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NUMERICAL SOLUTION OF FUZZY PARABOLIC DIFFERENTIAL EQUATIONS BY A FINITE DIFFERENCE METHODS

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Abstract. In this study, we consider the concept of under generalized differentiability for the fuzzy parabolic differential equations. When the fuzzy derivative is considered as generalization of the H-derivative, for our case, the fuzziness is in the coefficents as well as initial and boundary conditions. We analysed and applied to numerically solve a fuzzy parabolic equation by finite difference method. The applicability of presented algorithm is illustrated by solving an examples of fuzzy partial differential equations.

Keywords: Fuzzy partial differential equations, generalized differentiability, fuzzy heat equation, finite difference methods.

AMS Subject Classification: 34A07, 35R13.

1. Introduction

Fuzzy differential equations (FDEs) have been studied entensively in recent years to model in science and engineering problems [3, 4, 12]. Fuzzy partial differential equations were introduced by Buckley and Feuring in 1999 to study partial differential equations with uncertainty [8]. Also, they have studied solutions to elementary fuzzy partial differential equations [2]. First order linear fuzzy differential equations under generalized differentiability concept are studied in [15,16]. Recently, Bertone et al. [7] have considered fuzzy solutions to some partial differential equations by fuzzification of the deterministic solution. For instance, in [1, 5, 13, 17, 18] the authors proposed difference methods for solving fuzzy partial differential equations. Chen et al. presented a new inference method to find fuzzy solutions to PDE's [10]. See [23] for interpretation of used FPDEs to modeling hydrogeological systems. Also studying heat, wave and Poisson equations with uncertain parameters can be found in [23]. Alikhani et al. obtained the fuzzy solutions of hyperbolic equation with initial values in [25] and [24] studied a linear fuzzy partial differential equation under generalized Hukuhara differentiability concept. Then, this paper will obtain the use of finite difference methods for solving fuzzy heat equation. The paper is organized as

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follows. In section 2, we present some basic concepts of fuzzy calculus and useful theoretical information. Fuzzy parabolic differential equation under generalized differentiability, we study in section 3. Numerical algorithm for solving considered problem is introduced in section 4. At the end of the paper, we present some conclusions and topics for further research. To describe the behavior of imprecise phenomena in real world, fuzzy dynamical systems have been studied frequently in the past twenty years [1–4]. The study of the existence of solution for fuzzy dynamical equations using fixed point theorems, Banach's fixed point principle and monotone iterative method is carried out in [5–9].

2. Preliminaries

We give some definitions and useful results and introduce the necessary notation which will use throughout the paper see [3]. In the following, the space of fuzzy numbers is denoted R_F.
Given a fuzzy number $u\epsilon R_F$ and $0 < \alpha \leq 1$, we obtain the α-level set of u by $[u]^\alpha = \{x\epsilon R; u(x) \geq \alpha\}$ and the support of u as $[u]^0 = cl\{x \in R|u(x) > 0\}$. For any $\alpha\epsilon[0,1]$, $[u]_\alpha = [\underline{u}^\alpha, \overline{u}^\alpha]$ is a bounded closed interval. The length of the α-level set is defined by $len(u) = (\overline{u}^\alpha - \underline{u}^\alpha)$, $\forall\alpha\epsilon[0,1]$. In the special case $\alpha = 0$, $len([u]_\alpha) = diam(u)$.

Definition 2.1. *A fuzzy number in parametric form is presented by an ordered pair of functions [3] $[\underline{u}^\alpha, \overline{u}^\alpha]$ to define the parametric form of a fuzzy interval are the followings:*
(i) $\underline{u}^\alpha$ is a bounded monotonic increasing left-continuous function in $\alpha \in (0,1]$ and right-continuous at $\alpha = 0$.
(ii) $\overline{u}^\alpha$ is a bounded monotonic decreasing left-continuous function in $\alpha \in (0,1]$ and right-continuous at $\alpha = 0$.
(iii) $\underline{u}^\alpha \leq \overline{u}^\alpha, 0 \leq \alpha \leq 1$.

For $u, v \in R_F$ and $\lambda \in R$, we define the sum $u + v$ and scalar multiplication λu as $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]_\alpha = \lambda[u]^\alpha, \forall\alpha \in [0,1]$, where $[u]^\alpha + [v]^\alpha$ mean the usual addition of two intervals (subsets) of R and usual product between a scalar and a subset and interval of R respectively. Hausdorff distance between u and v is given by [11] $D : R_F \times R_F \to R_+ \cup \{0\}$ $D(u,v) = \sup_{\alpha\in[0,1]} \max\{|\underline{u}^\alpha - \underline{v}^\alpha|, |\overline{u}^\alpha - \overline{v}^\alpha|\}$, $u, v \in R_F$. The space (R_F, D) is a complete metric space.

