TWMS J. App. and Eng. Math. V.10, N.4, 2020, pp. 940-950

A STRONG CONVERGENCE FOR FINDING A COMMON FIXED POINT OF A REPRESENTATION OF NONEXPANSIVE MAPPINGS AND W-MAPPINGS IN BANACH SPACES

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Abstract. In this paper, a strong convergence for finding an element of the set of common fixed points of a representation and W-mappings of nonexpansive mappings is introduced. Then, the strong convergence of the proposed implicit scheme to the common fixed point of a representation of nonexpansive mappings and W-mappings will be proved.

Keywords: Fixed point; Nonexpansive mapping; Representation; Semigroup; W-mappings.

AMS Subject Classification: 47H10, 47H09

1. Introduction

Suppose that C is a nonempty closed and convex subset of a Banach space E and E^* is the dual space of E. Let $\langle ., . \rangle$ denotes is the pairing between E and E^*. The normalized duality mapping $J : E \to E^*$ is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}$$

for all $x \in E$. In this paper, J is used to show the single-valued normalized duality mapping. Suppose that $U = \{x \in E : \|x\| = 1\}$. E is called smooth or said to have a Gâteaux differentiable norm if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. A Banach space E is smooth if the duality mapping J of E is single valued. If E is smooth, then J is norm to weak-star continuous; for more details, see [9].

Suppose that C is a nonempty closed and convex subset of a Banach space E. A mapping T of C into itself is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$ and a mapping f is called α-contraction on E if $\|f(x) - f(y)\| \leq \alpha\|x - y\|$, $x, y \in E$ such that $0 \leq \alpha < 1$.

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§ Manuscript received: January 6, 2019; accepted: May 15, 2019.

In this paper, motivated by [5], the following strong convergence is studied for finding a common element of the set of fixed points of a representation $\mathcal{S} = \{T_t : t \in S\}$ of a semigroup S as nonexpansive mappings from C into itself and the set of fixed point of W-mappings, with respect to a left regular sequence of means defined on an appropriate subspace of bounded real-valued functions of the semigroup. On the other hand, our aim is to show that there exists a sunny nonexpansive retraction P from C onto $\mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix}(\mathrm{W_n})$ and $x \in C$ such that the following sequence $\{z_n\}$ converges strongly to Px.

$$z_n = \epsilon_n f(W_n z_n) + (1 - \epsilon_n) T_{\mu_n} W_n z_n \quad (n \in \mathbb{N}).$$

2. Preliminaries

Suppose that S is a semigroup. The Banach space of all bounded real-valued functions defined on S with supremum norm is denoted by $B(S)$. For each $s \in S$ and $f \in B(S)$, l_s and r_s are defined in $B(S)$ by $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$, $(t \in S)$.
Suppose that X is a subspace of $B(S)$ containing 1 and let X^* be its topological dual space. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. $\mu_t(f(t))$ is often written instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Suppose that X is left invariant (resp. right invariant), i.e. $l_s(X) \subset X$ (resp. $r_s(X) \subset X$) for each $s \in S$. A mean μ on X is called left invariant (resp. right invariant) if $\mu(l_s f) = \mu(f)$ (resp. $\mu(r_s f) = \mu(f)$) for each $s \in S$ and $f \in X$. X is called left (resp. right) amenable if X has a left (resp. right) invariant mean. X is amenable if X is both left and right amenable. As is well known, $B(S)$ is amenable when S is a commutative semigroup (see page 29 of [9]). A net $\{\mu_\alpha\}$ of means on X is said to be left regular if

$$\lim_{\alpha} \|l_s^* \mu_\alpha - \mu_\alpha\| = 0,$$

for each $s \in S$, where l_s^* is the adjoint operator of l_s.

Suppose that f is a function of semigroup S into a reflexive Banach space E such that the weak closure of $\{f(t) : t \in S\}$ is weakly compact and Suppose that X is a subspace of $B(S)$ containing all the functions $t \to \langle f(t), x^* \rangle$ where $x^* \in E^*$. It is concluded from [3] that for any $\mu \in X^*$, there exists a unique element f_μ in E such that $\langle f_\mu, x^* \rangle = \mu_t \langle f(t), x^* \rangle$ for all $x^* \in E^*$ and such f_μ is denoted by $\int f(t)\,\mathrm{d}\mu(t)$. Moreover, if μ is a mean on X then $\int f(t)\,\mathrm{d}\mu(t) \in \overline{\mathrm{co}}\,\{f(t) : t \in S\}$(see [4] for example).

