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EXISTENCE OF A POSITIVE SOLUTION FOR SUPERLINEAR LAPLACIAN EQUATION VIA MOUNTAIN PASS THEOREM

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Abstract. In this paper, we are going to show a nonlinear laplacian equation with the Dirichlet boundary value as follow has a positive solution:

$$\begin{cases} -\Delta u + V(x)u = g(x,u) & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

where, $\Delta u = \operatorname{div}(\nabla u)$ is the laplacian operator, Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$.

At first, we show the equation has a nontrivial solution. next, using strong maximal principle, Cerami condition and a variation of the mountain pass theorem help us to prove critical point of functional I is a positive solution.

Keywords: Laplacian equation; Postive solution; Cerami condition; Mountain pass theorem.

AMS Subject Classification: 83-02, 99A00

1. Introduction

In this paper, we consider the following nonlinear ellipitic equation of the Laplace type:

$$\begin{cases} -\Delta u + V(x)u = g(x,u) & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases} \tag{1.1}$$

where, $\Delta u = \operatorname{div}(\nabla u)$ is the laplacian operator, Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Also, the functions V and $g \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies the following conditions:

(V1) $V \in C(\Omega, \mathbb{R})$, $v_0 := \inf_{x\in\Omega} V(x) > 0$.

(F1) g subcritical with respect to t, and there exists $q \in (2,6)$ such that

$$\lim_{t\to+\infty} \frac{g(x,t)}{t^{q-1}} = 0,$$

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uniformly in a.e. $x \in \Omega$.

(F2)

$$b_0 \leq \lim \inf_{t \to 0^+} \frac{g(x,t)}{t} \leq \lim \inf_{t \to 0^+} \frac{g(x,t)}{t} \leq a(x),$$

where b_0 is constant, $a \in L^\infty$; for all $x \in \bar{\Omega}, a(x) < \lambda_1$ on some $\Omega_1 \subseteq \Omega$; with $|\Omega_1| > 0$, λ_1 is the first eigenvalue of $(-\Delta + v)$, $|\Omega_1|$ is measure of Ω_1.

(F3) $\inf_{x \in \mathbb{R}^3} \lim_{u \to \infty} \dfrac{g(x,u)}{u} > \Gamma := \inf \sigma(-\Delta + v)$ the infimum of the spectrum of the operator $(-\Delta + v)$.

(F4) $\lim_{t \to +\infty} \dfrac{g(x,t)}{t} = +\infty$ uniformly in a.e. $x \in \Omega$.

In this paper, we study the existence of positive solution for (1.1) under the above assumptions. Since $(F4)$ holds, problem (1.1) is called superlinear in t at $+\infty$. In many studies involving this superlinear problem, to obtain a nontrivial solution of (1.1), Mountain pass theorem is a common tool, but in using this theorem, usually, we have to suppose another condition, that is, for some $\mu > 2$, $M > 0$

$$0 < \mu F(x,t) \leq f(x,t)t \quad \text{for a.e. } x \in X \text{ and for all } |t| \geq M. \tag{1.2}$$

The condition (1.2) is convenient, but it is very restrictive, in particular, it implies $(F4)$. To overcome this difficulty, many efforts have been made. Wang and Tang [8] studied the following superlinear laplacian equation without condition (1.2).

$$-\Delta_p u = f(x,u), \qquad x \in \Omega,\ u = 0,\ x \in \partial\Omega \tag{1.3}$$

The authors by using the following assumption for f proved the existence theorem.

(F') There exists $\theta \geq 1$ such that $\theta G(x,t) \geq G(x,st)$ for all $x \in \bar{\Omega}$, $t \in \mathbb{R}$ and $s \in [0,1]$, where $G(x,t) = f(x,t)t - pF(x,t)$ and $F(x,t) = \int_0^t f(x,s)ds$.