Definition 2.2. *Let $u, v \in R_F$. If there exists $w \in R_F$ such that $u = v + w$ then w is called the H-difference of u, v and it is denoted $u \ominus v$ [11] .*

Definition 2.3. *Let $f : (a,b) \longrightarrow R_F$ and fix $x_0 \in (a,b)$. We say that if there is exists an element $f'(x_0) \in R_F$ such that one of the following statements is true:*
(1) For all $h > 0$ sufficiently near to 0, the H-differences $f(x_0+h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0-h)$ exist and the limits (in the metric D)

$$\lim_{h\to 0^+} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h\to 0^+} \frac{f(t_0) \ominus f(x_0-h)}{h} = f'(x_0)$$

or
(2) For all $h < 0$ sufficiently near to 0, the H-differences $f(x_0+h) \ominus f(x_0)$, $f(x_0) \ominus f(x_0-h)$ exist and the limits (in the metric D)

$$\lim_{h\to 0^-} \frac{f(x_0+h) \ominus f(x_0+)}{h} = \lim_{h\to 0^-} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0).$$

for all $x \in (a,b)$ [4].

Theorem 2.1. *Let $f:(a,b)\longrightarrow R_F$ be function where, $[f(x)]_\alpha=[\underline{f}_\alpha(x),\overline{f}_\alpha(x)]$ for each $\alpha\in[0,1]$. The following assertion are valid:*
(i) If f is (1)-differentiable in the first form, then $\underline{f}^\alpha$ and $\overline{f}^\alpha$ are differentiable functions and $[f'(x)]_\alpha=[(\underline{f}_\alpha)'(x),(\overline{f}_\alpha)'(x)], x\in(a,b)$.
(ii) If f is (2)-differentiable in the first form, then $\underline{f}_\alpha(x)$ and $\overline{f}_\alpha(x)$ are differentiable functions and $[f'(x)]_\alpha=[(\overline{f}_\alpha)'(x),(\underline{f}_\alpha)'(x)], x\in(a,b)$ [3].

Theorem 2.2. *Let fuzzy functions $f,g:(a,b)\longrightarrow R_F$ be (1)-differentiable on (a,b).*
1) *If the H- differences $f(x)\ominus g(x)$ and $f'(x)\ominus g'(x)$ exist for all $x\in(a,b)$, then $f\ominus g$ is (1)-differentiable and $(f\ominus g)'(x)=f'(x)\ominus g'(x),\forall x\in(a,b)$.*
2) *If the H - differences $f(x)\ominus g(x)$ and $g'(x)\ominus f'(x)$ exist for all $x\in(a,b)$, then $f\ominus g$ is 2-differentiable and $(f\ominus g)'(x)=(-1)(g'(x)\ominus f'(x))$, $\forall x\in(a,b)$ [4].*

Theorem 2.3. *Let fuzzy functions $f,g:(a,b)\longrightarrow R_F$ be (2)-differentiable on (a,b).*
1) If the H- differences $f(x)\ominus g(x)$ and $g'(x)\ominus f'(x)$ exist for all $x\in(a,b)$, then $f\ominus g$ is (1)-differentiable and $(f\ominus g)'(x)=f'(x)\ominus g'(x),\forall x\in(a,b)$.
2) If the H - differences $f(x)\ominus g(x)$ and $f'(x)\ominus g'(x)$ exist for all $x\in(a,b)$, then $f\ominus g$ is 2-differentiable and $(f\ominus g)'(x)=((f'(x)\ominus g'(x)),\forall x\in(a,b)$.

In the following, according to the previous definition, we present partial generalized Hukuhara derivatives for fuzzy functions [4,6,19,20].

Definition 2.4. *Let $u:R\times(0,l)\rightarrow R_F$ be a fuzzy function. If there exists an element $D_xu(x_0,y_0)\in R_F$ such that either*
(1) for all $h>0$ sufficiently near to 0, there exist $u(x_0+h,t_0)\ominus u(x_0,t_0)$, $u(x_0,t_0)\ominus u(x_0-h,t_0)$ and the limits
$\lim_{h\to 0^+}\frac{u(x_0+h,t_0)\ominus u(x_0,t_0)}{h}=\lim_{h\to 0^+}\frac{u(x_0,t_0)\ominus u(x_0-h,t_0)}{h}=D_xu(x_0,t_0)$,
or
(2) for all $h<0$ sufficiently near to 0, there exist $u(x_0,t_0)\ominus u(x_0+h,t_0)$, $u(x_0-h,t_0)\ominus u(x_0,t_0)$ and the limits
$\lim_{h\to 0^-}\frac{u(x_0,t_0)\ominus u(x_0+h,t_0)}{h}=\lim_{h\to 0^-}\frac{u(x_0-h,t_0)\ominus u(x_0,t_0)}{h}=D_xu(x_0,t_0)$.

We denote by $D^1_nf(x_0)$ the first derivative of f, if it is (n)-differentiable at x_0. $(n=1,2)$ in [24].

