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DEGREE OF APPROXIMATION IN THE GENERALIZED LIPSCHITZ CLASS VIA $(E, q)A$- PRODUCT SUMMABILITY MEANS OF FOURIER SERIES

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Abstract. Dealing with degree of approximation of Fourier series of functions of Lipchitz classes, some results have been established by Nigam [15] and Misra et al. [17]. In this paper, we have established a new theorem based on $(E, q)A$- product summability mean in order to estimate the degree of approximation of a function $f \in Lip(\xi(t), r)$.

Keywords: $(E, q)A$- product summability mean, Lipchitz classes, degree of approximation, Fourier series.

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1. Introduction

The Theory of Summability is a wide field of Mathematics as regards to the study of Analysis and Functional Analysis. It has many applications in Numerical Analysis (to study the rate of convergence), Operator Theory (approximation of functions of positive linear operators), Theory of Orthogonal Series and Approximation Theory, etc. Approximation Theory has been originated from a well known theorem of Weierstrass, while approximating a continuous function in a given interval by a polynomial. Later the study was extended to approximate the piecewise continuous periodic function by trigonometric polynomials. Next for estimating the errors out of the approximations, it was noticed that the error is minimum if the coefficients of n^{th} trigonometric polynomial are the Fourier coefficients. Thus, n^{th} partial sum of Fourier series is a better estimate for the approximation of a periodic function. Moreover, for accuracy of estimations to a certain degree, different linear summation methods of Fourier series of 2π periodic functions on real line

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$\mathbb{R}$ (that is, by Cesàro mean, Nörlund mean, Matrix mean methods etc.) were introduced. Much of advanced in the theory of trigonometric approximations has been studied by different investigators for periodic functions of $Lip(\alpha, r)$-class. The degree of approximation of functions belonging to $Lip(\alpha)$, $Lip(\alpha, r)$, $Lip(\xi(t), r)$, $(r \geq 1)$-classes of functions have been studied by various investigators like Pradhan *et al.* [20], [21], [23], [22], Parida *et al.* [23], Lal [7], Mishra *et al.* ([8], [9], [10], [11], [12] and [13]), Paikray *et al.* [18], Deepmala et al. [2] and Misra *et al.* [14]. Recently, Nigam [15] has established a result on product summability of a function of Lipschitz class for sharpening the estimate of errors. In an attempt to make an advance study in this direction, a new theorem on $(E, q)A$- product summability mean for a function f belongs to $Lip(\xi(t), r)$-class has been established. Some more fundamental results and current topics on summability theory and its applications can be found in [1], [3], [4], [5], etc.

2. Definition and Notations

Let $\sum u_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $A = (a_{mn})$ be a matrix. Then the sequence-to-sequence transformation,

$$t_n = \sum_{k=0}^{n} a_{mv} s_v, \quad n = 1, 2, 3 \cdots \tag{1}$$

defines the sequence $\{t_n\}$ of the A-mean of the sequence $\{s_n\}$. If

$$t_n \to s, \quad \text{as} \quad n \to \infty \tag{2}$$

then the series $\sum u_n$ said to be A-summable to s.

The necessary and sufficient conditions for regularity of A- summability is such that, it satisfies the following conditions:

(a) $\sup_{n\to\infty} \sum_{k=1}^{\infty} |a_{n,k}| < H$, where H as an absolute constant,

(b) $\lim_{n\to\infty} a_{n,k} = 0$ for every $n = 1, 2, 3, ...$, and

(c) $\lim_{n\to\infty} \sum_{n=0}^{\infty} a_{n,k} = 1$.

Let $\{s_n\}$ be a sequence. The sequence-to-sequence transformation,

$$T_n = \frac{1}{(1+q)^n} \sum_{v=0}^{n} \binom{n}{v} q^{n-v} s_v, \tag{3}$$

defines the sequence $\{T_n\}$ of the (E, q) mean of the sequence $\{s_n\}$. If

$$T_n \to s, \quad \text{as} \quad n \to \infty, \tag{4}$$

then the series $\sum u_n$ said to be (E, q) summable to s. Clearly (E, q) method is regular.

The (E, q)- transform of the A- transform of $\{s_n\}$ is defined by

$$\tau_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{v} q^{n-k} t_k = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{v} q^{n-k} \left\{ \sum_{v=0}^{k} a_{kv} s_v \right\}. \tag{5}$$

If

$$\tau_n \to s, \quad \text{as} \quad n \to \infty, \tag{6}$$

then the series $\sum u_n$ said to be $(E,q)A$- summable to s.