Suppose that C is a nonempty closed and convex subset of E. Then, a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings from C into itself is said to be a representation of S as nonexpansive mapping on C into itself if $\mathcal{S}$ satisfies the following:

(1) $T_{st}x = T_s T_t x$ for all $s, t \in S$ and $x \in C$;
(1) for every $s \in S$ the mapping $T_s : C \to C$ is nonexpansive.

The set of common fixed points of $\mathcal{S}$ is denoted by $\mathrm{Fix}(\mathcal{S})$, that is $\mathrm{Fix}(\mathcal{S}) = \bigcap_{\mathrm{s \in S}} \{\mathrm{x \in C : T_s x = x}\}$.

Definition 2.1. *Defined the mapping $W_n : C \longrightarrow C$ as follows:*

$$\begin{aligned}
U_{n,1} &= \lambda_{n,1} T_1 + (1 - \lambda_{n,1}) I, \\
U_{n,2} &= \lambda_{n,2} T_2 U_{n,1} + (1 - \lambda_{n,2}) I, \\
&\vdots \\
U_{n,N-1} &= \lambda_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \lambda_{n,N-1}) I, \\
W_n = U_{n,N} &= \lambda_{n,N} T_N U_{n,N-1} + (1 - \lambda_{n,N}) I,
\end{aligned}$$

where $\{\lambda_{n,i}\}_{i=1}^{N} \subset [0,1]$.

The following results hold for the mappings W_n.

Theorem 2.1. ([8]). *Let* C *be a nonempty closed convex subset of a strictly convex Banach space. Let* $\{\mathrm{T_i}\}_{\mathrm{i}\in\mathbb{N}}$ *be a sequence of nonexpansive mappings of* C *into itself such that* $\bigcap_{i=1}^{\infty} \mathrm{Fix}(\mathrm{T_i}) \neq \emptyset$ *and let* $\{\lambda_i\}$ *be a real sequence such that* $0 \leq \lambda_i \leq b < 1$ *for every* $i \in \mathbb{N}$. *Then*

(1) W_n *is nonexpansive and* $\mathrm{Fix}(\mathrm{W_n}) = \bigcap_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{Fix}(\mathrm{T_i})$ for each $\mathrm{n} \geq 1$,

(2) *for each* $x \in C$ *and for each positive integer* j, *the limit* $\lim_{n\to\infty} U_{n,j}x$ *exists.*

(3) *The mapping* $W : C \to C$ *defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} \quad (\mathrm{x} \in \mathrm{C}),$$

is a nonexpansive mapping satisfying $\mathrm{Fix}(\mathrm{W}) = \bigcap_{\mathrm{i=1}}^{\infty} \mathrm{Fix}(\mathrm{T_i})$ *and it is called the* W*-mapping generated by* $\{\mathrm{T_i}\}_{\mathrm{i}\in\mathbb{N}}$, *and* $\{\lambda_i\}_{i\in\mathbb{N}}$.

Theorem 2.2. ([10]). *Let* C *be a nonempty closed convex subset of* H, $\{\mathrm{T_i}\}_{\mathrm{i=1}}^{\infty}$ *be a sequence of nonexpansive mappings of* C *into itself such that* $\bigcap_{i=1}^{\infty} \mathrm{Fix}(\mathrm{T_i}) \neq \emptyset, \{\lambda_\mathrm{i}\}$ *be a real sequence such that* $0 < \lambda_i \leq b < 1$, $(i \geq 1)$. *If* D *is any bounded subset of* C, *then* $\lim_{n\to\infty} \sup_{x\in D} \|Wx - W_n x\| = 0$.