Theorem 2.4. *Denote $[u(x,t)]^\alpha=[\underline{u}^\alpha(x,t),\ \overline{u}^\alpha(x,t)]$, $\alpha\in[0,1]$, where $u:R\times(0,T)\rightarrow R_F$ in [24]. Then*
(1) If D^1_xu exists on $R\times(0,T)$, then $\underline{u}^\alpha$ and $\overline{u}^\alpha$ are differentiable functions with respect to x on $R\times(0,T)$, and
$[D^1_xu(x,t)]^\alpha=[D_x\underline{u}^\alpha(x,t),D_x\overline{u}^\alpha(x,t)]$, $\forall(x,t)\in R\times(0,T)$, $\alpha\in[0,1]$.
(2) If D^1_tu exists on $R\times(0,T)$, then $\underline{u}^\alpha$ and $\overline{u}^\alpha$ are differentiable functions with respect to t on $R\times(0,T)$, and
$[D^1_tu(x,t)]^\alpha=[D_t\underline{u}^\alpha(x,t),D_t\overline{u}(x,t)]^\alpha$, $\forall(x,t)\in R\times(0,T)$, $\alpha\in[0,1]$.
(3) If D^2_xu exists on $R\times(0,T)$, then $\underline{u}^\alpha$ and $\overline{u}^\alpha$ are differentiable functions with respect to x on $R\times(0,T)$, and
$[D^2_xu(x,t)]^\alpha=[D^2_x\underline{u}^\alpha(x,t),D^2_x\overline{u}^\alpha(x,t)]$, $\forall(x,t)\in R\times(0,T)$,$\alpha\in[0,1]$.
(4) If D^2_tu exists on $R\times(0,T)$, then $\underline{u}^\alpha$ and $\overline{u}^\alpha$ are differentiable functions with respect to t on $R\times(0,T)$, and
$[D^2_tu(x,t)]^\alpha=[D^2_t\underline{u}^\alpha(x,t),D^2_t\overline{u}^\alpha]$, $\forall(x,t)\in R\times(0,T)$, $\alpha\in[0,1]$.

3. The fuzzy parabolic differential equation

In this section we present the general form of heat equation in environment by using the basic concepts of fuzzy properties [21, 22]. Consider the one-dimensional fuzzy heat equation with the initial and boundary conditions

$$[D_t^1 u(x,t)]^\alpha = a^2 [D_x^2 u(x,t)]^\alpha, 0 < x < l, 0 < t < T, \tag{1}$$

$$[u(x,0)]^\alpha = [u_0(x)]^\alpha, 0 < x \leq l \tag{2}$$

$$[u(0,t)]^\alpha = [\mu_1(t)]^\alpha, 0 < t \leq T \tag{3}$$

$$[D_x^1 u(0,t)]^\alpha = [\mu_2(t)]^\alpha, 0 < t \leq T \tag{4}$$

where $[u(x,t)]^\alpha = [\underline{u}(x,t)^\alpha, \overline{u}(x,t)^\alpha]$ is a fuzzy function [21] of crisp variables t and x and all $\alpha \in [0,1]$.

Let the solution domain of the problem need to be partitioned uniformly. For some positive integers M and N, the grids sizes in space and time directions for the finite difference algorithm are defined as $h = \frac{l}{N}$ in x-direction and $\tau = \frac{T}{M}$ in t-direction. The grid points are given by $x_i = ih, i = 0, 1, ..., N$ and $t_j = j\tau, j = 0, 1, ..., M$. The values of the fuzzy function $[u(x,t)]^\alpha$ at the grid points are denoted as $[u_{i,j}(x,t)]^\alpha = [u(x_i, t_j)]^\alpha$.

Using the implicit finite difference scheme of Eq. (1) at $(x_i, t_j) = (ih, j\tau)$ for (i)-differentiability as follows:

$$\begin{aligned} \frac{\underline{u}_{i,j+1} - \underline{u}_{i,j}}{\tau} &= a^2 \frac{\underline{u}_{i+1,j} - 2\underline{u}_{i,j} + \underline{u}_{i,j-1}}{h^2} \\ \frac{\overline{u}_{i,j+1} - \overline{u}_{i,j}}{\tau} &= a^2 \frac{\overline{u}_{i+1,j} - 2\overline{u}_{i,j} + \overline{u}_{i,j-1}}{h^2} \end{aligned} \tag{5}$$

Eq.(5) can be rewritten as the following

$$\begin{aligned} -\underline{u}_{i,j-1} &= \frac{r\underline{u}_{i+1,j} - (2r+1)\underline{u}_{i,j} + r\underline{u}_{i,j-1}}{h^2}, \\ -\overline{u}_{i,j-1} &= \frac{r\overline{u}_{i+1,j} - (2r+1)\overline{u}_{i,j} + r\overline{u}_{i,j-1}}{h^2}, \\ & 1 \leq i \leq N-1, 0 \leq j \leq M-1, \end{aligned} \tag{6}$$

subject to initial and boundary conditions

$$u_{i,0} = [u_0(x)]^\alpha, 0 < i \leq N, \tag{7}$$

$$u_{0,m} = [\mu_1(t)]^\alpha, 0 < j \leq M, \tag{8}$$

$$\frac{u_{0,m} - u_{0,m-1}}{h} = [\mu_2(t)]^\alpha, 0 < j \leq M \tag{9}$$

where $r = \frac{\tau a^2}{h^2}$. Based on Eq. (6), it can be solved the tridiagonal linear system constructed a matrix form.

