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# SOME FIXED POINT RESULTS IN B-MULTIPLICATIVE METRIC SPACE

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Abstract. The desired outcome of this paper is to extend the result of Ali et.al (U.P.B. Sci. Bull. Series A, 79(3):107-116,2017) by applying contractive condition only on a closed ball. Moreover, we obtained some fixed point results satisfying contractive condition on closed ball in $b$-multiplicative metric space. Our results are improved and more generalized form of several recent results.

Keywords: Common fixed point; closed ball; $b$-multiplicative metric space; $b$-metric space; contractive condition.

AMS Subject Classification: 46S40; 47H10; 54H25

## 1. Introduction

Bakhtin [1] was the first who gave the idea of $b$-metric. After that Czerwik [2] gave an axiom and formally defined a $b$-metric space. For further results on $b$-metric space, see [3, 4]. Ozaksar and Cevical [5] investigated multiplicative metric space and proved its topological properties. Mongkolkeha et al. [6] described the concept of multiplicative proximal contraction mapping and proved best proximity point theorems for such mappings. Recently, Abbas et al. [7] proved some common fixed points results of quasi weak commutative mappings on a closed ball in the setting of multiplicative metric spaces. They also describes the main conditions for the existence of common solution of multiplicative boundary value problem. For further results on multiplicative metric space, see [8, 9, 10]. In 2017, Ali et al. [11] introduced the notion of $b$-multiplicative and proved some fixed point result. As an application, they established an existence theorem for the solution of a system of Fredholm multiplicative integral equations. Shoaib et al. [4], discussed the result for mappings satisfying contraction condition on a closed ball in a $b$-metric space. For further results on closed ball, see [12, 13, 14, 15, 16, 17, 18]. In this paper, we proved a result in [7] for b-multiplicative metric space. Moreover, we proved the result in [11] by applying contractive condition only on a closed ball. The following definitions and results will be used to understand this paper.

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**Definition 1.1** [11] Let $W$ be a non-empty set and let $s \geq 1$ be a given real number. A mapping $m : W \times W \to [1, \infty)$ is called a $b$-multiplicative metric with coefficient $s$, if the following conditions hold:

$(i)$ $m(w, y) > 1$ for all $w, y \in W$ with $w \neq y$ and $m(w, y) = 1$ if and only if $w = y$.

$(ii)$ $m(w, y) = m(y, w)$ for all $w, y \in W$.

$(iii)$ $m(w, z) \leq [m(w, y).m(y, z)]^s$ for all $w, y, z \in W$.

The triplet $(W, m, s)$ is called $b$-multiplicative metric space.

**Example 1.2** [11] Let $W = [0, \infty)$. Define a mapping $m_a : W \times W \to [1, \infty)$

$$m_a(w, y) = a^{(w-y)^2},$$

where $a > 1$ is any fixed real number. Then for each $a$, $m_a$ is $b$-multiplicative metric on $W$ with $s = 2$. Note that $m_a$ is a not multiplicative metric on $W$.

**Definition 1.3** [11] Let $(W, m)$ be a $b$-multiplicative metric space.

$(i)$ A sequence $\{w_n\}$ is convergent iff there exist $w \in W$ such that

$$m(w_n, w) \to 1, \text{ as } n \to +\infty.$$

$(ii)$ A sequence $\{w_n\}$ is called $b-$multiplicative Cauchy iff

$$m(w_m, w_n) \to 1, \text{ as } m, n \to +\infty.$$

$(iii)$ A $b$-multiplicative metric space $(W, m)$ is said to be complete if every multiplicative Cauchy sequence in $Y$ is convergent to some $y \in W$.

**Definition 1.4** [3] Let $W$ be a non-empty set and $s \geq 1$ be a real number. A mapping $d\colon W \times W \to \mathbb{R}^+ \cup \{0\}$ is said to be $b$-metric with coefficient "$s$", if for all $w, y, z \in W$, the following conditions hold:

$(i)$ $d(w, y) = 0$ if and only if $w = y$;

$(ii)$ $d(w, y) = d(y, w)$;

$(iii)$ $d(w, z) \leq s\left[d(w, y) + d(y, z)\right]$.

The pair $(W, d)$ is called $b$-metric space.

**Remark 1.5** [11] Every $b$-metric space $(W, d)$ generates a $b-$multiplicative metric space $(W, m)$ defined as

$$m\left(x, y\right) = e^{d(x,y)}.$$

## 2. Result for Ciric Type Contraction

**Theorem 2.1** Let $(X, d)$ be a complete $b$-multiplicative metric space with coefficient "$s$" such that

$$d(fx, fy) \leq \max\left\{d(x, y), d(x, fx), d(y, fy), d(x, fy)^{\frac{1}{2s}}, d(y, fx)\right\}^k, \tag{2.1}$$

for all $x, y \in \overline{B(x_0, r)}$ and

$$d(b_0, fb_0) \leq r^{\frac{1-sk}{s}}, \tag{2.2}$$

where $k \in [0, \frac{1}{s})$. Then $f$ has a unique common fixed point.

