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THE BOUNDEDNESS OF A CLASS OF FRACTIONAL TYPE ROUGH HIGHER ORDER COMMUTATORS ON VANISHING GENERALIZED WEIGHTED MORREY SPACES

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Abstract. This paper includes the new bounds that feature the vanishing generalized weighted Morrey spaces. In this regard, the article outlines the improved bounds about the class of fractional type rough higher order commutators on vanishing generalized weighted Morrey spaces.

Keywords: Fractional type higher order commutator operator, rough kernel, vanishing generalized weighted Morrey space.

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1. Introduction

Let $\Omega \in L_s(S^{n-1})$, $1 < s \leq \infty$. Ω is the function defined on $\mathbb{R}^n \setminus \{0\}$ satisfying the homogeneous condition of degree zero, that is,

$$\Omega(\lambda x) = \Omega(x) \text{ for any } \lambda > 0,\, x \in \mathbb{R}^n \setminus \{0\} \tag{1}$$

and the integral zero property (=the vanishing moment condition) over the unit sphere S^{n-1}, that is,

$$\int\limits_{S^{n-1}} \Omega(x')d\sigma(x') = 0, \tag{2}$$

where $x' = \frac{x}{|x|}$ for any $x \neq 0$.

In this paper we consider the following higher order (= k-th order) commutator operators of rough fractional integral and maximal operators,

$$\begin{aligned} T_{\Omega,\alpha}^{A,k} f(x) &= T_{\Omega,\alpha}\left(\left(A\left(x\right) - A\left(\cdot\right)\right)^k f\left(\cdot\right)\right)(x), \qquad k = 0,1,2,\dots, \\ &= \int\limits_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^{n-\alpha}} \left(A\left(x\right) - A\left(y\right)\right)^k f(y)dy \end{aligned} \tag{3}$$

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and

$$M_{\Omega,\alpha}^{A,k}f(x) = M_{\Omega,\alpha}\left(\left(A(x)-A(\cdot)\right)^k f(\cdot)\right)(x), \qquad k=0,1,2,\dots,$$
$$= \sup_{r>0}\frac{1}{r^{n-\alpha}}\int\limits_{|x-y|<r}|\Omega(x-y)|\,|A(x)-A(y)|^k\,|f(y)|\,dy, \tag{4}$$

as long as the integrals above make sense, where rough fractional integral operator $T_{\Omega,\alpha}$ and rough fractional maximal operator $M_{\Omega,\alpha}$ are defined by

$$T_{\Omega,\alpha}f(x) = \int\limits_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)dy \qquad 0<\alpha<n$$

and

$$M_{\Omega,\alpha}f(x) = \sup_{r>0}\frac{1}{r^{n-\alpha}}\int\limits_{|x-y|<r}|\Omega(x-y)|\,|f(y)|\,dy \qquad 0<\alpha<n.$$

For $k=1$ above, $T_{\Omega,\alpha}^{A,k}$ and $M_{\Omega,\alpha}^{A,k}$ are obviously reduced to the rough commutator operators of $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$, respectively:

$$[A,T_{\Omega,\alpha}]f(x) = A(x)\,T_{\Omega,\alpha}f(x) - T_{\Omega,\alpha}(Af)(x)$$
$$= \int\limits_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}\left(A(x)-A(y)\right)f(y)dy$$

and

$$[A,M_{\Omega,\alpha}]f(x) = A(x)\,M_{\Omega,\alpha}f(x) - M_{\Omega,\alpha}(Af)(x)$$
$$= \sup_{r>0}\frac{1}{r^{n-\alpha}}\int\limits_{|x-y|<r}|\Omega(x-y)|\,|A(x)-A(y)|\,|f(y)|\,dy.$$

Moreover, $T_{\Omega,\alpha}^{A,k}$ and $M_{\Omega,\alpha}^{A,k}$ are trivial generalizations of the above commutators, respectively.

Here and henceforth, $F\approx G$ means $F\gtrsim G\gtrsim F$; while $F\gtrsim G$ means $F\geq CG$ for a constant $C>0$; and also C stands for a positive constant that can change its value in each statement without explicit mention.