Using the implicit finite difference scheme of Eq. (1) at $(x_i, t_j) = (ih, j\tau)$ for (ii)-differentiability as follows:

$$\frac{u_{i,j+1} - u_{i,j}}{\tau} = a^2 \frac{v_{i+1,j} - 2v_{i,j} + v_{i,j-1}}{h^2}, 1 \leq i \leq N-1, 0 \leq j \leq M-1, \tag{10}$$

$$v_{i,0} = [\underline{u_0}(x_i)]^\alpha, 0 < i \leq N, \tag{11}$$

$$v_{0,m} = [\underline{\mu_1}(t_j)]^\alpha, 0 < j \leq M, \tag{12}$$

$$\frac{v_{0,m} - v_{0,m-1}}{h} = [\underline{\mu_2}(t_j)]^\alpha, 0 < j \leq M, \tag{13}$$

$$\frac{v_{i,j+1} - v_{i,j}}{\tau} = a^2 \frac{u_{i+1,j} - 2u_{i,j} + u_{i,j-1}}{h^2}, 1 \leq i \leq N-1, 0 \leq j \leq M-1, \tag{14}$$

$$u_{i,0} = [\overline{u_0}(x_i)]^\alpha, 0 < i \leq N, \tag{15}$$

$$u_{0,m} = [\overline{\mu_1}(t_j)]^\alpha, 0 < j \leq M, \tag{16}$$

$$\frac{u_{0,m} - u_{0,m-1}}{h} = [\overline{\mu_2}(t_j)]^\alpha, 0 < j \leq M. \tag{17}$$

Based on system (10)-(17), it can be solved the tridiagonal linear system (TDMA) constructed a matrix form.
We solve the problem (1)-(4) for its derivatives have valid sets according to the type of differentiability.

4. Numerical Examples

In this section, we implement the implicit finite difference approximations to solve fuzzy heat equation for different values of α.

Example 4.1. *Consider the fuzzy heat equation*

$$[D_t^1 u(x,t)]^\alpha = [D_x^2 u(x,t)]^\alpha, 0 \leq x \leq 1, 0 \leq t \leq 1, \tag{18}$$

subject to the initial condition

$$[u(x,0)]^\alpha = \tilde{k}e^x, 0 < x \leq 1 \tag{19}$$

and the boundary conditions

$$[u(0,t)]^\alpha = \tilde{k}e^t, 0 < t \leq 1 \tag{20}$$

$$[D_x^1 u(0,t)]^\alpha = \tilde{k}e^t, 0 < t \leq 1 \tag{21}$$

where $\tilde{k} = [\alpha - 1, 1 - \alpha]$*. At* $h = \tau = 0.1$ *we have the following results:*
Fig. 1-2 show that the len with $\alpha = 0.5$ *by the implicit finite difference scheme for* (i)*-differentiability and* (ii)*-differentiability, respectively. Fig. 3 shows that both the implicit finite difference and exact solutions satisfy the fuzzy number properties.*

Example 4.2. *Consider the fuzzy heat equation*

$$[D_t^1 u(x,t)]^\alpha = \frac{1}{2}x^2 [D_x^2 u(x,t)]^\alpha, 0 \leq x \leq 1, 0 \leq t \leq 1, \tag{22}$$

subject to the initial condition

$$[u(x,0)]^\alpha = \tilde{k}x^2, \ 0 < x \leq 1 \tag{23}$$

and the boundary conditions

$$[u(0,t)]^\alpha = \tilde{k}e^t, \ 0 < t \leq 1 \tag{24}$$

$$[D_x^1 u(0,t)]^\alpha = 2\tilde{k}xe^t, \ 0 < t \leq 1 \tag{25}$$

where $\tilde{k} = [\alpha - 1, 1 - \alpha]$*. At* $h = \tau = 0.1$ *we have the following results:*
Fig. 4-5 show that the len with $\alpha = 0.5$ *by the implicit finite difference scheme for* (i)*-differentiability and* (ii)*-differentiability, respectively. Fig. 6 shows that both the implicit finite difference and exact solutions satisfy the fuzzy number properties.*