*Proof.* Suppose $b_0 \in X$ and $b_1 \in X$ such that $b_1 = f(b_0), b_2 = f(b_1), ...b_{n+1} = f\left(b_n\right)$. Now, by inequality (2.2), we have

$$d(b_0, b_1) = d(b_0, fb_0) \leq r^{\frac{1-sk}{s}} \leq r.$$

This implies that $b_1 \in \overline{B(b_0,r)}$. Suppose that $b_2, b_{3,}...b_j \in \overline{B(b_0,r)}$. Now, if $j = 2i+1, i = 1,2,3,..., i = \frac{j-1}{2}$

$$\begin{aligned}
d(b_{2i+1}, b_{2i+2}) &= d(fb_{2i}, fb_{2i+1}) \\
&\leq \max\left\{ \begin{array}{c} d(b_{2i}, b_{2i+1}), d(b_{2i}, fb_{2i}), d(b_{2i+1}, fb_{2i+1}) \\ , d(b_{2i}, fb_{2i+1})^{\frac{1}{2s}}, d(b_{2i+1}, fb_{2i}) \end{array} \right\}^k \\
&\leq \max\left\{ \begin{array}{c} d(b_{2i}, b_{2i+1}), d(b_{2i}, b_{2i+1}), d(b_{2i+1}, b_{2i+2}) \\ , d(b_{2i}, b_{2i+2})^{\frac{1}{2s}}, d(b_{2i+1}, b_{2i+1}) \end{array} \right\}^k \\
&\leq \max\left\{ d(b_{2i}, b_{2i+1}), d(b_{2i+1}, b_{2i+2}) \right\}^k
\end{aligned}$$

$$d(b_{2i+1}, b_{2i+2}) \leq d(b_{2i}, b_{2i+1})^k. \tag{2.3}$$

If $j = 2i$ , $i = 1,2,3,..., i = \frac{j-1}{2}$

$$\begin{aligned}
d(b_{2i}, b_{2i+1}) &= d(fb_{2i-1}, fb_{2i}) \\
&\leq \max\left\{ \begin{array}{c} d(b_{2i-1}, b_{2i}), d(b_{2i-1}, fb_{2i-1}), d(b_{2i}, fb_{2i}) \\ , d(b_{2i-1}, fb_{2i})^{\frac{1}{2s}}, d(b_{2i}, fb_{2i-1}) \end{array} \right\}^k \\
&\leq \max\left\{ \begin{array}{c} d(b_{2i-1}, b_{2i}), d(b_{2i-1}, b_{2i}), d(b_{2i}, b_{2i+1}) \\ , d(b_{2i-1}, b_{2i+1})^{\frac{1}{2s}}, d(b_{2i}, b_{2i}) \end{array} \right\}^k \\
&\leq \max\left\{ d(b_{2i-1}, b_{2i}), d(b_{2i}, b_{2i+1}) \right\}^k
\end{aligned}$$

So, we have

$$d(b_{2i}, b_{2i+1}) \leq d(b_{2i-1}, b_{2i})^k. \tag{2.4}$$

From (2.3), (2.4) and by induction, we have

$$d(b_j, b_{j+1}) \leq d(b_0, b_1)^{k^j}. \tag{2.5}$$

$$\begin{aligned}
d(b_0, b_{j+1}) &\leq d(b_0, b_1)^s . d(b_1, b_2)^{s^2} . d(b_2, b_3)^{s^3} ... d(b_j, b_{j+1})^{s^{j+1}} \\
d(b_0, b_{j+1}) &\leq d(b_0, b_1)^s . d(b_0, b_1)^{s^2 k} . d(b_2, b_3)^{s^3 k^2} ... d(b_j, b_{j+1})^{s^{j+1} k^j} \\
d(b_0, b_{j+1}) &\leq d(b_0, b_1)^{s\left(1 + sk + s^2k^2 + ... + s^j k^j\right)} \\
d(b_0, b_{j+1}) &\leq d(b_0, b_1)^{s\left(1 + sk + s^2k^2 + ...\right)} \\
d(b_0, b_{j+1}) &\leq d(b_0, b_1)^{s(\frac{1}{1-sk})},
\end{aligned}$$

since $b_1 \in \overline{B(b_0,r)}$

$$d(b_0, b_{j+1}) \leq r^{\frac{1(1-sk)}{s} . \frac{s(1)}{1-sk}} \leq r.$$

This is implies that $b_{j+1} \in \overline{B(b_0,r)}$. By mathematical induction $b_n \in \overline{B(b_0,r)}$.

