}(x,t)\big)|| + 2||\big(q(x,t)q_x(x,t) - \rho(x,t)\rho_x(x,t)\big)|| \\
& + \xi||\big(q_{xxx}(x,t) - \rho_{xxx}(x,t)\big)||\big] \\
& \leq \frac{\nu}{B(\nu)\Gamma(\nu)}a^\nu\big[\eta||\partial_{xx}\big(\rho(x,t) - q(x,t)\big)|| + 2||\partial_x\big(q(x,t)^2 - \rho(x,t)^2\big)|| \\
& + \xi||\partial_{xxx}\big(q(x,t) - \rho(x,t)\big)||\big].
\end{aligned} \tag{19}$$

Because of assumption that $\rho$ and $q$ are bounded, there is a positive constant $c > 0$ such that $||\rho|| \leq c$ and $||q|| \leq c$. Then, their first order derivative function $\partial x$ satisfies the Lipschitz condition and there is a number $L_1 \geq 0$ such that

$$\begin{aligned}
& \leq \eta L_1^2||\rho(x,t) - q(x,t)|| + 2L_1||\rho(x,t) - q(x,t)||\big(\rho(x,t) + q(x,t)\big) \\
& + \xi L_1^3||\rho(x,t) - q(x,t)|| \\
& \leq \big[nL_1^2 + 2L_1c + \xi L_1^3\big]||\rho(x,t) - q(x,t)||.
\end{aligned} \tag{20}$$

Then, we will solve the first part of the equation (19).

$$
\begin{aligned}
&\left|\left|\frac{1-\nu}{B(\nu)}\big((s(x,t,\rho(x,t)) - (s(x,t,q(x,t))\big)\right|\right| \\
&= \left|\left|\frac{1-\nu}{B(\nu)}\big(\eta\rho_{xx}(x,t) - 2\rho(x,t)\rho_x(x,t) - \xi\rho_{xxx}(x,t) \right. \\
&\left. - \eta q_{xx}(x,t) + 2q(x,t)q_x(x,t) + \xi q_{xxx}(x,t)\big)\right|\right| \\
&\leq \frac{1-\nu}{B(\nu)}\Big[\eta\big(\rho_{xx}(x,t) - q_{xx}(x,t)\big)|| + 2||q(x,t)q_x(x,t) - \rho(x,t)\rho_x(x,t)|| \\
&+ \xi||\big(q_{xxx}(x,t) - \rho_{xxx}(x,t)\big)||\Big] \\
&\leq \frac{1-\nu}{B(\nu)}\Big[\eta L_1^2||\rho(x,t) - q(x,t)|| + 2L_1||\rho(x,t) - q(x,t)||\big(\rho(x,t) + q(x,t)\big) \\
&+ \xi L_1^3||\rho(x,t) - q(x,t)||\Big] \\
&= \frac{1-\nu}{B(\nu)}\big[\eta L_1^2 + 2L_1 c + \xi L_1^3\big]||\rho(x,t) - q(x,t)||.
\end{aligned}
\tag{21}
$$

So that,

$$
\begin{aligned}
||T\rho(x,t) - Tq(x,t)|| &\leq \frac{1-\nu}{B(\nu)}\big(\eta L_1^2 + 2L_1 c + \xi L_1^3\big) \\
&+ \frac{\nu}{B(\nu)\Gamma(\nu)}a^\nu\big(\eta L_1^2 + 2L_1 c + \xi L_1^3\big) \\
&\leq \frac{1-\nu}{B(\nu)}N + \frac{\nu}{B(\nu)\Gamma(\nu)}a^\nu N,
\end{aligned}
\tag{22}
$$

with $N < 1$ since $T$ is a contraction, which implies fixed point, the equation (9) has a unique solution.

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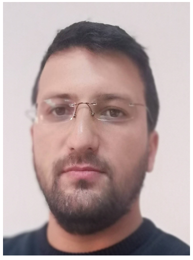

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