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AN APPLICATION OF (p,q)-CALCULUS IN HARMONIC UNIVALENT FUNCTIONS

POONAM SHARMA[1], SAURABH PORWAL[2], OMENDRA MISHRA[1], §

Abstract. The main aim of this paper is to define a (p,q)-multiplier transformation for harmonic functions. Using that we obtain coefficient inequality, extreme points, distortion theorem and covering results for the new subclass of harmonic functions. Several known results may be derived as special cases of our main results.

Keywords: (p,q)-calculus, q-calculus, univalent harmonic functions.

AMS Subject Classification: 30C45.

1. Introduction

The theory of q-calculus has motivated the researchers due to its application in mathematics and physics. Jackson [7] was the first to give some applications of q-calculus with the introduction of q-analogue of derivative and integral. Recently, Chakrabarti and Jagannathan [4] introduced (p,q)-integer in order to generalize several forms of q-oscillator algebras, well known in the physics literature related to the theory of single parameter quantum algebra. Post quantum calculus denoted by (p,q)-calculus is an extension of q-calculus at $p=1$. Let $0<q<p\leq 1$. The (p,q)-integer or (p,q)-bracket or twin-basic number $[k]_{p,q}$ is defined by

$$[k]_{p,q}=\frac{p^k-q^k}{p-q}, k\in\mathbb{N}_0=\mathbb{N}\cup\{0\}=\{0,1,2.\dots\}.$$

Notice that

$$\lim_{q\to p}[k]_{p,q}=kp^{k-1}.$$

In case $p=1, 0<q<1$, (p,q)-bracket $[k]_{p,q}$ reduces to the q-bracket $[k]_q$ and for $k=0,1,2.\dots$ it is given by

$$[k]_q=\frac{1-q^k}{1-q} \quad \text{and} \quad \lim_{q\to 1}[k]_q=k.$$

[1] Department of Mathematics and Astronomy, University of Lucknow, Lucknow-226007, U. P. India.
e-mail: sharma_poonam@lkouniv.ac.in; ORCID: https://orcid.org/0000-0002-7738-8010.
e-mail: mishraomendra@gmail.com; ORCID: https://orcid.org/0000-0001-9614-8656.
[2] Sri Radhey Lal Arya Inter College, Ehan, Hathras, India.
e-mail: saurabhjcb@rediffmail.com; ORCID: https://orcid.org/0000-0003-0847-3550.
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The (p,q)-derivative operator $\partial_{p,q}$ of a function f is defined by

$$\partial_{p,q}f(z)=\frac{f(pz)-f(qz)}{(p-q)z}\qquad (z\neq 0)$$

$\partial_{p,q}f(0)=1$ and $\lim_{q\to p=1}\partial_{p,q}f(z)=f'(z)$. For more detailed study one may refer [4, 2, 1]. When $p=1$, the (p,q)-derivative operator $\partial_{p,q}f(z)$ is known as q-derivative operator $\partial_q f(z)$ defined in [8] by

$$\partial_q f(z)=\frac{f(z)-f(qz)}{(1-q)z}\qquad (z\neq 0).$$

Let $\mathcal{H}$ denotes the class of complex-valued functions $f=u+iv$ which are harmonic in the unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$, where u and v are real-valued harmonic functions in $\mathbb{D}$. Functions $f\in\mathcal{H}$ can also be expressed as $f=h+\overline{g}$, where h and g are analytic in $\mathbb{D}$, called the analytic and co-analytic parts of the function f, respectively. The Jacobian of the function $f=h+\overline{g}$ is given by $J_f(z)=|h'(z)|^2-|g'(z)|^2$. According to the Lewy, every harmonic function $f=h+\overline{g}\in\mathcal{H}$ is locally univalent and sense preserving in $\mathbb{D}$ if and only if $J_f(z)>0$ in $\mathbb{D}$ which is equivalent to the existence of an analytic function $\omega(z)=g'(z)/h'(z)$ in $\mathbb{D}$ such that