$$d(b_n, b_{n+1}) \leq d(b_0, b_1)^{k^n}. \tag{2.6}$$

Now, we show that $\{b_n\}$ is a Cauchy sequence for $m > n$.

$$\begin{aligned}
d(b_n, b_m) &\leq d(b_n, b_{n+1})^s . d(b_{n+1}, b_{n+2})^{s^2} . d(b_{n+2}, b_{n+3})^{s^3} ... d(b_{m-1}, b_m)^{s^m} \\
&\leq d(b_0, b_1)^{sk^n} . d(b_0, b_1)^{s^2 k^{n+1}} . d(b_0, b_1)^{s^3 k^{n+2}} ... d(b_0, b_1)^{s^m k^{m-1}} \\
&< d(b_0, b_1)^{\left(sk^n + s^2 k^{n+1} + ...\right)} = d(b_0, b_1)^{\frac{sk^n}{1-sk}}.
\end{aligned}$$

Applying $\lim_{n\to\infty}$, we get $d(b_n, b_m) \leq 1$. This implies that $\{b_n\}$ is a multiplicative Cauchy sequence in $X$. Since $X$ is complete so $b_n \to b^* \in X$. Now,

$$\begin{aligned} d(b^*, fb^*) &\leq d(b^*, b_{n+1})^s.d(fb_n, fb^*)^s \\ d(b^*, fb^*) &\leq d(b^*, b_n)^s.\left(\max\left\{\begin{array}{c} d(b_n, b^*), d(b_n, b_{n+1}), d(b^*, fb^*), \\ d(b_n.fb^*)^{\frac{1}{2s}}, d(b^*, b_{n+1}) \end{array}\right\}^k\right)^s, \end{aligned}$$

Taking $\lim_{n\to\infty}$, we get

$$\begin{aligned} d(b^*, fb^*) &\leq 1.\max\{1, 1, d(b^*, fb^*), 1\}^{ks} \\ d(b^*, fb^*) &\leq d(b^*, fb^*)^{ks} \\ d(b^*, fb^*)^{1-ks} &\leq 1. \end{aligned}$$

This is implies that

$$d(b^*, fb^*) \leq 1. \tag{2.7}$$

So $b^* = fb^*$. Hence $b^*$is a fixed point of $f$. Let $z$ be another fixed point of $f$ such that $fz = z$.

$$\begin{aligned} d(b^*, z) &= d(fb^*, fz) \\ &\leq \max\left\{d(b^*, z), d(b^*, b^*), d(z, z), d(b^*, z)^{\frac{1}{2s}}, d(z, b^*)\right\}^k \\ d(b^*, z)^{1-k} &\leq 1, \end{aligned}$$

So $b^* = z$. Hence $b^*$is a unique fixed point of $f$. □

**Corollary 2.2** Let $(X, d)$ be a complete $b$-multiplicative metric space with coefficient "$s$" such that

$$d(fx, fy) \leq (d(x, y))^k,$$

for all $x, y \in \overline{B(x_0, r)}$ and

$$d(b_0, fb_0) \leq r^{\frac{1-sk}{s}},$$

where $k \in [0, \frac{1}{s})$. Then $f$ has a unique common fixed point.

## 3. Fixed Point Result for Four Mappings

**Theorem 3.1** Let $S, T, f$ and $g$ be self-mappings of a complete multiplicative $b$-metric space $(X, d)$ with coefficient "$s$" and $(f, S)$ and $(g, T)$ weakly commutative with $SX \subset gX$, $TX \subset fX$, and one of $S, T, f$ and $g$ is continuous. Let $b_0 \in X$ and $Sb_0 = gb_1 = y_0$. If there exists $\lambda \in (0, \frac{1}{2})$ such that

$$d(Sx, Ty) \leq (M(x, y))^\lambda, \text{ for any } x, y \in \overline{B(y_0, r)} \tag{3.1}$$

holds, where

$$M(x, y) = \max\{d(fx, gy), d(fx, Sx), d(gy, Ty), d(Sx, gy), d(fx, Ty)^{\frac{1}{s}}\}. \tag{3.2}$$

Then there exists a unique common fixed point of $f, T, S$ and $g$ in $\overline{B(y_0, r)}$ provided that

$$d(y_0, Tb_1) \leq r^{(1-sh)/s}, \text{ where } h = \lambda/(1-\lambda) \text{ and } sh < 1. \tag{3.3}$$