Now, let us list some definitions that we need in the proof of following Theorem 2.1:

Definition 1.1. (***Bounded Mean Oscillation (BMO)***) *We denote the mean value of f on $B=B(x,r)\subset\mathbb{R}^n$ by*

$$f_B = M(f,B) = M(f,x,r) = \frac{1}{|B|}\int\limits_B f(y)dy,$$

and the mean oscillation of f on $B=B(x,r)$ by

$$MO(f,B) = MO(f,x,r) = \frac{1}{|B|}\int\limits_B |f(y)-f_B|dy.$$

We also define for a non-negative function ϕ on $\mathbb{R}^n$

$$MO_\phi(f,B) = MO_\phi(f,x,r) = \frac{1}{\phi(|B|)\,|B|}\int\limits_B |f(y)-f_B|dy.$$

Now, we define

$$BMO_\phi = \left\{ f \in L_1^{loc}(\mathbb{R}^n) : \sup_B MO_\phi(f,B) < \infty \right\}$$

and

$$\|f\|_{BMO_\phi} = \sup_B MO_\phi(f,B).$$

The real importance comes when $\phi = 1$, in which case $BMO_\phi = BMO$.

Definition 1.2. [1, 3] (***Weighted Lebesgue space***) *Let $1 \le p \le \infty$ and given a weight $w(x) \in A_p(\mathbb{R}^n)$, we shall define weighted Lebesgue spaces as*

$$L_p(w) \equiv L_p(\mathbb{R}^n, w) = \left\{ f : \|f\|_{L_{p,w}} = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} < \infty \right\}, \qquad 1 \le p < \infty.$$

$$L_{\infty,w} \equiv L_\infty(\mathbb{R}^n, w) = \left\{ f : \|f\|_{L_{\infty,w}} = \operatorname*{esssup}_{x\in\mathbb{R}^n} |f(x)| w(x) < \infty \right\}.$$

Here and later, we refer to A_p as the the Muckenhoupt classes. That is, $w(x) \in A_p(\mathbb{R}^n)$ for some $1 < p < \infty$ if $\left(\frac{1}{|B|} \int_B w(y) dy \right) \left(\frac{1}{|B|} \int_B w(y)^{-\frac{1}{p-1}} dy \right)^{p-1} \le C$ for all balls B (see [1] for more details).

Now, let us consider the Muckenhoupt-Wheeden class $A(p,q)$ defined in [5]. One says that $w(x) \in A(p,q)$ for $1 < p < q < \infty$ if and only if

$$[w]_{A(p,q)} := \sup_B \left(|B|^{-1} \int_B w(x)^q dx \right)^{\frac{1}{q}} \left(|B|^{-1} \int_B w(x)^{-p'} dx \right)^{\frac{1}{p'}} < \infty, \tag{5}$$

where the supremum is taken over all the balls B. Note that, by Hölder's inequality, for all balls B we have:

$$[w]_{A(p,q)} \ge [w]_{A(p,q)(B)} = |B|^{\frac{1}{p} - \frac{1}{q} - 1} \|w\|_{L_q(B)} \|w^{-1}\|_{L_{p'}(B)} \ge 1.$$

By (5), we have:

$$\left(\int_B w(x)^q dx \right)^{\frac{1}{q}} \left(\int_B w(x)^{-p'} dx \right)^{\frac{1}{p'}} \lesssim |B|^{\frac{1}{q} + \frac{1}{p'}}.$$

On the other hand, let $\mu(x) = w(x)^{s'}$, $\tilde{p} = \frac{p}{s'}$ and $\tilde{q} = \frac{q}{s'}$. If $w(x)^{s'} \in A\left(\frac{p}{s'}, \frac{q}{s'}\right)$, then we get $\mu(x) \in A(\tilde{p}, \tilde{q})$.