$$|\omega(z)|<1\quad\text{for all}\quad z\in\mathbb{D}.$$

The function $\omega(z)$ is called the dilatation of f. By requiring harmonic function to be sense-preserving, we retain some basic properties exhibited by analytic functions, such as the open mapping property, the argument principle, and zeros being isolated (see for detail [6]). The class of all univalent, sense preserving harmonic functions $f=h+\overline{g}\in\mathcal{H}$ with the normalized conditions $h(0)=0=g(0)$ and $h'(0)=1$ is denoted by $S_{\mathcal{H}}$. If $f=h+\overline{g}\in S_{\mathcal{H}}$, then h and g are of the form:

$$h(z)=z+\sum_{k=2}^{\infty}a_kz^k\qquad\text{and}\qquad g(z)=\sum_{k=1}^{\infty}b_kz^k. \tag{1}$$

A subclass of functions $f=h+\overline{g}\in S_{\mathcal{H}}$ with the condition $g'(0)=0$ is denoted by $S^0_{\mathcal{H}}$. If $f=h+\overline{g}\in S^0_{\mathcal{H}}$, then h and g are of the form:

$$h(z)=z+\sum_{k=2}^{\infty}a_kz^k\qquad\text{and}\qquad g(z)=\sum_{k=2}^{\infty}b_kz^k. \tag{2}$$

Further, if $g(z)\equiv 0$, the class $S_{\mathcal{H}}$ reduces to the class $\mathcal{A}$ of normalized univalent analytic functions.

The convolution of two harmonic functions $f=h+\overline{g}$ and $F=H+\overline{G}$ is defined by $(f*F)(z)=g*G+\overline{(h*H)}$.

For a function h analytic in $\mathbb{D}$, (p,q)-Salagean operator of $h(z)$, denoted by $D^m_{p,q}h(z)$, is defined by (see Ahuja et al. [1]):

$$\begin{aligned}D^0_{p,q}h(z)&=h(z),D^1_{p,q}h(z)=z\partial_{p,q}h(z)\\ D^m_{p,q}h(z)&=z\partial_{p,q}(D^{m-1}_{p,q}h((z)),m\in\mathbb{N}.\end{aligned}$$

For h and g of the form (2), we get

$$D^m_{p,q}h(z)=z+\sum_{k=2}^{\infty}([k]_{p,q})^m a_kz^k\ \text{ and }\ D^m_{p,q}g(z)=\sum_{k=1}^{\infty}([k]_{p,q})^m b_kz^k.$$

The generalized (p,q)-Salagean operator for harmonic function $f = h + \overline{g}$ is defined by (see Ahuja et al. [1]):

$$\mathcal{D}^m_{p,q} f(z) = D^m_{p,q} h(z) + (-1)^m \overline{D^m_{p,q} g(z)}.$$

Using the generalized (p,q)-Salagean operator $\mathcal{D}^m_{p,q}$ for $m = 0,1$, we define a modified multiplier transformation $I^m_{p,q,\gamma,\beta} : \mathcal{H} \to \mathcal{H}$ for $m \in \mathbb{N}_0, \beta \geq \gamma \geq 0, \gamma+\beta > 0$, by

$$I^0_{p,q,\gamma,\beta} f(z) = \mathcal{D}^0_{p,q} f(z) = h(z) + \overline{g(z)},$$
$$I^1_{p,q,\gamma,\beta} f(z) = \frac{\gamma \mathcal{D}^0_{p,q} f(z) + \beta \mathcal{D}^1_{p,q} f(z)}{\gamma+\beta} \tag{3}$$

and

$$I^m_{p,q,\gamma,\beta} f(z) = I^1_{p,q,\gamma,\beta}(I^{m-1}_{p,q,\gamma,\beta} f(z)). \tag{4}$$

In case $p = 1$ and $q \to 1$, the transformation $I^m_{p,q,\gamma,\beta} = I^m_{\gamma,\beta}$ was studied by Bayram and Yalcin [3] and further, $I^m_{\gamma,1}$ was studied by Cho and Srivastava [5] for the functions $f \in \mathcal{A}$.