*Proof.* Let $b_0$ be a given point in $X$. Since $SX \subset gX$, we can choose a point $b_1$in $X$ such that $Sb_0 = gb_1 = y_0$. Similarly, there exists a point $b_2 \in X$ such that $Tb_1 = fb_2 = y_1$. Indeed, it follows from the assumption that $TX \subset fX$. Thus we can construct sequences $\{b_n\}$ and $\{y_n\}$ in $X$ such that

$$y_{2n} = Sb_{2n} = gb_{2n+1},\ y_{2n+1} = Tb_{2n+1} = fb_{2n+2},\ n = 0, 1, 2, ...$$

Now, we show that $\{y_n\}$ is a sequence in $\overline{B(y_0, r)}$. By (3.3), $d(y_0, y_1) = d(y_0, Tb_1) \leq r^{(1-sh)/s} < r$. Hence $y_1 \in \overline{B(y_0, r)}$. Assume $y_2, y_3..., y_j \in \overline{B(y_0, r)}$ for some $j \in \mathbb{N}$. Then if $j = 2k$, it follows from conditions (3.1) and (3.2), that

$$\begin{aligned}
d(y_{2k},\ y_{2k+1}) &= d(Sb_{2k}, Tb_{2k+1}) \leq (M(b_{2k}, y_{2k+1}))^{\lambda} \\
&\leq \max\{d(fb_{2k}, gy_{2k+1}), d(fb_{2k}, Sb_{2k}), d(gb_{2k+1}, Tb_{2k+1}), \\
&\quad d(Sb_{2k}, gb_{2k+1}), d(fb_{2k}, Tb_{2k+1})^{1/s}\}^{\lambda} \\
&\leq \max\{d(y_{2k-1}, y_{2k}), d(y_{2k-1}, y_{2k}), d(y_{2k}, y_{2k+1}) \\
&\quad , d(y_{2k}, y_{2k}), d(y_{2k-1}, y_{2k+1})^{1/s}\}^{\lambda} \\
&\leq \max\{d(y_{2k-1}, y_{2k}), d(y_{2k}, y_{2k+1}), 1, \\
&\quad , d(y_{2k-1}, y_{2k})^{1/s}.d(y_{2k}, y_{2k+1})^{1/s}\}^{\lambda} \\
&\leq d(y_{2k-1}, y_{2k})^{\lambda}, d(y_{2k}, y_{2k+1})^{\lambda} \\
d(y_{2k},\ y_{2k+1})^{1-\lambda} &\leq d(y_{2k-1}, y_{2k})^{\lambda} \\
d(y_{2k},\ y_{2k+1}) &\leq d(y_{2k-1}, y_{2k})^{\lambda/1-\lambda}
\end{aligned}$$

$$d(y_{2k},\ y_{2k+1}) \leq d(y_{2k-1}, y_{2k})^{h}. \tag{3.4}$$

Thus Similarly, if $j = 2k + 1$, then

$$d(y_{2k+1},\ y_{2k+2}) \leq d(y_{2k}, y_{2k+1})^{h}. \tag{3.5}$$

Hence from(3.4)and (3.5) ,we have

$$d(y_k,\ y_{k+1}) \leq d(y_{k-1}, y_k)^{h}. \tag{3.6}$$

From (3.4), (3.5) and (3.6),we have

$$d(y_k,\ y_{k+1}) \leq d(y_{k-1}, y_k)^{h} \leq d(y_{k-2}, y_{k-1})^{h^2} \cdots d(y_0, y_1)^{h^k} \quad \forall k \in \mathbb{N}. \tag{3.7}$$

Thus, from (3.7),we have

$$\begin{aligned}
d(y_0,\ y_{k+1}) &\leq d(y_0, y_1)^{s}.d(y_1, y_2)^{s^2}.d(y_2, y_3)^{s^3} \cdots d(y_k, y_{k+1})^{s^{k+1}} \\
&\leq d(y_0, y_1)^{s}.d(y_0, y_1)^{s^2 h}.d(y_2, y_3)^{s^3 h^2} \cdots d(y_k, y_{k+1})^{s^{k+1} h^k} \\
&\leq d(y_0, y_1)^{s(1+sh+s^2h^2+\cdots+s^k h^k)} \\
&\leq d(y_0, y_1)^{s(1-(sh)^{k+1})/1-sh}.
\end{aligned}$$