Assumption (F') was first introduced in [3] for $p = 2$, Liu and his coworker in [5] extended it for every $p > 1$. Also, Gao and Tang [2] proved the existance of postive solutions for (1.3) with following condition

(F5) There exists two constants $\theta \geq 1, \theta_0 > 0$;

$$\theta H(x,s) \geq H(x,t) - \theta_0 \text{ for all } x \in \bar{\Omega}, 0 \leq t \leq s.$$

where $H(x,t) = g(x,t)t - 2G(x,t)$ and $G(x,t) = \int_0^t g(x,s)ds$.

Now, we want to find a solution for equation(1.1).

Theorem 1.1. *Let $(F1) - (F5)$ and $(V1)$ hold. Then, (1.1) has at least one postive solution.*

2. Preliminaries

In this section, we present some important lemma which will be applied to prove our theorem. Let

$$E = \{u \in H^1(\Omega) : \int_\Omega V(x)u^2 dx < \infty\},$$

by $(V1)$, E is a Hilbert space with the inner product

$$< u, v >= \int_\Omega (\nabla u \nabla v + V(x)uv)dx,$$

and the norm

$$\|u\|^2 = \int_\Omega (|\nabla u|^2 + V(x)u^2)dx.$$

Now it is easy to verify that $u \in E$ is a solution of (1.1) if and only if $u \in E$ is a critical point of the functional $I : E \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + V(x)u^2)dx - \int G(x, u^+)dx,$$

where $G(x,t) = \int_0^t g(x,s)ds$ and t^+ denotes positive part of $t.I$ is a C^1 functional with derivative given by

$$< I'(u), v >= \int_\Omega (\nabla u \nabla v + V(x)uv)dx - \int_\Omega g(x,u)vdx.$$

Definition 2.1. *We say a C^1 functional I satisfies Palais-Smale condition $\big($Cerami condition$\big)$ if any sequence $\{u_n\} \subset H^1(\Omega)$ such that*

$$I(u_n) \text{ being bounded, } I'(u_n) \to 0, \text{ as } n \to 0 \tag{2.1}$$

$$\Big(I(u_n) \text{ being bounded, } (1 + \|u_n\|)I'(u_n) \to 0, \text{ as } n \to 0\Big)$$

admits a convergent subsequence, and such a sequence is called a palais-smale sequence$\big($cerami sequence$\big)$.

Lemma 2.1. *Let $(F1) - (F4)$ and $(V1)$ hold, then the functional I satisfies the Cerami condition.*

Proof. Let $\{u_n\} \subseteq E$ be Cerami sequence;

$$\begin{cases} I(u_n) = \frac{1}{2} \parallel u_n \parallel^2 - \int_\Omega G(x,u_n)dx \to c & \text{as } n \to \infty \\ (1+ \parallel u_n \parallel) \parallel I'(u_n) \parallel \to 0 & \text{as } n \to \infty. \end{cases} \tag{2.2}$$

On the other hand

$$\frac{1}{2}I'(u_n)u_n = \frac{1}{2}\int_\Omega (|\nabla u_n|^2 + v(x)u_n^2)dx - \frac{1}{2}\int_\Omega g(x,u_n)dx \to 0, \tag{2.3}$$

we notice

$$0 > \int (| \nabla u | \nabla u^- + v(x)u^- - \parallel u^- \parallel^2)dx - \int_\Omega g(x,u^+)u^- dx = \parallel u^- \parallel^2 \geq 0. \tag{2.4}$$

so now, (2.2)-(2.4) implies

$$\frac{1}{2}\int_\Omega g(x,{u_n}^+){u_n}^+dx - \int_\Omega G(x,{u_n}^+)dx = c + O(1). \tag{2.5}$$

Next, we prove the sequence $\{u_n\}$ is bounded. in the otherwise , there is a subsequence of u_n satisfies in $\|u_n\| \to \infty$ as $n \to \infty$.