If $f = h + \overline{g}$, where h and g are of the form (1), then from (3) and (4) we see that

$$I^m_{p,q,\gamma,\beta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\beta[k]_{p,q} + \gamma}{\gamma+\beta} \right)^m a_k z^k + (-1)^m \sum_{k=1}^{\infty} \left(\frac{\beta[k]_{p,q} - \gamma}{\gamma+\beta} \right)^m \overline{b_k z^k}.$$

In this paper, we introduce a modified multiplier transformation $I^m_{p,q,\gamma,\beta}$ defined as above and with the help of this transformation, a new subclass of harmonic functions is defined by (7). For functions in this class, results on coefficient estimates, distortion bounds, covering theorems, radius of convexity, convolution and convex combination are obtained. It is mentioned that the results in [1] are the special cases of our main results.

2. MAIN RESULTS

Definition 2.1. Suppose $i, j \in \{0,1\}$. Let the function ϕ_i,ψ_j given by

$$\phi_i(z) = z + \sum_{k=2}^{\infty} \lambda_k z^k + (-1)^i \sum_{k=1}^{\infty} \mu_k \overline{z}^k \tag{5}$$

and

$$\psi_j(z) = z + \sum_{k=2}^{\infty} u_k z^k + (-1)^j \sum_{k=1}^{\infty} v_k \overline{z}^k \tag{6}$$

be harmonic in $\mathbb{D}$ with $\lambda_k > u_k \geq 0$ $(k \geq 2)$ and $\mu_k > v_k \geq 0$ $(k \geq 1)$.

For $\alpha \in [0,1)$, $0 < q < p \leq 1, m \in \mathbb{N}, n \in \mathbb{N}_0, m > n$ and $z \in \mathbb{D}$, let $S_H(m,n,\phi_i,\psi_j,p,q,\alpha,\beta,\gamma)$ denote a family of harmonic functions $f \in \mathcal{H}$ that satisfy the condition

$$Re \left\{ \frac{(I^m_{p,q,\gamma,\beta} f(z) * \phi_i)(z)}{(I^n_{p,q,\gamma,\beta} f(z) * \psi_j)(z)} \right\} > \alpha. \tag{7}$$

For $f = h + \overline{g}$, where h and g are of the form (1), we obtain

$$(I^m_{p,q,\gamma,\beta} f * \phi_i)(z) = z + \sum_{k=2}^{\infty} \lambda_k \left(\frac{\beta[k]_{p,q} + \gamma}{\gamma+\beta} \right)^m a_k z^k \tag{8}$$
$$+ (-1)^{m+i} \sum_{k=1}^{\infty} \mu_k \left(\frac{\beta[k]_{p,q} - \gamma}{\gamma+\beta} \right)^m \overline{b_k z^k}$$

and

$$\begin{aligned}(I^n_{p,q,\gamma,\beta} f * \psi_j)(z) &= z + \sum_{k=2}^{\infty} u_k \left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^n a_k z^k \\ &\quad +(-1)^{n+j} \sum_{k=1}^{\infty} v_k \left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^n \overline{b_k z^k}.\end{aligned}$$

Remark 1. If we choose $\gamma = 0$ in our main results, we get the results proved in [1] which include several results proved earlier.

Definition 2.2. Let $\mathcal{T}S_H(m,n,\phi_i,\psi_j,p,q,\alpha,\beta,\gamma)$ be the family of harmonic functions $f_m = h + \overline{g_m} \in \mathcal{T}S_H(m,n,\phi_i,\psi_j,p,q,\alpha,\beta,\gamma)$ such that h and g_m are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k \qquad g_m(z) = (-1)^{m+i-1} \sum_{k=1}^{\infty} |b_k| z^k \qquad (|b_1| < 1; z \in \mathbb{D}) \tag{9}$$

Theorem 2.1. *Let $f = h + \overline{g}$ be such that h and g are of the form* (1). *Also let*

$$\sum_{k=2}^{\infty} \Omega(m,n,\alpha)|a_k| + \sum_{k=1}^{\infty} \Theta(m,n,\alpha)|b_k| \le 1 \tag{10}$$

where

$$\Omega(m,n,\alpha) = \frac{\lambda_k \left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^m - \alpha u_k \left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^n}{1-\alpha} \tag{11}$$

$$\Theta(m,n,\alpha) = \frac{\mu_k \left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^m - (-1)^{n+j-(m+i)} \alpha v_k \left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^n}{1-\alpha} \tag{12}$$