Since $y_1 \in \overline{B(y_0, r)}$, we have

$$\begin{aligned}
d(y_0,\ y_{k+1}) &\leq (r)^{\frac{1-sh}{s} \cdot s \frac{(1-(sh)^{k+1})}{1-sh}} \\
&\leq r^{1-(sh)^{k+1}} \leq r
\end{aligned}$$

$$d(y_0, y_{k+1}) \leq r \text{ for all } k \in \mathbb{N}. \tag{3.8}$$

Hence $y_{k+1} \in \overline{B(y_0, r)}$. By induction on $n$, we conclude that $\{y_n\} \in \overline{B(y_0, r)}$ for all $n \in \mathbb{N}$. We claim that the sequence $\{y_n\}$ satisfies the multiplicative Cauchy criterion for convergence in $(\overline{B(y_0,r)},d)$. To see this let $m,n\in\mathbb{N}$ be such that $m>n$ and let $m=n+p$, then

$$\begin{aligned}
d(y_n,\ y_{n+p}) &\le d(y_n,y_{n+1})^s.d(y_{n+1},y_{n+2})^{s^2}\cdots d(y_{n+p-1},y_{n+p})^{s^p}\\
&\le d(y_0,y_1)^{sh^n}.d(y_0,y_1)^{s^2h^{n+1}}\cdots d(y_{n+p-1},y_{n+p})^{s^ph^{n+p-1}}\\
&< d(y_0,y_1)^{sh^n+s^2h^{n+1}+\cdots}\le r^{\frac{1-sh}{s}\cdot\frac{sh^n}{1-sh}}=r^{h^n}.
\end{aligned}$$

Letting $n\to\infty$., we have $d(y_m,y_n)\to 1$, as $n,m\to\infty$. Hence the sequence $\{y_n\}$ is a multiplicative Cauchy sequence. As $(X,d)$ is complete, so $(\overline{B(y_0,r)},d)$ is complete. Hence $\{y_n\}$ has a limit, say $u$ in $\overline{B(y_0,r)}$. The fact that $\{Sb_{2n}\}=\{gb_{2n+1}\}=\{y_{2n}\}$ and $\{Tb_{2n+1}\}=\{fb_{2n+2}\}=\{y_{2n+1}\}$ are subsequences of $\{y_n\}$ makes $\lim_{n\to\infty}Sb_{2n}=\lim_{n\to\infty}gb_{2n+1}=\lim_{n\to\infty}T_{2n+1}=\lim_{n\to\infty}f_{2n+2}=u$. Suppose $f$ is continuous, then

$$\lim_{n\to\infty}f(Sb_{2n})=f(\lim_{n\to\infty}Sb_{2n})=f(\lim_{n\to\infty}fb_{2n+2})=f(u). \tag{3.9}$$

By weak commutativity of a pair $\{f,S\}$ , we have

$$d(f(Sb_{2n})),S(fb_{2n}))\le d(fb_{2n},Sb_{2n}). \tag{3.10}$$

Taking limit as $n\to\infty$, on both sides of (3.10) and by (3.9), we get

$$d(f(u),\lim_{n\to\infty}S(fb_{2n}))\le d(u,u), \tag{3.11}$$

which further implies that $\lim_{n\to\infty}S(fb_{2n})=f(u)$. Now, by conditions (3.1) and (3.2), we have

$$\begin{aligned}
d(S(fb_{2n}),Tb_{2n+1})\le \max\{&d(f^2b_{2n},gy_{2n+1}),d(f^2b_{2n},Sfb_{2n}),d(gb_{2n+1},Tb_{2n+1}),\\
&d(Sfb_{2n},gb_{2n+1}),d(f^2b_{2n},Tb_{2n+1})^{1/s}\}^\lambda.
\end{aligned} \tag{3.12}$$

Taking limit as $n\to\infty$, on both sides of (3.12), we obtain

$$d(fu),u)\le\max\{d(fu,u),d(fu,fu),d(u,u),d(fu,u),d(fu,u)^{1/s}\}^\lambda, \tag{3.13}$$

that is $d(fu,u)\le d(fu,u)^\lambda$. Hence $d(fu,u)=1$, and $u$ is a fixed point of $f$ in $\overline{B(y_0,r)}$. In similar way by conditions (3.1) and (3.2), we have

$$\begin{aligned}
d(S(u),Tb_{2n+1})\le\max\{&d(fu,gu_{2n+1}),d(fu,Su),d(gu_{2n+1},Tu_{2n+1}),\\
&d(Su,gu_{2n+1}),d(fu,Tu_{2n+1})^{1/s}\}^\lambda.
\end{aligned} \tag{3.14}$$

Taking limit as $n\to\infty$, on both sides of (3.14), we obtain

$$\begin{aligned}
d(Su,u) &\le \max\{d(fu,u),d(u,Su),d(u,u)\\
&\quad ,d(Su,u),d(u,u)^{1/s}\}^\lambda\\
d(Su,u) &\le d(Su,u)^\lambda\\
d(Su,u) &\le 1.
\end{aligned}$$

Hence $d(Su,u)=1$, and $u$ is a fixed point of $S$ in $\overline{B(y_0,r)}$. Because of the fact that $u=S(u)\in S\overline{B(y_0,r)}\subseteq g\overline{B(y_0,r)}$. Let $u^*$ in $\overline{B(y_0,r)}$, be such that $u=g(u^*)$.