Let f be a periodic function with period 2π and L- integrable over $(-\pi,\pi)$. The Fourier series of the function f at any point x is defined by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)=\sum_{n=0}^{\infty}A_n(x), \tag{7}$$

where

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)dt,\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos ntdt\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin ntdt.$$

Furthermore, the L_∞-norm of a function $f:\mathbb{R}\to\mathbb{R}$ is defined by

$$\|f\|_\infty=\sup\{|f(x)|:x\in\mathbb{R}\} \tag{8}$$

and L_r-norm of a function $f:\mathbb{R}\to\mathbb{R}$ is defined by

$$\|f\|_r=\left(\int_0 2\pi|f(x)|^r dx\right)^{\frac{1}{r}},\quad r\geq 1. \tag{9}$$

The degree of approximation of a function $f:\mathbb{R}\to\mathbb{R}$ by a trigonometric polynomial $t_n(x)$ of order n under $\|.\|_\infty$ is defined by

$$\|t_n-f(x)\|_\infty=\sup\{|t_n(x)-f(x)|:\ x\in\mathbb{R}\} \tag{10}$$

and the degree of approximation $E_n(f)$ of a function $f\in L_r$ is defined by

$$E_n(f)=\min_n\|t_n-f\|_r. \tag{11}$$

Here, as regards to the functions belonging to various Lipschitz classes, we may recall that

(a) $f\in Lip(\alpha)$, if

$$|f(x+t)-f(x)|=O(|t|^\alpha)\text{ for }0<\alpha\leq 1,\ t>0 \tag{12}$$

(b) $f\in Lip(\alpha,r)$, for $0\leq x\leq 2\pi$, if

$$\left(\int_{[0,2\pi]}|f(x+t)-f(x)|^r dx\right)^{\frac{1}{r}}=O(|t|^\alpha)\text{ for }0<\alpha\leq 1,\ t>0,\ r\geq 1 \tag{13}$$

(c) $f\in Lip(\xi(t),r)$, if

$$\|f(x+t)-f(x)\|_r=\left(\int_{[0,2\pi]}|f(x+t)-f(x)|^r dx\right)^{\frac{1}{r}}=O(\xi(t)),\ r\geq 1,\ t>0, \tag{14}$$

where, $\xi(t)$ is any positive increasing function.

We use the following notations throughout this paper

$$\Phi(t)=f(x+t)+f(x-t)-2f(x) \tag{15}$$

and

$$K_n(t)=\frac{1}{2\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{v=0}^{k}a_{kv}\frac{\sin(v+\frac{1}{2})t}{\sin\frac{t}{2}}\right\}. \tag{16}$$

Also, the method $(E,q)A$ is assumed to be regular.

3. Known Theorems

Dealing with the $(C,1)(E,1)$- product summability mean, in 2013 Nigam [15] proved the following theorem.

Theorem 3.1. *If f be a 2π periodic function belonging to the class Lip α, then its degree of approximations by $(E,1)(C,1)$ means of Fourier series $\sum_{n=0}^{\infty} A_n(t)$ is given by*

$$\|E_n^1 C_n^1 - f\|_\infty = O\left\{\frac{1}{(n+1)^\alpha}\right\} \quad 0 \le \alpha \le 1, \tag{17}$$

where $E_n^1 C_n^1$ represents the $(E,1)$ transform of $(C,1)$ transform of $s_n(f;x)$.

Subsequently, Padhy et al. [16] proved the following theorem based on degree of approximation by the product $(E,1)A$- summability mean of the Fourier series.

Theorem 3.2. *Let $A = (a_{nm})$ be an infinite matrix. If f is a 2π-periodic function of class $Lip(\alpha, r)$, the degree of approximation of product $(E,q)A$- summabiliy means of its Fourier series is given by*

$$\|\tau_n - f\|_\infty = O\left\{\frac{1}{(n+1)^\alpha}\right\} \quad 0 \le \alpha \le 1, \tag{18}$$

where τ_n is defined as (5).

Next, Misra et. al [17] proved the following theorem.

Theorem 3.3. *If f is a 2π-periodic function of class $Lip(\alpha, r)$, the degree of approximation by product $(E,q)A$- summabiliy means of its Fourier series is given by*

$$\|\tau_n - f\|_\infty = O\left\{\frac{1}{(n+1)^{\alpha-\frac{1}{l}}}\right\} \quad 0 \le \alpha \le 1,\ l \ge 1. \tag{19}$$

4. Main Theorem

The objective of this paper is to prove the following theorem.