Set $w_n = \dfrac{u_n}{\|u_n\|}$, then $\|w_n\| = 1$. Up to a subsequence, we assume that

$$w_n \to w \text{ in} E;\ w_n \to w \text{ in } L^r(2 \leq r \leq 6); \tag{2.6}$$
$$w_n(x) \to w(x) \text{ a.e. } x \in \Omega.$$

for some $w \in E$ as $n \to \infty$. It is easy to see that w^+ and w^- have the same convergence like (2.3), where $w^\pm = max\{\pm w, 0\}$ for $w \in E$.

We claim that $w^+ \equiv 0$. Let $\Omega_0 = \{x \in \Omega; w^+(x) = 0\}$, $\Omega^+ = \{x \in \Omega : w^+(x) > 0\}$. since $\|u_n\| \to +\infty$, then, $u_n^+ \to +\infty$ as $n \to +\infty$ for a.e. $x \in \Omega^+$.

Since $\lim_{t\to+\infty}\frac{g(x,t)}{t}=+\infty$ by $(F4)$, one has

$$\lim_{n\to+\infty}\frac{g(x,{u_n}^+)}{{u_n}^+}=+\infty \text{ a.e. } x\in\Omega^+.$$

From (2.2) we obtain

$$|\langle I'(u_n,u)\rangle|\le\varepsilon_n \tag{2.7}$$

where $\varepsilon_n=(1+\|\ u_n\|)\|I'(u_n)\|\to 0$ as $n\to\infty$. It follows from (2.7) that

$$\big|\|u_n^+\|^2-\int_\Omega g(x,u_n^+)u_n^+dx\big|\le\varepsilon_n,$$

which implies

$$\begin{aligned}\big|\|u_n^+\|^2-\int_\Omega g(x,u_n^+)u_n^+dx\big|&\le\varepsilon_n\\ &\le|\frac{g(x,u_n^+)u_n^+}{\|u_n^+\|^2}|\\ &\le\frac{\varepsilon_n}{\|u_n^+\|^2}+1.\end{aligned}$$

Then,

$$\int\frac{g(x,u_n^+)}{u_n^+}(w_n^+)^2dx\le 1+\frac{\varepsilon_n}{\|u_n^+\|^2}. \tag{2.8}$$

If $|\Omega^+|>0$, since $\|w_n^+\|=1$ from (2.8) one obtains

$$+\infty\leftarrow\int\frac{g(x,u_n^+)}{u_n^+}(w_n^+)^2dx\le 1+\frac{\varepsilon_n}{\|u_n^+\|^2}\to 1,$$

which is a contradiction, so $|\Omega^+|=0$ and $w\equiv 0$.

By $(F1)$ and $(F2)$, we have

$$g(x,t)\le(a(x)+\varepsilon)|t|+A|t|^{q-1};\qquad \text{for all}\qquad (x,t)\in\bar\Omega\times R,$$

where $A>0$ is a constant, thus

$$G(x,t^+)\le\frac{1}{2}(a(x)+\varepsilon)|t|^2+A|t|^q;\qquad \text{for all}\qquad (x,t)\in\bar\Omega\times R. \tag{2.9}$$

Now, set a sequence $\{t_n\}$ of real numbers such that $I(t_nu_n^+)=\max_{t\in[0,1]}I(tu_n^+)$. For any integer $m>0$, since $w^+\equiv 0$, then by $(F2)$, (2.9) and the convergence of w_n^+ one has

$$\begin{aligned}\limsup_{n\to\infty}\int_\Omega G(x,(4m)^{\frac12}w_n^+)dx&\le\limsup_{n\to\infty}(\int 2m(\lambda_1+\epsilon)(w_n^+)dx+\int A(4m)^{\frac{q}{2}}(w_n^+)^{\frac{q}{2}}dx)\\ &=\lim_{n\to\infty}(C_1\parallel w_n^+\parallel_2^2+C_2\parallel w_n^+\parallel_q^q)\\ &=(C_1\parallel w^+\parallel_2^2+C_2\parallel w^+\parallel_q^q)\\ &=0,\end{aligned}$$