$$\begin{aligned}
d(u,Tu^*) &= d(S(u),Tu^*)\\
&\le \max\{d(fu,gu^*),d(fu,Su),d(gu^*,Tu^*)\\
&\quad ,d(Su,gu^*),d(fu,Tu^*)^{1/s}\}^\lambda\\
d(u,Tu^*) &\le 1\\
Tu^* &= u.
\end{aligned}$$

Since the pair $\{f, T\}$ weakly commutative from our assumptions, thus

$$d(gu, Tu) = d(gT(u), Tgu^*) \leq d(gu^*, Tu^*) = d(u, u) = 1.$$

Hence $gu = Tu$. By (3.1) and (3.2), we obtain

$$\begin{aligned} d(u, Tu) &= d(S(u), Tu) \\ &\leq \max\{d(fu, gu), d(fu, Su), d(gu, Tu) \\ &\quad , d(Su, gu), d(fu, Tu)^{1/s}\}^{\lambda}, \end{aligned}$$

which implies $u = T(u)$. Hence $u$ is a common fixed point of $f, g, S$ and $T$ in $\overline{B(y_0, r)}$. If $g$ is continuous, then following arguments similar to those given above, we obtain that $u = S(u) = f(u) = T(u) = g(u)$. Now suppose that $S$ is continuous, Thus

$$\lim_{n\to\infty} S(fb_{2n}) = S(\lim_{n\to\infty} Sb_{2n}) = S(u). \tag{3.15}$$

As the pair $\{f, S\}$ is weakly commuting, we have

$$d(f(Sb_{2n}), S(fb_{2n}) \leq d(fb_{2n}, Sb_{2n}). \tag{3.16}$$

Taking limit as $n \to \infty$, on both sides of (3.16), we have

$$d(\lim_{n\to\infty} f(Sb_{2n}),\ Su \leq d(u, u) = 1, \text{ and } \lim_{n\to\infty} f(Sb_{2n}) = S(u).$$

By contractive condition (3.1), we get

$$\begin{aligned} d(S(Sb_{2n}), Tb_{2n+1}) \leq \max\{&d(fSb_{2n}, gb_{2n+1}), d(fSb_{2n}, fSb_{2n}), d(gb_{2n+1}, Tb_{2n+1}), \\ &d(SSb_{2n}, gb_{2n+1}), d(fSb_{2n}, Tb_{2n+1})^{1/s}\}^{\lambda}. \end{aligned} \tag{3.17}$$

Taking limit as $n \to \infty$, on both sides of (3.17), implies that

$$d(Su, u) \leq d(Su, u)^{\lambda}.$$

Hence $d(Su, u) = 1$, and $u$ is a fixed point of $S$ in $\overline{B(y_0, r)}$. Since $u = S(u) \in S(\overline{B(y_0, r)}) \subseteq g(\overline{B(y_0, r)})$, let $u^*$in $\overline{B(y_0, r)}$ be such that $u = g(u^*)$. It follows from condition (3.1), that

$$\begin{aligned} d(S(Sb_{2n}), Tu^*) \leq \max\{&d(fSb_{2n}, gu^*), d(fSb_{2n}, SSb_{2n}), d(gu^*, Tu^*), \\ &d(SSb_{2n}, gu^*), d(fSb_{2n,}Tu^*)^{1/s}\}^{\lambda}. \end{aligned} \tag{3.18}$$

taking limit as $n \to \infty$, on both sides of (3.18), implies that

$$d(u, Tu^*) \leq d(u, Tu^*)^{\lambda}. \tag{3.19}$$

Thus, $Tu^* = u$. Since the pair $\{T, g\}$ is weakly commutative from our hypothesis, then

$$d(Tu, gu) = d(Tgu^*, gTu^*) \leq d(Tu^*, gu^*) = d(u, u) = 1, \tag{3.20}$$

which implies that $gu = Tu$. From (3.1), we have

$$\begin{aligned} d(Sb_{2n}, Tu) \leq \max\{&d(fb_{2n}, gu), d(fb_{2n}, Sb_{2n}), d(gu, Tu), \\ &d(Sb_{2n}, gu), d(fb_{2n,}Tu)^{1/s}\}^{\lambda}, \end{aligned} \tag{3.21}$$

taking limit as $n \to \infty$, on both sides of (3.21), gives

$$d(u, Tu) \leq d(u, Tu)^{\lambda} \quad \text{and} \quad u = T(u).$$

However, $u = T(u) \in T(\overline{B(y_0, r)}) \subseteq f(\overline{B(y_0, r)})$, so let $v \in (\overline{B(y_0, r)})$ be such that $u = f(v)$. It follows from (3.1), that