Theorem 4.1. *Let $\xi(t)$ be a positive increasing function and let $f(x)$ be a 2π periodic function of the class $Lip(\xi(t), r)$, $p \ge 1, t \ge 0$, and is integrable in the Lebesgue sense in $[0, 2\pi]$, then its degree of approximation by $(E,q)A$- summability means of its Fourier series is given by*

$$\|\tau_n - f\|_r = O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}, \tag{20}$$

provided $\xi(t)$ is such that $\left\{\frac{\xi(t)}{t}\right\}$ be a decreasing sequence.

To prove the above theorem, first we need to prove the following lemmas.

Lemma 4.1. $|K_n(t)| = O(n)$, *for* $0 \le t \le \frac{1}{n+1}$.

Proof. For $0 \leq t \leq \frac{1}{n+1}$, we have $\sin nt \leq n \sin t$.

$$\begin{aligned}
|K_n(t)| &= \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{v=0}^{k}a_{kv}\frac{\sin(v+\frac{1}{2})t}{\sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{v=0}^{k}a_{kv}(2v+1)\frac{\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}(2k+1)\left\{\sum_{v=0}^{k}a_{kv}\right\}\right| \\
&\leq \frac{(2k+1)}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\right| \\
&= O(n).
\end{aligned}$$

□

Lemma 4.2. $|K_n(t)| = O(\frac{1}{t})$, *for* $\frac{1}{n+1} < t \leq \pi$.

Proof. For $\frac{1}{n+1} < t \leq \pi$ and by using Jordan lemma, $\sin\frac{t}{2} \geq \frac{t}{\pi}$ and $\sin nt \leq 1$.

$$\begin{aligned}
|K_n(t)| &= \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{v=0}^{k}a_{kv}\frac{\sin(v+\frac{1}{2})t}{\sin\frac{t}{2}}\right\}\right| \\
&\leq \frac{1}{2\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{v=0}^{k}a_{kv}(\frac{\pi}{t})\right\}\right| \\
&\leq \frac{1}{2t(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{v=0}^{k}a_{kv}\right\}\right| \\
&\leq \frac{1}{2t(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\right| \\
&= O\left(\frac{1}{t}\right).
\end{aligned}$$

□

5. Proof of the Theorem 4.1

Proof. Using Riemann-Lebesgue theorem and considering the n^{th} partial sum of the Fourier series of $f(x)$ as $s_n(f;x)$ and following Titchmarsh [6], we have

$$s_n(f;x) - f(x) = \frac{1}{2\pi}\int_{[0,\pi]}\Phi(t)\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}dt. \tag{21}$$

Further, under the A-transform of $s_n(f;x)$, we have

$$t_n - f(x) = \frac{1}{2\pi}\int_{[0,\pi]}\Phi(t)\sum_{k=0}^{n}a_{nk}\frac{\sin(v+\frac{1}{2})t}{\sin\frac{t}{2}}dt. \tag{22}$$

Then, considering τ_n as the $(E,q)A$-transform of $s_n(f;x)$, we obtain

$$\begin{aligned}\|\tau_n - f(x)\| &= \frac{1}{2\pi(1+q)^n}\int_{[0,\pi]}\Phi(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{v=0}^{k}a_{kv}\frac{\sin(v+\frac{1}{2})t}{\sin\frac{t}{2}}\right\}\\ &= \int_{[0,\pi]}\Phi(t)K_n(t)dt\\ &= \left\{\int_{[0,\frac{1}{n+1}]}+\int_{[\frac{1}{n+1},\pi]}\right\}\Phi(t)K_n(t)dt\\ &= I_1 + I_2\ (say).\end{aligned}$$

Now

$$|\Phi(x,t)-\Phi(x)| \le |f(u+x+t)-f(u+x)| + |f(u-x-t)-f(u-x)|,$$

so, by using Minkowski's inequality,

$$\begin{aligned}\left[\int_0^{2\pi}|\{\Phi(x+t)-\Phi(x)\}\|^r\,dx\right]^{\frac{1}{r}} &\le \left[\int_0^{2\pi}|\{f(u+x+t)-f(u+x)\}|^r\,dx\right]^{\frac{1}{r}}\\ &+ \left[\int_0^{2\pi}|\{f(u-x-t)-f(u-x)\}|^r\,dx\right]^{\frac{1}{r}}\\ &= O\left(\xi(t)\right).\end{aligned}$$

$$\begin{aligned}\text{Now, } |I_1| &= \frac{1}{2\pi(1+q)^n}\left|\int_0^{\frac{1}{n+1}}\phi(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\sum_{v=0}^{k}a_{kv}\frac{\sin(v+\frac{1}{2})t}{\sin\frac{t}{2}}\right\}\right|\\ &\le \left|\int_0^{\frac{1}{n+1}}\phi(t)K_n(t)dt\right|\\ &\le \left|\int_0^{\frac{1}{n+1}}\frac{\Phi(t)}{\xi(t)}\xi(t)K_n(t)dt\right|.\end{aligned}$$