$\lambda_k > u_k \ge 0$ $(k \ge 2)$, $\beta \ge \gamma \ge 0$ *and* $\mu_k > v_k \ge 0$ $(k \ge 1)$. *For* $\alpha \in [0,1)$, $0 < q < p \le 1, m \in \mathbb{N}, n \in \mathbb{N}_0, m > n$. *Then*

(i) *the function f is sense preserving and harmonic univalent in $\mathbb{D}$ and $f \in S_H(m,n,\phi_i,\psi_j,p,q,\alpha,\beta,\gamma)$ if the inequality* (10) *is satisfied.*

(ii) *$f_m = h + \overline{g_m} \in \mathcal{T}S_H(m,n,\phi_i,\psi_j,p,q,\alpha,\beta,\gamma)$ if and only if* (10) *is satisfied.*

Proof (i)It is clear that the theorem is true for the function $f(z) \equiv z$. Let $f = h + \bar{g}$, where h and g of the form (2) and assume that there exist $k \in \{2,3,\dots\}$ such that $a_k \neq 0$ or $b_k \neq 0$. we have $\Omega(m,n,\alpha) \ge k_{p,q}$, $\Theta(m,n,\alpha) \ge k_{p,q}$ Observe that the condition (10) implies

$$\sum_{k=2}^{\infty} [k]_{p,q} |a_k| + \sum_{k=1}^{\infty} [k]_{p,q} |b_k| \le 1. \tag{13}$$

Hence, we have

$$\begin{aligned}|\partial_{p,q} h(z)| - |\partial_{p,q} g(z)| &\ge 1 - \sum_{k=2}^{\infty} [k]_{p,q} |a_k|\, |z|^{k-1} - \sum_{k=1}^{\infty} [k]_{p,q} |b_k|\, |z|^{k-1} \\ &> 1 - |z| \left(\sum_{k=2}^{\infty} [k]_{p,q} |a_k| + \sum_{k=1}^{\infty} [k]_{p,q} |b_k| \right) \\ &\ge 1 - |z| > 0\end{aligned}$$

f is sense-preserving in $\mathbb{D}$ when $p = 1, q \to 1^-$. Moreover, if $z_1, z_2 \in \mathbb{D}$ and $(0 < q < p \leq 1)$ such that $pz_1 \neq qz_2$,

$$\left|\frac{(pz_1)^k - (qz_2)^k}{(pz_1) - (qz_2)}\right| = \left|\sum_{l=1}^{k}(pz_1)^{l-1}(qz_2)^{k-l}\right| \leq \sum_{l=1}^{k}|z_1|^{l-1}p^{l-1}q^{k-l}|z_2|^{k-l} < [k]_{p,q}$$

Hence, by (13), we have

$$\begin{aligned}|f(pz_1) - f(qz_2)| &\geq |h(pz_1) - h(qz_2)| - |g(pz_1) - g(qz_2)| \\ &\geq \left|pz_1 - qz_2 - \sum_{k=2}^{\infty}((pz_1)^k - (qz_2)^k)a_k\right| - \left|\sum_{k=1}^{\infty}\overline{((pz_1)^k - (qz_2)^k)b_k}\right| \\ &\geq |pz_1 - qz_2|\left(1 - \sum_{k=2}^{\infty}\left|\frac{(pz_1)^k - (qz_2)^k}{pz_1 - qz_2}\right||a_k| - \sum_{k=1}^{\infty}\left|\frac{(pz_1)^k - (qz_2)^k}{pz_1 - qz_2}\right||b_k|\right) \\ &> |pz_1 - qz_2|\left(1 - \sum_{k=2}^{\infty}[k]_{p,q}|a_k| - \sum_{k=1}^{\infty}[k]_{p,q}|b_k|\right) \geq 0\end{aligned}$$

which proves that f is univalent in $\mathbb{D}$. Using the fact $Re(w) > \alpha$ if and only if $|1 - \alpha + w| \geq |1 + \alpha - w|$, it suffices to show that

$$\left|1 - \alpha + \frac{(I^m_{p,q,\gamma,\beta}f * \phi_i)(z)}{(I^n_{p,q,\gamma,\beta}f * \psi_j)(z)}\right| - \left|1 + \alpha - \frac{(I^m_{p,q,\gamma,\beta}f * \phi_i)(z)}{(I^n_{p,q,\gamma,\beta}f * \psi_j)(z)}\right| \geq 0. \quad (14)$$