$$\begin{aligned} d(Sv, u) &= d(Sv, Tu) \\ &\leq \max\{d(fv, gu), d(fv, Sv), d(gu, Tu), d(Sv, gu), d(fv, Tu)^{1/s}\}^{\lambda}, \end{aligned}$$

which implies that $d(S(v), u) \leq d(S(v), u)^{\lambda}$. Hence $S(v) = u$. Since $S$ and $f$ are weakly commutative, so

$$d(fu, Su) = d(fSv, Sfv) \leq d(fv, Sv) = d(u, u) = 1, \tag{3.22}$$

gives $f(u) = S(u)$. Applying condition (3.1), we obtain

$$\begin{aligned} d(Su, u) &= d(Su, Tu) \\ &\leq \max\{d(fu, gu), d(fu, Su), d(gu, Tu), d(Su, gu), d(fu, Tu)^{1/s}\}^{\lambda} \\ &= \max\{d(Su, u), d(Su, Su), d(gu, gu), d(Su, u), d(Su, u)^{1/s}\}^{\lambda}, \end{aligned}$$

which implies that $u = S(u)$. Hence $u$ is a common fixed point of $f, S, T$ and $g$ in $\overline{B(y_0, r)}$. If $T$ is continuous, then by using arguments similar to those given above, we can easily obtain a common fixed point of $f, S, T$ and $g$ in $\overline{B(y_0, r)}$. We proceed to show the uniqueness of the common fixed point of the mappings $f, T, S$ and $g$. So let $z \in \overline{B(y_0, r)}$ be another common fixed point of $f, T, S$ and $g$. By (3.1), we have

$$\begin{aligned} d(u, z) &= d(Su, Tz) \\ &\leq \max\{d(fu, gz), d(fu, Su), d(gz, Tz), d(Su, gz), d(fu, Tz)^{1/s}\}^{\lambda}. \end{aligned}$$

That is, $d(u, z) \leq d(u, z)^{\lambda}$. Hence $u = z$ and this implies that the common fixed point of $f, T, S$ and $g$ is unique. □

**Example 3.2** Let $b = \mathbb{R}^{+} \cup \{0\}$ and $d : X \times X \to [1, \infty)$ be a $b-$multiplicative metric defined by $d(x, y) = 2^{(x-y)^2}$. Note that $(X, d)$ is complete $b-$multiplicative metric space, define mappings $f, g, S, T : X \to X$ by

$$f(x) = \begin{cases} x \text{ if } x \leq 2 \\ 30x \text{ if } x > 2 \end{cases}, \quad g(x) = \begin{cases} 3x \text{ if } x \leq 2 \\ 20x \text{ if } x > 2 \end{cases}$$

$$S(x) = \begin{cases} 2x \text{ if } x \leq 2 \\ 100x \text{ if } x > 2 \end{cases}, \quad T(x) = \begin{cases} \frac{1}{3}x \text{ if } x \leq 2 \\ 4x \text{ if } x > 2 \end{cases}$$

Obviously, maps are continuous, $(f, S)$ and $(T, g)$ are weak commutative with $S(X) \subset g(X)$, and $T(X) \subset f(X)$. First we construct a closed ball, such that $x_0 = \frac{1}{7}$ and $\epsilon = 16$

$$\begin{aligned} \overline{B(x_o, \epsilon)} &= \left\{ y \in X : d(y, \frac{1}{7}) \leq 16 \right\} \\ &= \left\{ y \in X : 2^{(y-\frac{1}{7})^2} \leq 16 \right\} \\ &= \left\{ y \in X : \left(y - \frac{1}{7}\right)^2 \leq 4 \right\} \\ &= \left\{ y \in X : y \leq 2 + \frac{1}{7} \right\} = \left[0, \frac{15}{7}\right]. \end{aligned}$$

Now, we will show that mappings are weakly commutative. We know that

$$\begin{aligned} 1 &\leq 2^{4x^2} \text{ for } x \in R^{+} \\ &\Rightarrow 2^{(3x-3x)^2} \leq 2^{(x-3x)^2} \\ &\Rightarrow d(fgx, gfx) \leq d(fx, gx). \end{aligned}$$

This implies that $f$ and $g$ are weakly commutative. Similarly we can show that $S$ and $T$ are also weakly commutative. Choose $x_0 = \frac{1}{7}$ then there exist $x_1 \in \overline{B(\frac{1}{7}, \epsilon)}$, such that