Now, we introduce some spaces which play important roles in PDE. Except the weighted Lebesgue space $L_p(w)$, the weighted Morrey space $L_{p,\kappa}(w)$, which is a natural generalization of $L_p(w)$ is another important function space. Then, the definition of generalized weighted Morrey spaces $M_{p,\varphi}(w)$ which can be viewed as extension of $L_{p,\kappa}(w)$ has been given as follows:

Definition 1.3. ***(Generalized weighted Morrey spaces)*** *For $1 \le p < \infty$, positive measurable function $\varphi(x,r)$ on $\mathbb{R}^n \times (0,\infty)$ and nonnegative measurable function w on $\mathbb{R}^n$, $f \in M_{p,\varphi}(w) \equiv M_{p,\varphi}(\mathbb{R}^n, w)$ if $f \in L^{loc}_{p,w}(\mathbb{R}^n)$ and*

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} \|f\|_{L_p(B(x,r),w)} < \infty.$$

Note that for $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa}{p}}$, $0 < \kappa < 1$ and $\varphi(x,r) \equiv 1$, we have $M_{p,\varphi}(w) = L_{p,\kappa}(w)$ and $M_{p,\varphi}(w) = L_p(w)$, respectively.

Moreover, Gürbüz [2] proved that the operators $T^{A,k}_{\Omega,\alpha}$ and $M^{A,k}_{\Omega,\alpha}$ from one generalized weighted Morrey space $M_{p,\varphi_1}(w^p, \mathbb{R}^n)$ to another $M_{q,\varphi_2}(w^q, \mathbb{R}^n)$ are bounded.

The following definition was introduced by Gürbüz [4].

Definition 1.4. ***(Vanishing generalized weighted Morrey spaces)*** *For $1 \le p < \infty$, $\varphi(x,r)$ is a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and nonnegative measurable function w on $\mathbb{R}^n$, $f \in VM_{p,\varphi}(w) \equiv VM_{p,\varphi}(\mathbb{R}^n, w)$ if $f \in L^{loc}_{p,w}(\mathbb{R}^n)$ and*

$$\lim_{r\to 0^+} \sup_{x\in\mathbb{R}^n} \frac{1}{\varphi(x,r)} \|f\|_{L_p(B(x,r),w)} = 0. \tag{6}$$

Inherently, it is appropriate to impose on $\varphi(x,t)$ with the following circumstances:

$$\lim_{t\to 0^+} \sup_{x\in\mathbb{R}^n} \frac{(w(B(x,t)))^{\frac{1}{p}}}{\varphi(x,t)} = 0, \tag{7}$$

and

$$\inf_{t>1} \sup_{x\in\mathbb{R}^n} \frac{(w(B(x,t)))^{\frac{1}{p}}}{\varphi(x,t)} > 0. \tag{8}$$

From (7) and (8), we easily know that the bounded functions with compact support belong to $VM_{p,\varphi}(w)$. On the other hand, the space $VM_{p,\varphi}(w)$ is Banach space with respect to the following finite quasi-norm

$$\|f\|_{VM_{p,\varphi}(w)} = \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{\varphi(x,r)} \|f\|_{L_p(B(x,r),w)},$$

such that

$$\lim_{r\to 0^+} \sup_{x\in\mathbb{R}^n} \frac{1}{\varphi(x,r)} \|f\|_{L_p(B(x,r),w)} = 0,$$

we omit the details. Moreover, we have the following embeddings:

$$VM_{p,\varphi}(w) \subset M_{p,\varphi}(w), \qquad \|f\|_{M_{p,\varphi}(w)} \le \|f\|_{VM_{p,\varphi}(w)}.$$

Henceforth, we write $\varphi \in \mathcal{B}(w)$ if $\varphi(x,r)$ is a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and positive for all $(x,r) \in \mathbb{R}^n \times (0,\infty)$ and satisfies (7) and (8).

Inspired of [2], the aim of the present paper is to study the boundedness of the operators $T^{A,k}_{\Omega,\alpha}$ and $M^{A,k}_{\Omega,\alpha}$ generated by $T_{\Omega,\alpha}$ and $M_{\Omega,\alpha}$ with a BMO functions on vanishing generalized weighted Morrey spaces, respectively. That is, in this paper we will consider the following problem.

2. Main results

Let us state our main result as follows.