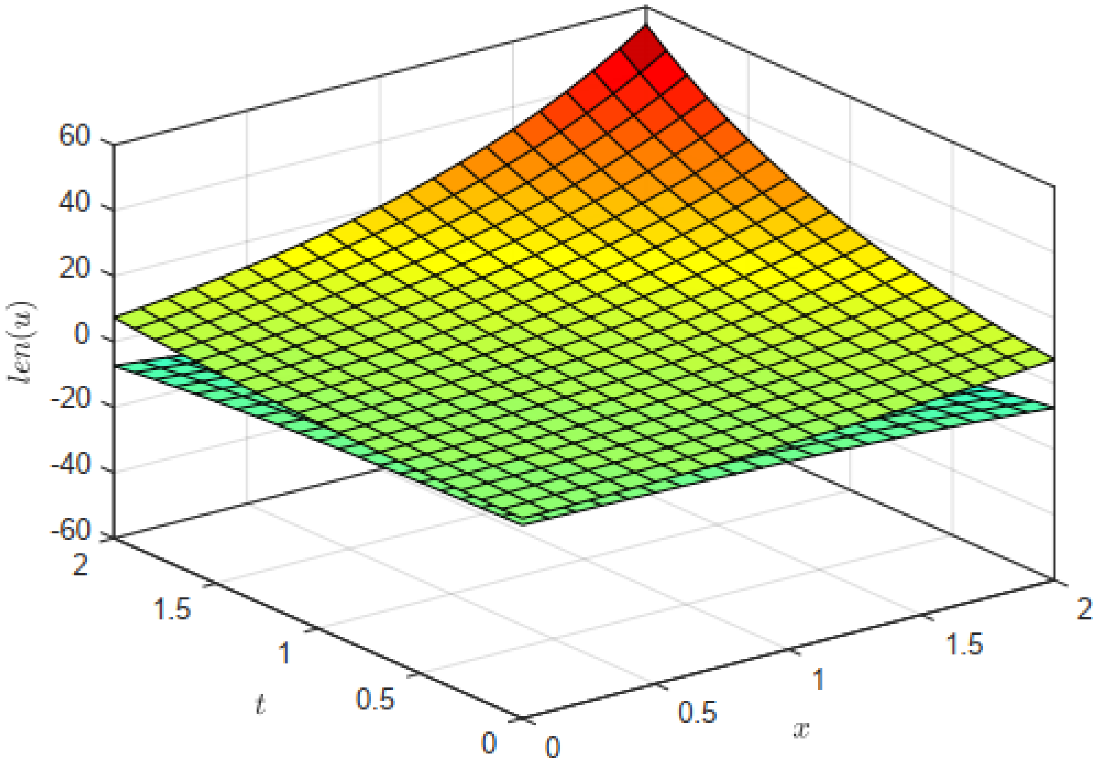


FIGURE 1. The approximate solution of Ex. 1 for (i)-differentiability.

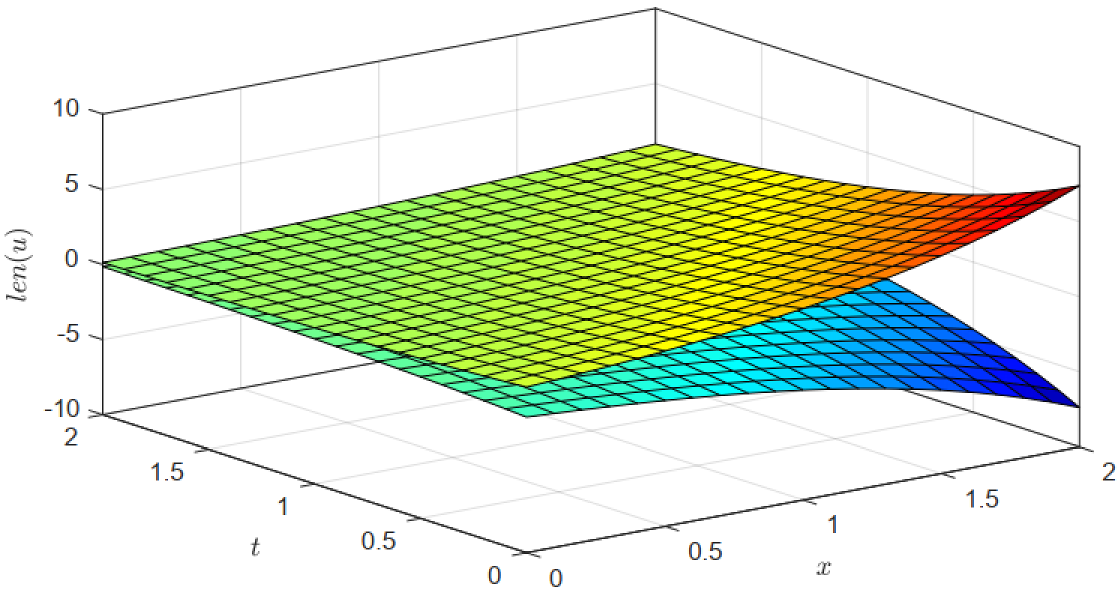


FIGURE 2. The approximate solution of Ex. 1 for (ii)-differentiability.