Using Hölder's inequality $\frac{1}{l}+\frac{1}{m}=1$ and Lemma 1, we have

$$
\begin{aligned}
|I_1| &\le \left(\int_0^{\frac{1}{n+1}} \left|\frac{\phi(t)}{\xi(t)}\right|^l dt\right)^{\frac{1}{l}} \left(\int_0^{\frac{1}{n+1}} |\xi(t)K_n|^m \, dt\right)^{\frac{1}{m}} \\
&\le O(1)\left(\int_0^{\frac{1}{n+1}} \xi(t).n^m dt\right)^{\frac{1}{m}} \\
&= O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left(\frac{n^m}{n+1}\right)^{\frac{1}{m}} \\
&= O\left\{\xi\left(\frac{1}{n+1}\right)\left(\frac{1}{(n+1)^{\frac{1}{m}-1}}\right)\right\} \\
&= O\left\{\xi\left(\frac{1}{n+1}\right)\left(\frac{1}{(n+1)^{\frac{-1}{l}}}\right)\right\} \\
&= O\left\{\xi\left(\frac{1}{n+1}\right)(n+1)^{\frac{1}{l}}\right\}.
\end{aligned}
$$

Next,

$$
\begin{aligned}
|I_2| &= \left|\int_{\frac{1}{n+1}}^{\pi} \phi(t)K_n(t)dt\right| \\
&= \left|\int_{\frac{1}{n+1}}^{\pi} \frac{\Phi(t)}{\xi(t)}.\xi(t)K_n(t)dt\right| \\
&\le \left(\int_{\frac{1}{n+1}}^{\pi} \left|\frac{\phi(t)}{\xi(t)}\right|^l dt\right)^{\frac{1}{l}} \left(\int_{\frac{1}{n+1}}^{\pi} |\xi(t)K_n(t)|^m \, dt\right)^{\frac{1}{m}} \\
&\le O(1)\left(\int_{\frac{1}{n+1}}^{\pi} \left(\frac{\xi(t)}{t}\right)^m dt\right)^{\frac{1}{m}} \quad \text{using Lemma 4.2} \\
&\le O(1)\left(\int_{\frac{1}{\pi}}^{n+1} \left(\frac{\xi(\frac{1}{y})}{\frac{1}{y}}\right)^m dy\right)^{\frac{1}{m}}.
\end{aligned}
$$

Since $\xi(t)$ is a positive increasing function, so is $\left(\frac{\xi(\frac{1}{y})}{\frac{1}{y}}\right)$. Using second mean value theorem, we have

$$
\begin{aligned}
|I_2| &\le O\left\{(n+1)\xi\left(\frac{1}{n+1}\right)\right\}\left(\int_{\delta}^{n+1}\frac{dy}{y^2}\right)^{\frac{1}{m}}, \quad \text{for some } \frac{1}{\pi}\le\delta\le n+1 \\
&\le O\left\{(n+1)^{\frac{1}{l}}\xi\left(\frac{1}{n+1}\right)\right\}.
\end{aligned}
$$

Now, combining (21) and (22), we have

$$
|\tau_n - f(x)| \;=\; O\left\{(n+1)^{\frac{1}{l}}\xi\left(\frac{1}{n+1}\right)\right\}, \quad l\ge 1 \tag{23}
$$

Now, using L_r-norm, we have

$$\begin{aligned}\|\tau_n - f(x)\|_r &= \left\{\int_0^{2\pi} O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}^r dx\right\}^{\frac{1}{r}} \\ &= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\}\left(\int_0^{2\pi} dx\right)^{\frac{1}{r}} \\ &= O\left\{(n+1)^{\beta+\frac{1}{r}}\xi\left(\frac{1}{n+1}\right)\right\},\end{aligned}$$

which completes the proof of the theorem. □

6. Conclusions

The result established here is more general than some earlier existing results in the sense that, for $q = 1$ and $A = \frac{1}{n+1}$ our proposed mean reduces to $(E, 1)(C, 1)$ mean. Moreover, for $q = 1$ and $A = \frac{p_n}{P_n}$ our proposed mean reduces to $(E, 1)(N, p_n)$ mean.

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