Theorem 2.1. *Suppose that* $0 < \alpha < n$, $1 \le s' < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $1 < q < \infty$, $\Omega \in L_s(S^{n-1})\,(s > 1)$ *satisfies (1) such that* $k \in \mathbb{N}$, $w\,(x)^{s'} \in A\left(\frac{p}{s'}, \frac{q}{s'}\right)$, $A \in BMO\,(\mathbb{R}^n)$, $T_{\Omega,\alpha}^{A,k}$, $M_{\Omega,\alpha}^{A,k}$ *are defined as (3), (4) and* $T_{\Omega,\alpha}^{A,k}$ *satisfies (13) in* [2]. *If* $\varphi_1 \in \mathcal{B}\,(w^p)$, $\varphi_2 \in \mathcal{B}\,(w^q)$ *and the pair* (φ_1, φ_2) *satisfies the conditions*

$$C_\delta := \int_\delta^\infty \left(1 + \ln \frac{t}{r}\right)^k \sup_{x \in \mathbb{R}^n} \frac{\varphi_1\,(x,t)}{(w^q\,(B\,(x,t)))^{\frac{1}{q}}} \frac{1}{t} dt < \infty, \tag{9}$$

for every $\delta > 0$, *and*

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right)^k \frac{\varphi_1\,(x,t)}{(w^q\,(B\,(x,t)))^{\frac{1}{q}}} \frac{1}{t} dt \le C_0 \frac{\varphi_2(x,r)}{(w^q\,(B\,(x,t)))^{\frac{1}{q}}}, \tag{10}$$

where C_0 *does not depend on* $r > 0$, *then the operators* $T_{\Omega,\alpha}^{A,k}$ *and* $M_{\Omega,\alpha}^{A,k}$ *are bounded from* $VM_{p,\varphi_1}\,(w^p)$ *to* $VM_{q,\varphi_2}\,(w^q)$. *Moreover,*

$$\left\| T_{\Omega,\alpha}^{A,k} f \right\|_{VM_{q,\varphi_2}(w^q,\mathbb{R}^n)} \lesssim \|A\|_*^k \, \|f\|_{VM_{p,\varphi_1}(w^p,\mathbb{R}^n)}\,, \tag{11}$$

$$\left\| M_{\Omega,\alpha}^{A,k} f \right\|_{VM_{q,\varphi_2}(w^q,\mathbb{R}^n)} \lesssim \|A\|_*^k \, \|f\|_{VM_{p,\varphi_1}(w^p,\mathbb{R}^n)}\,.$$

For $\alpha = 0$, from Theorem 2.1, we get the following:

Corollary 2.1. *Suppose that* $1 < p < \infty$, $s' < p$, $\Omega \in L_s(S^{n-1})\,(s > 1)$ *satisfies (1) and (2) such that* $k \in \mathbb{N}$, $w\,(x)^{s'} \in A_{\frac{p}{s'}}$, $A \in BMO\,(\mathbb{R}^n)$, $T_\Omega^{A,k}$, $M_\Omega^{A,k}$ *are defined as*

$$\begin{aligned} T_\Omega^{A,k} f(x) &= T_\Omega \left((A\,(x) - A\,(\cdot))^k f\,(\cdot) \right)(x)\,, \qquad k = 0,1,2,\ldots, \\ &= p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} (A\,(x) - A\,(y))^k f(y) dy \end{aligned}$$

and the corresponding higher order (= k-th order) commutator operator of M_Ω :

$$\begin{aligned} M_\Omega^{A,k} f(x) &= M_\Omega \left((A\,(x) - A\,(\cdot))^k f\,(\cdot) \right)(x)\,, \qquad k = 0,1,2,\ldots, \\ &= \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega\,(x-y)|\,|A\,(x) - A\,(y)|^k\,|f(y)|\,dy \end{aligned}$$

and $T_\Omega^{A,k}$ satisfies (11) in [3]. If $\varphi \in \mathcal{B}\,(w)$ and the pair (φ_1, φ_2) satisfies the conditions