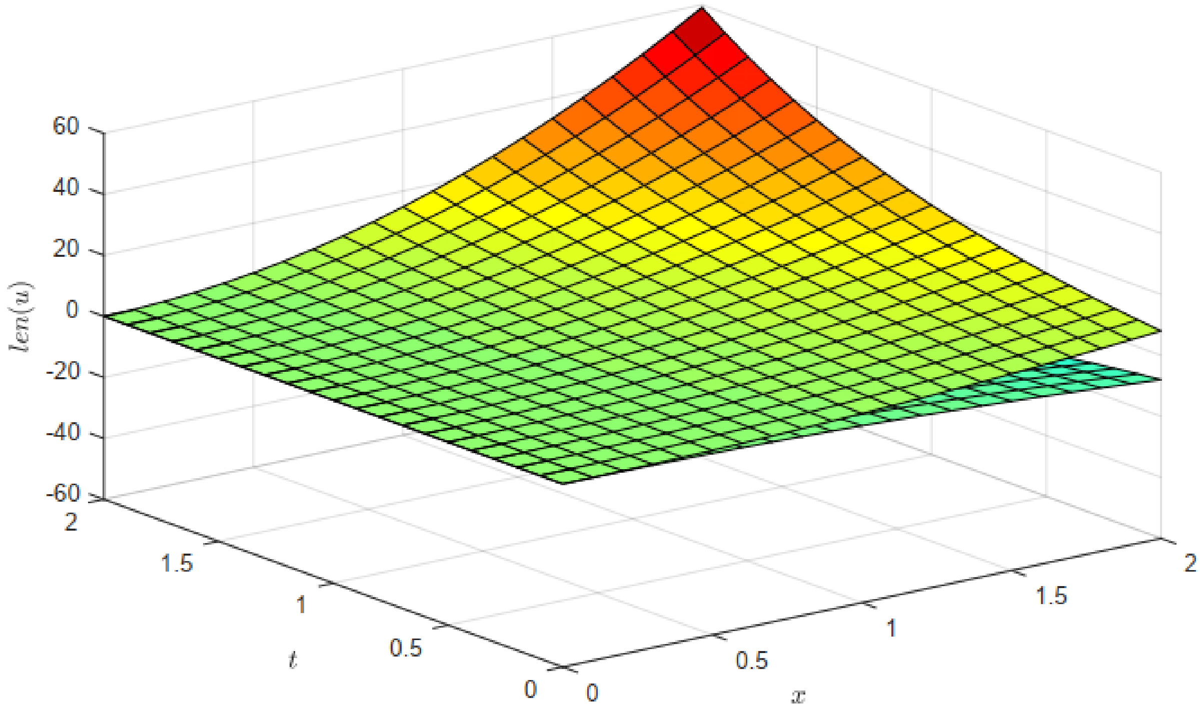


FIGURE 3. The exact solution of Ex. 1 .

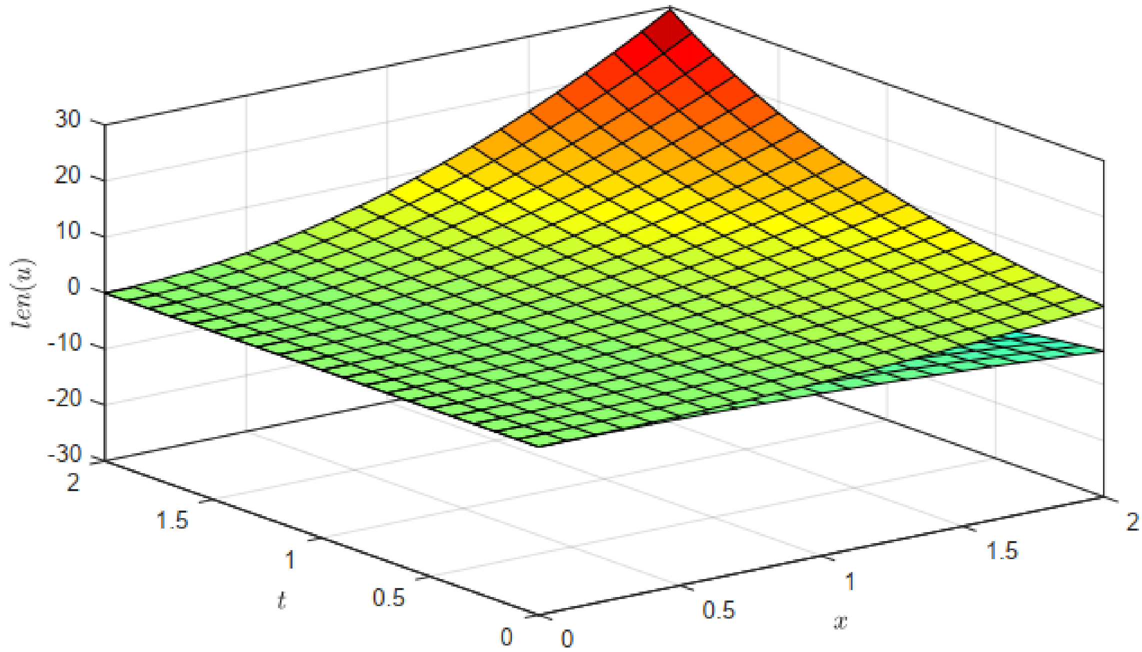


FIGURE 4. The approximate solution of Ex. 2 for (i)-differentiability.