We have

$$
\begin{aligned}
&|(I^m_{p,q,\gamma,\beta}f*\phi_i)(z)+(1-\alpha)(I^n_{p,q,\gamma,\beta}f*\psi_j)(z)|-|(I^m_{p,q,\gamma,\beta}f*\phi_i)(z)-(1+\alpha)(I^n_{p,q,\gamma,\beta}f*\psi_j)(z)|\\
&\quad=\Bigg|(2-\alpha)z+\sum_{k=2}^{\infty}\left\{\lambda_k\left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^m+(1-\alpha)u_k\left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^n\right\}a_kz^k\\
&\qquad+(-1)^{m+i}\sum_{k=1}^{\infty}\left(\mu_k\left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^m+(-1)^{n+j-(m+i)}(1-\alpha)v_k\left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^n\right)b_k\bar{z}^k\Bigg|\\
&\quad-\Bigg|-\alpha z+\sum_{k=2}^{\infty}\left(\lambda_k\left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^m-(1+\alpha)u_k\left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^n\right)a_kz^k\\
&\qquad+(-1)^{m+i}\sum_{k=1}^{\infty}\left(\mu_k\left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^m-(-1)^{n+j-(m+i)}(1+\alpha)v_k\left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^n\right)b_k\bar{z}^k\Bigg|\\
&\quad\geq 2(1-\alpha)|z|-2\sum_{k=2}^{\infty}\left\{\lambda_k\left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^m-\alpha u_k\left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^n\right\}|a_k||z|^k\\
&\qquad-\sum_{k=1}^{\infty}\left\{\mu_k\left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^m+(-1)^{n+j-(m+i)}(1-\alpha)v_k\left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^n\right\}|b_k||z|^k\\
&\quad-\sum_{k=1}^{\infty}\left\{\mu_k\left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^m-(-1)^{n+j-(m+i)}(1+\alpha)v_k\left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^n\right\}|b_k||z|^k\\
&\quad\geq 2(1-\alpha)|z|\left[1-\sum_{k=2}^{\infty}\Omega(m,n,\alpha)|a_k||z|^{k-1}-\sum_{k=1}^{\infty}\Theta(m,n,\alpha)|b_k||z|^{k-1}\right]\\
&\quad> 2(1-\alpha)|z|\left[1-\left(\sum_{k=2}^{\infty}\Omega(m,n,\alpha)|a_k|+\sum_{k=1}^{\infty}\Theta(m,n,\alpha)|b_k|\right)\right].
\end{aligned}
$$

This last expression is non negative by (10). This completes the proof .
(ii) Since $\mathcal{T}S_H(m,n,\phi_i,\psi_j,p,q,\alpha,\beta,\gamma)\subset S_H(m,n,\phi_i,\psi_j,p,q,\alpha,\beta,\gamma)$, the sufficient part of part (ii) follows from (i).To poof the necessary part of part (ii) , we assume that f_m $\in\mathcal{T}S_H(m,n,\phi_i,\psi_j,p,q,\alpha,\beta,\gamma)$. We have for all $z\in\mathbb{D}$, in particular, choosing $z=r$

$(0 < r < 1)$

$$Re\left\{\frac{(I^m_{p,q,\gamma,\beta}f_m*\phi_i)(z)}{(I^n_{p,q,\gamma,\beta}f_m*\psi_j)(z)}-\alpha\right\}>0$$

$$= Re\left\{\frac{(1-\alpha)z-\sum_{k=2}^{\infty}\left\{\lambda_k\left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^m-\alpha u_k\left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^n\right\}|a_k|\,z^k}{z-\sum_{k=2}^{\infty}u_k\left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^n|a_k|\,z^k+(-1)^{m+i+n+j-1}\sum_{k=1}^{\infty}v_k\left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^n|b_k|\,\bar{z}^k}\right.$$