$$C_{\delta'} := \int_\delta^\infty \left(1 + \ln \frac{t}{r}\right)^k \sup_{x \in \mathbb{R}^n} \frac{\varphi_1\,(x,t)}{(w^p\,(B\,(x,t)))^{\frac{1}{p}}} \frac{1}{t} dt < \infty,$$

for every $\delta' > 0$, and

$$\int_r^\infty \left(1 + \ln \frac{t}{r}\right)^k \frac{\varphi_1\,(x,t)}{(w^p\,(B\,(x,t)))^{\frac{1}{p}}} \frac{1}{t} dt \lesssim \frac{\varphi_2(x,r)}{(w^p\,(B\,(x,t)))^{\frac{1}{p}}},$$

then

$$\left\|T_{\Omega}^{A,k}f\right\|_{VM_{p,\varphi_2}(w,\mathbb{R}^n)} \lesssim \|A\|_*^k \|f\|_{VM_{p,\varphi_1}(w,\mathbb{R}^n)},$$

$$\left\|M_{\Omega}^{A,k}f\right\|_{VM_{p,\varphi_2}(w,\mathbb{R}^n)} \lesssim \|A\|_*^k \|f\|_{VM_{p,\varphi_1}(w,\mathbb{R}^n)}.$$

3. Proof of the main result

Proof of Theorem 2.1.

Proof. By Definition 1.4, (13) in [2] and (10) we get

$$\begin{aligned}\left\|T_{\Omega,\alpha}^{A,k}f\right\|_{VM_{q,\varphi_2}(w^q,\mathbb{R}^n)} &= \sup_{x\in\mathbb{R}^n,r>0}\frac{\left\|T_{\Omega,\alpha}^{A,k}f\right\|_{L_q(w^q,B(x,r))}}{\varphi_2(x,r)}\\ &\lesssim \sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi_2(x,r)}\|A\|_*^k\left(w^q\left(B(x,r)\right)\right)^{\frac{1}{q}}\\ &\times\int_r^\infty\left(1+\ln\frac{t}{r}\right)^k\|f\|_{L_p(w^p,B(x,t))}\left(w^q\left(B(x,t)\right)\right)^{-\frac{1}{q}}\frac{1}{t}dt\\ &\lesssim \sup_{x\in\mathbb{R}^n,r>0}\frac{1}{\varphi_2(x,r)}\|A\|_*^k\left(w^q\left(B(x,r)\right)\right)^{\frac{1}{q}}\\ &\times\int_r^\infty\left(1+\ln\frac{t}{r}\right)^k\frac{\varphi_1(x,t)}{\left(w^q\left(B\left(x,t\right)\right)\right)^{\frac{1}{q}}}\frac{1}{t}\|f\|_{L_p(w^p,B(x,t))}\varphi_1(x,t)^{-1}dt\\ &\lesssim \|A\|_*^k\|f\|_{VM_{p,\varphi_1}(w^p,\mathbb{R}^n)}.\end{aligned}$$

At last, we need to prove that

$$\lim_{r\to0^+}\sup_{x\in\mathbb{R}^n}\frac{1}{\varphi_2(x,r)}\left\|T_{\Omega,\alpha}^{A,k}f\right\|_{L_q(w^q,B(x_0,r))} \lesssim \lim_{r\to0^+}\sup_{x\in\mathbb{R}^n}\frac{1}{\varphi_1(x,r)}\|f\|_{L_p(w^p,B(x_0,r))}=0.$$

Indeed, for any $\epsilon>0$, let $0<r<\psi$. By (13) in [2], we have:

$$\frac{\left\|T_{\Omega,\alpha}^{A,k}f\right\|_{L_q(w^q,B(x,r))}}{\varphi_2(x,r)} \le C\left[\mathcal{F}_\psi\left(x,r\right)+\mathcal{G}_\psi\left(x,r\right)\right], \tag{12}$$