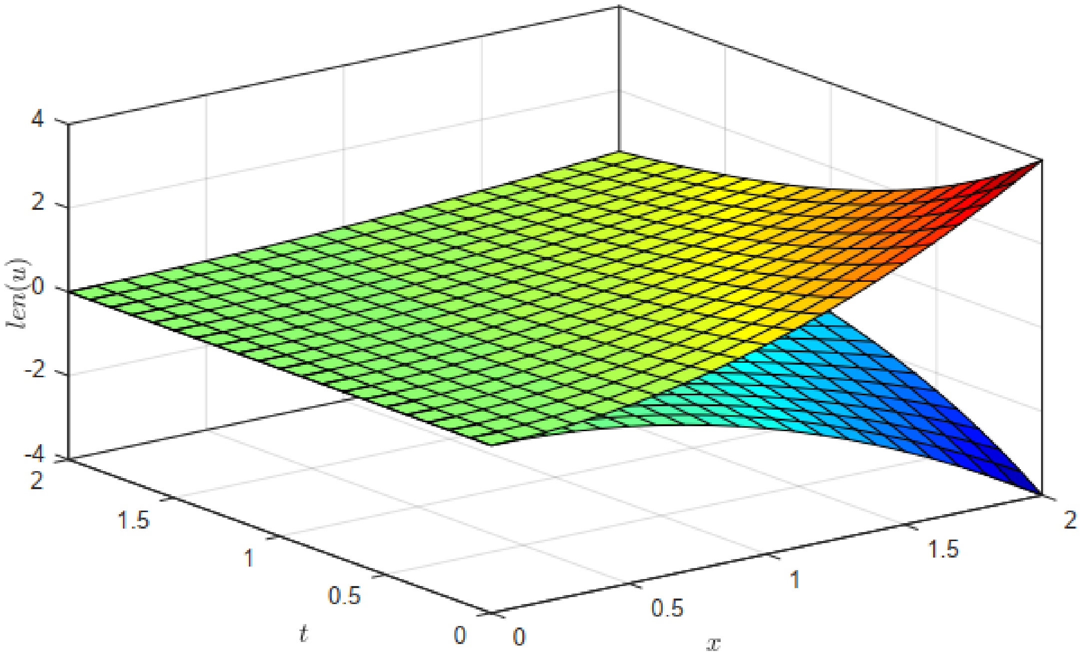


FIGURE 5. The approximate solution of Ex. 2 for (ii)-differentiability.

5. Conclusion

In this paper, a numerical procedure for a fuzzy parabolic equation is proposed. Comparison of numerical and exact solution at different values of α are also made by finite difference method. The numerical examples are given to demonstrate the efficiency of the results.

References

[1] Tofigh, A.,(2006), Difference methods for fuzzy partial differential equation, Comput. Methods Appl. Math.,3, pp.233-242.

[2] Tofigh, A., Gouyandeh,Z., Armand, A. and Hasanoglu, A., (2015), On fuzzy solutions for heat equation based on generalized Hukuhara differentiability, Fuzzy Sets and Systems, 265, pp.1-23.

[3] Bede, B.,(2013), Mathematics of Fuzzy Sets and Fuzzy Logic, Springer, London.

[4] Bede,B. and Gal,S.G.,(2005), Generalizations of the differentibility of fuzzy-number-valued functions with applications to FDEs, Fuzzy Sets and Syst., 151, pp.581-599.

[5] Gazanfari, B. and Ebrahimi, P., (2015), Differential transformation method for solving fuzzy fractional heat equation, Int. J. Math. Model. Comput., 5, pp.81-89.

[6] Bede,B. and Stefanini,L.,(2013), Generalized differentiability of fuzzy-valued functions, Fuzzy Sets and Syst., 230, pp.119-141.

[7] Bertone, A.M., Jafelice,R.M., Barros, L.C. and Bassanezi, R.C.,(2013), On fuzzy solutions for partial differential equations, Fuzzy Sets and Syst., 219, pp.68-80.

[8] Buckley, J.J. and Feuring, T., (1999),Introduction to fuzzy partial differential equations, Fuzzy Sets and Syst., 105, pp.241-248.

[9] Chalco-Cano,Y. and Roman-Flores, H.,(2008),On new solutions of FDEs, Chaos, Solitons and Fractals, 38, pp.112-119.

[10] Chen, Y.Y., Chang,Y.T.and Chen, B.S.(2009), Fuzzy solutions to partial differential equations: adaptive approach, IEEE Trans. Fuzzy Syst., 17 , pp.116-127.

[11] Diamond, P. and Kloeden, P.,(1994), Metric Spaces of Fuzzy Sets, World Scientific, Singapore.

[12] Georgiou, D.N., Nieto,J.J.and Rodriguez-Lopez, R.,(2005),Initial value problems for higher-order fuzzy differential equations, Nonlin. Analy., 63 pp.587-600.

[13] Gouyandeh, Z., Allahviranloo, T., Abbasbandy,S. and Armand, A.,(2017),A fuzzy solution of heat equation under generalized Hukuhara differentiability by fuzzy Fourier transform, Fuzzy Sets and Syst., 309, pp.81-97.