$$\left.+\frac{(-1)^{2m+2i-1}\sum_{k=1}^{\infty}\left\{\mu_k\left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^m-(-1)^{n+j-(m+i)}\alpha v_k\left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^n\right\}|b_k|\,\bar{z}^k}{z-\sum_{k=2}^{\infty}u_k\left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^n|a_k|\,z^k+(-1)^{n+j+m+i-1}\sum_{k=1}^{\infty}v_k\left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^n|b_k|\,\bar{z}^k}\right\}$$

$$=\frac{(1-\alpha)-\sum_{k=2}^{\infty}\left\{\lambda_k\left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^m-\alpha u_k\left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^n\right\}|a_k|\,r^{k-1}}{1-\sum_{k=2}^{\infty}u_k\left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^n|a_k|\,r^{k-1}-(-1)^{m+i+n+j}\sum_{k=1}^{\infty}v_k\left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^n|b_k|\,r^{k-1}}$$

$$-\frac{\sum_{k=1}^{\infty}(\mu_k\left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^m-(-1)^{n+j-(m+i)}\alpha v_k\left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^n|b_k|\,r^{k-1}}{1-\sum_{k=2}^{\infty}u_k\left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^n|a_k|\,r^{k-1}-(-1)^{n+j+m+i}\sum_{k=1}^{\infty}v_k\left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^n|b_k|\,r^{k-1}}$$

$$\geq 0$$

as $r\to 1^-$. This proves the required result. The harmonic univalent function

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1}{\Omega(m,n,\alpha)}x_kz^k+\sum_{k=1}^{\infty}\frac{1}{\Theta(m,n,\alpha)}y_k\bar{z}^k, \tag{15}$$

where

$$\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$$

shows that the coefficient inequality given by (10) is sharp.

Corollary 2.1. *For $f_m=h+\overline{g_m}$ given by (9), we have*

$$|a_k|\leq\frac{1}{\Omega(m,n,\alpha)},\quad k\geq 2\qquad and\qquad |b_k|\leq\frac{1}{\Theta(m,n,\alpha)},\quad k\geq 1.$$

The result is sharp.

Theorem 2.2. *Let $f_m=h+\overline{g_m}$ be given by (9) . Then $f_m\in clco\,\mathcal{T}S_H(m,n,\phi_i,\psi_j,p,q,\alpha,\beta,\gamma)$ if and only if $f_m(z)=\sum_{k=1}^{\infty}(x_kh_k(z)+y_kg_{mk}(z))$, where*

$$h_1(z)=z,\qquad h_k(z)=z-\frac{1}{\Omega(m,n,\alpha)}z^k\qquad(k\geq 2)$$

and

$$g_{mk}(z)=z+(-1)^{m+i-1}\frac{1}{\Theta(m,n,\alpha)}\bar{z}^k\qquad(k\geq 1)$$

and $\sum_{k=1}^{\infty}(x_k+y_k)=1$, where $x_k\geq 0$, $y_k\geq 0$. In particular the extreme point of $\mathcal{T}S_H(m,n,\phi_i,\psi_j,p,q,\alpha,\beta,\gamma)$ are $\{h_k\}$ and $\{g_{mk}\}$.

Proof: Let $f_m(z)=\sum_{k=1}^{\infty}(x_kh_k(z)+y_kg_{mk}(z))$, where $\sum_{k=1}^{\infty}(x_k+y_k)=1$. Then, we have

$$f_m(z)=z-\sum_{k=2}^{\infty}\frac{1}{\Omega(m,n,\alpha)}x_kz^k+(-1)^{m+i-1}\sum_{k=1}^{\infty}\frac{1}{\Theta(m,n,\alpha)}y_k\bar{z}^k$$

which proves that $f_m \in clco\mathcal{T}S_H(m,n,\phi_i,\psi_j,p,q,\alpha,\beta,\gamma)$ because

$$\sum_{k=2}^{\infty}\Omega(m,n,\alpha)\left(\frac{1}{\Omega(m,n,\alpha)}x_k\right)+\sum_{k=1}^{\infty}\Theta(m,n,\alpha)\left(\frac{1}{\Theta(m,n,\alpha)}y_k\right)$$