$S(\frac{1}{7}) = g(x_1) = y_0 = \frac{1}{7}$ and $y_0 = g(x_1) = 3x_1$. Now,

$$y_0 = \frac{1}{7} = 3x_1 = g(x_1) \Rightarrow x_1 = \frac{1}{21}$$

Also,

$$T(x_1) = T(\frac{1}{21}) = \frac{1}{63} = y_1$$

Thus,

$$d(y_0, T(x_1)) = 2^{(\frac{1}{7}-\frac{1}{63})^2} = 2^{(\frac{54}{63})^2} = 2^{(\frac{36}{49})}.$$

Where $\epsilon = 16$, $\lambda = \frac{3}{16}$ and $s = 2$ with $h = \frac{3}{13}$, $sh < 1$, then

$$\epsilon^{\frac{1-sh}{s}} = 16^{\frac{1-(2)(\frac{3}{13})}{2}} = 16^{\frac{7}{26}} \leq 16.$$

So

$$d(y_0, Tx_0) \leq \epsilon^{\frac{1-sh}{s}},$$

holds. Also, for $x, y \in \overline{B(x_o, \epsilon)}$

$$\begin{aligned}
(x-\frac{1}{3}y)^2 &\leq \left[\max\left\{(3x-y)^2,\ x^2,\ \left(\frac{8}{3}x\right)^2,\ (3x-2y)^2,\ \left(x-\frac{1}{3}y\right)\right\}\right]^{\lambda} \\
2^{(x-\frac{1}{3}y)^2} &\leq \left[\max\left\{2^{(3x-y)^2},\ 2^{x^2},\ 2^{(\frac{8}{3}x)^2},\ 2^{(3x-2y)^2},\ 2^{(x-\frac{1}{3}y)}\right\}\right]^{\lambda} \\
d(Sx, Ty) &= \left[\max\{d(fx, gy), d(fx, Sx), d(gy, Ty), d(Sx, gy), d(fx, Ty)^{\frac{1}{s}}\}\right]^{\lambda}.
\end{aligned}$$

holds for $x, y \in \overline{B(x_o, \epsilon)}$. Thus all the conditions of Theorem 3.1 hold. Hence $x = 0$ is the unique common fixed point of $f, T, S$ and $g$ in $\overline{B(0, \frac{15}{2})}$.

Note that, if $x = 4$ and $y = 3$, then above contractive condition of Theorem 3.1 does not hold on the whole space $X$.

**Corollary 3.3** Let $S, T, f,$and $g$ be self-mappings of a complete multiplicative $b$-metric space $(X, d)$ with coefficient "$s$" and $(f, S)$ and $(g, T)$ weakly commutative with $SX \subset gX$, $TX \subset fX$, and one of $S, T, f$ and $g$ is continuous. If $Sx_0 = y_0$ for some given point $x_0$ in $X$ and there exists $\lambda \in (0, \frac{1}{2})$ with $h = \lambda/(1-\lambda)$, such that

$$d(Sx, Ty) \leq (M(x, y))^{\lambda} \text{ for any } x, y \in \overline{B(y_0, r).}$$

holds, where

$$M(x, y) = \max\{d(Sx, gy), d(fx, Ty)\}.$$

Then there exists a unique common fixed point of $f, T, S$ and $g$ in $\overline{B(y_0, r)}$ provided that

$$d(y_0, Tx_1) \leq r^{(1-sh)/s} \text{ for some } x_1 \text{ in } X \text{ and } s \geq 1, sh < 1.$$

**Corollary 3.4** Let $S, T, f,$and $g$ be self-mappings of a complete multiplicative $b$-metric space $(X, d)$ with coefficient "$s$" and $(f, S)$ and $(g, T)$ weakly commutative with $SX \subset gX$, $TX \subset fX$, and one of $S, T, f,$and $g$ is continuous. If $Sx_0 = y_0$ for some given point $x_0$ in $X$ and there exists $\lambda \in (0, \frac{1}{2})$ with $h = \lambda/(1-\lambda)$ such that

$$d(Sx, Ty) \leq (M(x, y))^{\lambda} \text{ for any } x, y \in \overline{B(y_0, r).}$$

holds, where

$$M(x, y) = \max\{d(fx, gy), d(Sx, gy), d(fx, Ty)\}.$$

Then there exists a unique common fixed point of $f, T, S$ and $g$ in $\overline{B(y_0, r)}$ provided that

$$d(y_0, Tx_1) \leq r^{(1-sh)/s} \text{ for some } x_1 \text{ in } X \text{ and } s \geq 1,\ sh < 1.$$

## 4. Conclusions

In the present paper, we have achieved fixed point results satisfying contraction only on a closed ball. So, we conclude that our results can be used in those situations where the corresponding results have been failed to give guarantee of a fixed point. Example is also given to demonstrate the variety of our result. Moreover, we investigate our results in a more better recent framework. New results in multiplicative metric space, $b$-metric space and metric space can be obtained as corollaries of our results.

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