where

$$\begin{aligned}\mathcal{F}_\psi\left(x,r\right) &:= \frac{\|A\|_*^k\left(w^q\left(B(x,r)\right)\right)^{\frac{1}{q}}}{\varphi_2(x,r)}\int_r^\psi\left(1+\ln\frac{t}{r}\right)^k\varphi_1(x,t)\left(w^q\left(B(x,t)\right)\right)^{-\frac{1}{q}}\\ &\times\sup_{0<r<t}\left[\frac{\|f\|_{L_p(w^p,B(x,t))}}{\varphi_1(x,t)}\right]\frac{1}{t}dt\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}_\psi\left(x,r\right) &:= \frac{\|A\|_*^k\left(w^q\left(B(x,r)\right)\right)^{\frac{1}{q}}}{\varphi_2(x,r)}\int_\psi^\infty\left(1+\ln\frac{t}{r}\right)^k\varphi_1(x,t)\|f\|_{L_p(w^p,B(x,t))}\left(w^q\left(B(x,t)\right)\right)^{-\frac{1}{q}}\\ &\times\sup_{0<r<t}\left[\frac{\|f\|_{L_p(w^p,B(x,t))}}{\varphi_1(x,t)}\right]\frac{1}{t}dt.\end{aligned}$$

For any $\epsilon > 0$, now we can choose any fixed $\psi > 0$ such that whenever $r \in (0, \psi)$,

$$\sup_{x\in\mathbb{R}^n} \sup_{0<r<\psi} \frac{\|f\|_{L_p(w^p,B(x,r))}}{\varphi_1(x,r)} < \frac{\epsilon}{2CC_0 \|A\|_*^k}$$

where C and C_0 are constants from (10) and (12), which is possible since $f \in VM_{p,\varphi_1}(w^p, \mathbb{R}^n)$.

This allows to guess the first term properly from the type $r \in (0, \psi)$ such that

$$\sup_{x\in\mathbb{R}^n} C\mathcal{F}_{\psi}(x,r) < \frac{\epsilon}{2}.$$

For the second term, in view of (9), we obtain

$$\mathcal{G}_{\psi}(x,r) \lesssim \|A\|_*^k \|f\|_{VM_{p,\varphi}(w^p,\mathbb{R}^n)} \frac{(w^q(B(x,r)))^{\frac{1}{q}}}{\varphi_2(x,r)}.$$

Since $\varphi_2 \in \mathcal{B}(w^q)$, it gets along to select r minor sufficient such that

$$\sup_{x\in\mathbb{R}^n} \frac{w^q(B(x,r))}{\varphi_2^q(x,r)} \lesssim \left(\frac{\epsilon}{2CC_0 \|f\|_{VM_{p,\varphi}(w^p,\mathbb{R}^n)}}\right)^q.$$

Hence,

$$\sup_{x\in\mathbb{R}^n} C\mathcal{G}_{\psi}(x,r) < \frac{\epsilon}{2}.$$

Thus,

$$\frac{\left\|T_{\Omega,\alpha}^{A,k} f\right\|_{L_q(w^q,B(x,r))}}{\varphi_2(x,r)} < \epsilon,$$

which means that

$$\lim_{r\to 0^+} \sup_{x\in\mathbb{R}^n} \frac{1}{\varphi_2(x,r)} \left\|T_{\Omega,\alpha}^{A,k} f\right\|_{L_q(w^q,B(x_0,r))} = 0,$$

which completes the proof of (11). On the other hand, since $M_{\Omega,\alpha}^{A,k} f(x) \leq \widetilde{T}_{|\Omega|,\alpha}^{A,k}(|f|)(x)$, $x \in \mathbb{R}^n$ (see Lemma 6 in [2]) we can also use the same method for $M_{\Omega,\alpha}^{A,k}$, so we omit the details. As a result, we complete the proof of Theorem 2.1. □

4. Conclusions

In this paper, the author has researched the higher order commutators of rough fractional integrals and maximal operators with BMO functions. The boundedness of these operators on vanishing generalized weighted Morrey spaces has been established, respectively. The Morrey type spaces play important roles both in harmonic analysis and PDE. The results obtained in this paper extend some known results such as the boundedness on generalized weighted Morrey spaces in [2]. Thus, the research in meaningful.

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Ferit Gürbüz for the photography and short autobiography, see TWMS J. App. Eng. Math., V.10, current issue

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