[14] Kaleva, O.,(2006), A note on FDE's, Nonlin. Anal., 64, pp.895-900.

[15] Khastan, A., Bahrami, F. and Ivaz, K.,(2009), New Results on Multiple Solutions for Nth-order Fuzzy Differential Equations under Generalized Differentiability, Boundary Value Prob., Article ID 395714.

[16] Khastan, A. and Rodriguez-Lopez,R.,(2016) On the solutions to first order linear fuzzy differential equations, Fuzzy Sets and Syst., 295, pp.114-135.

[17] Long,H.V., Son, N.T.K. and Tam,H.T.T., (2017),The solvability of fuzzy fractional partial differential equations under Caputo gH-differentiability, Fuzzy Sets and Syst., 309, pp.35-63.

[18] Hamzeh,Z., Izani Ismail,A. and Sathasivam,S., (2018),Numerical solution of Fuzzy fractional partial differential equations by an implicit finte difference scheme, Neural Comput. and Appl., DOI: 10.1007/s00521-017-3299-7.

[19] Stefanini,L. and Bede,B., (2009),Generalized Hukuhara differentiability of interval-valued functions and interval differential equations, Nonlin. Anal., 71, pp.1311-1328.

[20] Stefanini, L., (2010),A generalization of Hukuhara difference and division for interval and fuzzy arithmetic, Fuzzy Sets and Syst., 161, pp.1564-1584.

[21] Seikkala, S.,(1987), On the fuzzy initial value problem, Fuzzy Sets Syst. 24,(3), pp.319–330.

[22] Dubois , D. and Prade , H.,(1982),Towards fuzzy differential calculus part 3: differentiation, Fuzzy Sets Syst. 8,(3), pp.225-233.

[23] Faybishenko, B.A., (2004),Introduction to modeling of hydrogeologic systems using fuzzy partial differential equation, Fuzzy Partial Diff. Eq. and Relational Eq., pp.267-284.

[24] Khastan,A. and Rodriguez-Lopez,R.,(2019), An existence and uniqueness result for fuzzy Goursat partial differential equation, Fuzzy Sets Syst.,375 , pp.141-160.

[25] Alikhani,R., Bahrami,F. and Parvizi,S.,(2019), Differential calculus of fuzzy multi-variable functions and its applications to fuzzy partial differential euations, Fuzzy Sets Syst.,375 , pp.100-120.

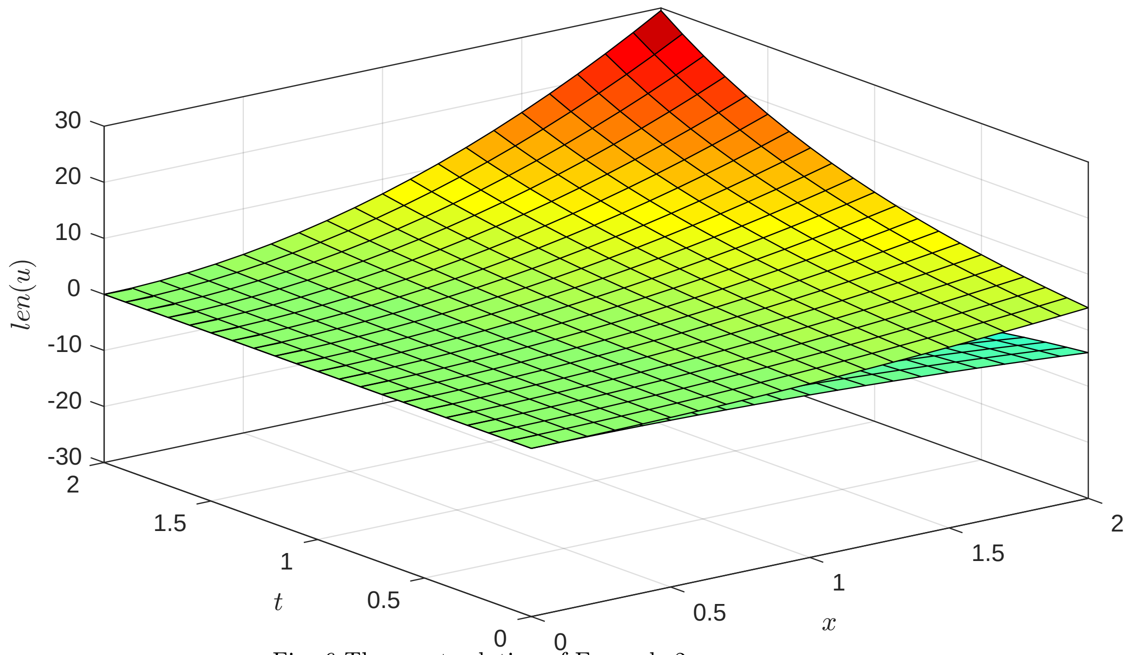
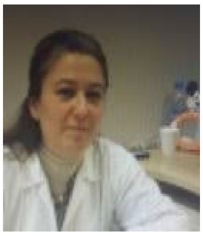


Fig. 6 The exact solution of Example 2.



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