where $C_1,C_2>0$ are constant. Since $\|u_n\|\to+\infty$ as $n\to\infty$. One has $0\le\frac{(4m)^{\frac12}}{\|u_n\|}\le 1$ when n is big enough. By definition of t_n, we obtain

$$I(t_nu_n^+)\ge I((4m)^{\frac12}w_n^+)\ge 2m-\int G(x,(4m)^{\frac12}w_n^+)dx\ge m,$$

which implies

$$I(t_nu_n^+)\to+\infty \text{ as } n\to\infty. \tag{2.10}$$

Notice that $I(0) = 0, I(u_n) \to C$, so $0 < t_n < 1$ when n is big enough. It follows that

$$\begin{aligned}&\int_\Omega | \nabla(t_n u_n^+) |^2 \, dx + \int_\Omega v(x)(t_n u_n^+)^2 - dx \int g(x, t_n u_n^+) t_n u_n^+ dx \qquad (2.11)\\ &= \langle I'(t_n u_n^+) u_n^+, t_n u_n^+ \rangle \\ &= t_n \frac{dI(t_n^+)}{dt}|_{t=t_n} \\ &= 0.\end{aligned}$$

But for $0 \le t_n \le 1, |t_n u_n| \le |u_n|$, then $(F5)$,(2.10) and (2.11) give

$$\begin{aligned}\int_\Omega (\frac{1}{2} g(x, u_n^+) u_n^+ - G(x, u_n^+) dx &= \frac{1}{2} \int H(x, u_n^+) dx \\ &\ge \frac{1}{2\theta} \int H(x, t_n u_n^+) - \theta_0) dx \\ &= \frac{1}{\theta} \int (\frac{1}{2} g(x, t_n u_n^+) t_n u_n - G(x, t_n u_n^+)) dx - \frac{\theta_0}{2\theta} |\Omega| \\ &= \frac{1}{\theta} I(t_n u_n^+) - \frac{\theta_0}{2\theta} |\Omega| \to +\infty, (n \to \infty),\end{aligned}$$

which contradicts to (2.5), so $\{u_n\}$ is bounded. By the compactness of Sobolev embedding and the standard procedures, we know $\{u_n\}$ has a convergence subsequence. So, the functional I satisfies the Cerami condition. □

Lemma 2.2. *Under the assumptions of the Theorem 1.1, there exist $\rho > 0$ such that for all $u \in E$ with $\| u \| = \rho$ we have $I(u) > 0$.*

Proof. Since $(F2)$ holds, there exist a positive constant $\alpha < 1$ such that

$$\int_\Omega a(x)|u| < \alpha \int_\Omega (|\nabla u|^2 + v(x) u^2) dx \qquad \text{for} u \in E,$$

see [8]. Let $\varepsilon > 0$ be the small enough such that $\alpha + \frac{\varepsilon}{\lambda_1} < 1$. By (2.9), together with the Poincare inequality and Sobolev inequality one obtains:

$$\begin{aligned} I(u) &\ge \frac{1}{2}\|u\|^2 - \frac{1}{2} \int_\Omega (a(x) + \varepsilon)|u|^2 - A \int_\Omega |u|^q dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{1}{2} \int_\Omega (\alpha + \frac{\varepsilon}{\lambda_1})(|\nabla u|^2 + v(x)u^2) dx - C\|u\|^q \\ &= \frac{1}{2}(1 - \alpha - \frac{\varepsilon}{\lambda_1})\|u\|^2 - C\|u\|^q \end{aligned}$$

where $C > 0$ is a constant, since $1 - \alpha - \frac{\varepsilon}{\lambda_1} > 0$ and $q > 2$, when $\rho > 0$ be small enough by $\|u\| = \rho$ we obtain

$$\beta = \frac{1}{2}(1 - \alpha - \frac{\varepsilon}{\lambda_1})\rho^2 - C\rho^4 > 0$$

$$I|_{\partial B_\rho} \ge \beta > 0$$

. □

Lemma 2.3. *Under the assumptions of the Theorem 1.1, there exists $e \in E$ with $\|e\| > \rho$ such that $I(e) < 0$, where ρ is given by the Lemma 2.2.*