$= \sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k = 1 - x_1 \leq 1$.
Conversely, suppose $f_m \in clco\mathcal{T}S_H(m,n,\phi_i,\psi_j,p,q,\alpha,\beta,\gamma)$. Then

$$|a_k| \leq \frac{1}{\Omega(m,n,\alpha)} \qquad \text{and} \qquad |b_k| \leq \frac{1}{\Theta(m,n,\alpha)}.$$

Set

$$x_k = \Omega(m,n,\alpha)|a_k| \qquad y_k = \Theta(m,n,\alpha)|b_k|$$

By Theorem (2.1), $\sum_{k=2}^{\infty} x_k + \sum_{k=1}^{\infty} y_k \leq 1$. Therefore we define $x_1 = 1 - \sum_{k=2}^{\infty} x_k - \sum_{k=1}^{\infty} y_k \geq 0$. Consequently, we obtain $f_m(z) = \sum_{k=1}^{\infty}[x_k h_k(z) + y_k g_{mk}(z)]$ as required.

Theorem 2.3. *Let* $f_m \in \mathcal{T}S_H(m,n,\phi_i,\psi_j,p,q,\alpha,\beta,\gamma)$ *and let* $\sigma_k = \lambda_k\left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^m - \alpha u_k\left(\frac{\beta[k]_{p,q}+\gamma}{\gamma+\beta}\right)^n$ $(k \geq 2)$, $\phi_k = \mu_k\left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^m - (-1)^{n+j-(m+i)}\alpha v_k\left(\frac{\beta[k]_{p,q}-\gamma}{\gamma+\beta}\right)^n$ $(k \geq 1)$ *be non decreasing sequence. Then*

$$|f_m(z)| \leq (1+|b_1|)|z| + \frac{1-\alpha}{\beta}\left(1 - \frac{\mu_1\left(\frac{\beta-\gamma}{\gamma+\beta}\right)^m - (-1)^{n+j-(m+i)}\left(\frac{\beta-\gamma}{\gamma+\beta}\right)^n \alpha v_1}{\beta}|b_1|\right)|z|^2 \tag{16}$$

and

$$|f_m(z)| \geq (1-|b_1|)|z| - \frac{1-\alpha}{\beta}\left(1 - \frac{\mu_1\left(\frac{\beta-\gamma}{\gamma+\beta}\right)^m - (-1)^{n+j-(m+i)}\left(\frac{\beta-\gamma}{\gamma+\beta}\right)^n \alpha v_1}{\beta}|b_1|\right)|z|^2 \tag{17}$$

for all $z \in \mathbb{D}$, *where* $b_1 = \left(\frac{\partial f_m(z)}{\partial \bar{z}}\right)_{z=0}$ *and* $\beta = min\{\sigma_2, \phi_2\}$.

Proof Let $f_m = h + \overline{g_m} \in \mathcal{T}S_H(m,n,\phi_i,\psi_j,p,q,\alpha,\beta,\gamma)$ be given by (9). Taking the absolute value of f_m, we obtain

$$\begin{aligned}
|f_m(z)| &\leq (1+|b_1|)|z| + \sum_{k=2}^{\infty}(|a_k|+|b_k|)|z|^k \\
&\leq (1+|b_1|)|z| + \sum_{k=2}^{\infty}(|a_k|+|b_k|)|z|^2 \\
&\leq (1+|b_1|)|z| + \frac{1-\alpha}{\beta}\sum_{k=2}^{\infty}\left(\frac{\beta}{1-\alpha}|a_k| + \frac{\beta}{1-\alpha}|a_k|\right)|z|^2 \\
&\leq (1+|b_1|)|z| + \frac{1-\alpha}{\beta}\sum_{k=2}^{\infty}\left(\Omega(m,n,\alpha)|a_k| + \Theta(m,n,\alpha)|b_k|\right)|z|^2 \\
&\leq (1+|b_1|)|z| + \frac{1-\alpha}{\beta}\left(1 - \frac{\mu_1\left(\frac{\beta-\gamma}{\gamma+\beta}\right)^m - (-1)^{n+j-(m+i)}\left(\frac{\beta-\gamma}{\gamma+\beta}\right)^n \alpha v_1}{\beta}|b_1|\right)|z|^2.
\end{aligned}$$

This proves (16). The proof of (17) is similar to to the proof of (16).

Theorem 2.4. *If $f_m \in \mathcal{T}S_H(m,n,\phi_i,\psi_j,p,q,\alpha,\beta,\gamma)$, then f_m is convex in the disc*

$$|z| \le \min_{k\ge 2}\left\{\frac{1-b_1}{k\left[1-\frac{\mu_1\left(\frac{\beta-\gamma}{\gamma+\beta}\right)^m-(-1)^{n+j-(m+i)}\left(\frac{\beta-\gamma}{\gamma+\beta}\right)^n\alpha v_1}{\beta}|b_1|\right]}\right\}^{\frac{1}{k-1}}.$$

Proof. Let $f_m \in \mathcal{T}S_H(m,n,\phi_i,\psi_j,p,q,\alpha,\beta,\gamma)$ and let $r, 0<r<1$, be fixed. Then $r^{-1}f_m(rz) \in \mathcal{T}S_H(m,n,\phi_i,\psi_j,p,q,\alpha,\beta,\gamma)$ and we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} k^2(|a_k|+|b_k|)r^{k-1} = \sum_{k=2}^{\infty} k(|a_k|+|b_k|)kr^{k-1}\\
&\le \sum_{k=2}^{\infty}\left(\Omega(m,n,\alpha)|a_k|+\Theta(m,n,\alpha)|b_k|\right)kr^{k-1}\\
&\le \sum_{k=2}^{\infty}\left(1-\frac{\mu_1\left(\frac{\beta-\gamma}{\gamma+\beta}\right)^m-(-1)^{n+j-(m+i)}\left(\frac{\beta-\gamma}{\gamma+\beta}\right)^n\alpha v_1}{1-\alpha}|b_1|\right)kr^{k-1}\\
&\le 1-b_1
\end{aligned}$$

Provided

$$kr^{k-1} \le \frac{1-b_1}{1-\frac{\mu_1\left(\frac{\beta-\gamma}{\gamma+\beta}\right)^m-(-1)^{n+j-(m+i)}\left(\frac{\beta-\gamma}{\gamma+\beta}\right)^n\alpha v_1}{1-\alpha}|b_1|}$$

which is true if

$$r \le \min_{k}\left\{\frac{1-b_1}{k\left[1-\frac{\mu_1\left(\frac{\beta-\gamma}{\gamma+\beta}\right)^m-(-1)^{n+j-(m+i)}\left(\frac{\beta-\gamma}{\gamma+\beta}\right)^n\alpha v_1}{\beta}|b_1|\right]}\right\}^{\frac{1}{k-1}} \qquad k\ge 2$$

□

The proof of the following theorems are much akin to the corresponding results of [9], therefore we only state the results.

Theorem 2.5. *For $0 \le \alpha_1 \le \alpha_2 < 1$, let*

$$f \in \mathcal{T}S_H(m,n,\phi_i,\psi_j,p,q,\alpha_1,\beta,\gamma)$$

and $F \in \mathcal{T}S_H(m,n,\phi_i,\psi_j,p,q,\alpha_2,\beta,\gamma)$. Then

$$f * F \in S_H(m,n,\phi_i,\psi_j,p,q,\alpha_2,\beta,\gamma) \subseteq S_H(m,n,\phi_i,\psi_j,p,q,\alpha_1,\beta,\gamma).$$

Theorem 2.6. *The class $\mathcal{T}S_H(m,n,\phi_i,\psi_j,p,q,\alpha,\beta,\gamma)$ is closed under convex combination.*

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Dr.Poonam Sharma is a professor in the Department of Mathematics and Astronomy, University of Lucknow, Lucknow India. She has 35 years experience of research in the field of complex variables and Geometric Function Theory.

Dr. Saurabh Porwal is currently working as a lecturer of mathematics in Sri Radhey Lal Arya Inter College, Ehan, Hathras. Formely, he was an assistant professor in the Department of Mathematics, UIET, CSJM University, Kanpur, India. He has published several research papers in various reputed national and international journals.

Omendra Mishra is currently a research scholar in the Department of Mathematics and Astronomy, University of Lucknow, India. His area of interests is Geometric Function Theory.