Proof. We follow the arguments in [9]. We find e for I by $(F3)$. In fact $a := \inf_{x\in\Omega} \lim_{n\to\infty} \inf \dfrac{g(x,u)}{u}$. then by $(F3)$ and definition of Γ there exists a nonnegative function $u_0 \in E$ such that

$$\int_\Omega (|\nabla u_0|^2 + v(x)u_0^2)dx < a\int_\Omega u_0^2 dx.$$

Hence, by Fatou's lemma, we have

$$\begin{aligned}\lim_{t\to+\infty} \sup \frac{I(tu_0)}{t^2} &= \frac{1}{2}\int_\Omega (|\nabla u_0|^2 + v(x)u_0^2) - \lim_{t\to\infty} \inf \int_\Omega \frac{G(x,tu_0)}{t^2}dx \\ &\le \frac{1}{2}\int_\Omega (|\nabla u_0|^2 + v(x)u_0^2) - \int_\Omega \liminf \frac{G(x,tu_0)u_0^2}{t^2u_0^2}dx \\ &\le \frac{1}{2}\int_\Omega (|\nabla u_0|^2 + v(x)u_0^2)dx - \frac{1}{2}\int_\Omega a u_0^2 dx \\ &< 0.\end{aligned}$$

Hence, $\limsup_{t\to+\infty} I(tu_0) = -\infty$. Then, there exists $e \in E$ with $\|e\| > \rho$ such that $I(e) < 0$. □

3. The proof of main result

Lemmas 2.1, 2.2 and 2.3 permit the application of a variant of mountain pass theorem (see [1]). So, we get a critical point u of the function I with $I(u) \ge \beta$. But, from $(F2)$, $g(x,0) = 0$. Then $I(0) = 0$, that is $u \ne 0$. Since

$$0 = \langle I'(u), u^-\rangle = \|u^-\|^2 - \int_\Omega g(x,u^+)u^- dx = \|u^-\|^2 \ge 0,$$

which implies that $\|u^-\| = 0$, so $u \ge 0$. By the regularity results(see[4]), $u \in L^\infty(\Omega)$ and hence $u \in C^1(\Omega)$ (see[6]). Since $u \in L^\infty(\infty)$, it is easy to see that $\Delta u + v(x)u = -g(x,u) \in L^2_{loc}(\Omega)$. From $b_0 \le \lim_{t\to 0^+} inf \dfrac{g(x,t)}{t}$ by $(F2)$ there exist a constant $\delta > 0$ such that

$$g(x,t) \ge (b_0 - 1)t, \qquad \text{for all} \qquad 0 \le t \le \delta.$$

By $(F4)$, we can find a positive constant M such that $g(x,t) \ge 0$ for all $t \ge M$. Because $g \in C(\bar\Omega \times \mathbb{R}, \mathbb{R})$, then

$$|\, g(x,t)| \le B = B\delta^{-1}\delta \le B\delta^{-1}t, \qquad \text{for all} \qquad \delta \le t \le M,$$

where $B > 0$ is a constant, hence

$$g(x,t) \ge (-|b_0 - 1| - B\delta^{-1})t, \qquad \text{for all} \qquad t \ge 0,$$

since $u \ge 0$, it follows that

$$g(x,u) \ge (-|b_0 - 1| - B\delta^{-1})u = -Du,$$

where $D = |b_0 - 1| + B\delta^{-1} > 0$. Therefore, $\Delta u + v(x)u = -g(x,u) \le Du$. Hence by the strong maximum principle for $\Delta + v$ in [7] with $\beta(u) = D$, one has $u > 0$ a.e. on Ω. That is u is a positive solution of problem (1.1). The proof is completed. □

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