Let K be a nonempty subset of a Banach space X and $\{x_n\}$ be a sequence in K. The set of the asymptotic center of $\{x_n\}$ with respect to K, defined by

$$A(\{x_n\}) = \left\{ x \in K : \limsup_{n\to\infty} \|x_n - x\| = \inf_{y\in K} \limsup_{n\to\infty} \|x_n - y\| \right\}.$$

Lemma 2.1. ([1]). *Let* X *be a uniformly convex Banach space satisfying the Opial's condition and* K *be a nonempty closed convex subset of* X. *If a sequence* $\{z_n\} \subset K$ *converges weakly to a point* z_0, *then* $\{z_0\}$ *is the asymptotic center of* $\{z_n\}$ *with respect to* K.

Theorem 2.3. ([6]). *Suppose that* S *is a semigroup, let* C *be a closed, convex subset of a reflexive Banach space* E, $\mathcal{S} = \{T_s : s \in S\}$ *be a representation of* S *as nonexpansive mapping from* C *into itself such that weak closure of* $\{T_t x : t \in S\}$ *is weakly compact for each* $x \in C$ *and* X *be a subspace of* $B(S)$ *such that* $1 \in X$ *and the mapping* $t \to \langle T(t)x, x^*\rangle$ *be an element of* X *for each* $x \in C$ *and* $x^* \in E$, *and* μ *be a mean on* X. *If we write* $T_\mu x$ *instead of* $\int T_t x \, d\mu(t)$, *then the following statements are held:*

(i) T_μ *is a nonexpansive mapping from* C *into* C,

(ii) $T_\mu x = x$ *for each* $x \in \mathrm{Fix}(\mathcal{S})$,

(iii) $T_\mu x \in \overline{\mathrm{co}}\, \{T_t x : t \in S\}$ *for each* $x \in C$,

(iv) *If* X *is* r_s*-invariant for each* $s \in S$ *and* μ *is right invariant, then* $T_\mu T_t = T_\mu$ *for each* $t \in S$.

Remark 2.1. *Each uniformly convex Banach space is strictly convex and reflexive (see for example, Theorem 4.1.6 in* [9]*).*

Suppose that D is a subset of B where B is a subset of a Banach space E and let P be a retraction of B onto D, that is, $Px = x$ for each $x \in D$. Then P is said to be sunny, if for each $x \in B$ and $t \geq 0$ with $Px + t(x - Px) \in B$, $P(Px + t(x - Px)) = Px$. A subset D of B is said to be a sunny nonexpansive retract of B if there exists a sunny nonexpansive retraction P of B onto D. If E is smooth and P is a retraction of B onto D, then P is sunny and nonexpansive if and only if for each $x \in B$ and $z \in D$, $\langle x - Px\,,\, J(z - Px)\rangle \leq 0$, for more details, see [9].

Throughout the rest of this paper, the open ball of radius r centered at 0 is denoted by B_r. Let C be a nonempty closed convex subset of a Banach space E. For $\epsilon > 0$ and a mapping $T : C \to C$, $F_\epsilon(T)$ is the set of ϵ-approximate fixed points of T, i.e. $F_\epsilon(T) = \{x \in C : \|x - Tx\| \leq \epsilon\}$.

3. Main results

In this section, a strong convergence approximation scheme for finding a common element of the set of common fixed points of a representation of nonexpansive mappings and fixed points of W-mappings will be studied. First, a Lemma that will be used in the sequel is proved.

Lemma 3.1. *Let S be a semigroup, C be a nonempty compact convex subset of a real strictly convex, reflexive and smooth Banach space E, $\{T_i\}_{i\in\mathbb{N}}$ be family of nonexpansive self mappings on C. Suppose that $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as nonexpansive mapping from C into itself, and $T_i(\mathrm{Fix}(\mathcal{S})) \subseteq \mathrm{Fix}(\mathcal{S})$ for all $i \in \mathbb{N}$. If X is left amenable and C is a compact convex subset of E, then $\mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix}(\mathrm{W})$ is a sunny nonexpansive retract of C and the sunny nonexpansive retraction of C onto $\mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix}(\mathrm{W})$ is unique.*

Proof. Suppose that $x \in C$ is fixed and let μ be a left invariant mean on X. Then, by the Banach contraction principle, a sequence $\{x_n\}$ in C is found such that,

$$x_n = \frac{1}{n}x + \left(1 - \frac{1}{n}\right) T_\mu W_n x_n \tag{1}$$

for each $n \in \mathbb{N}$, then the strong convergence of the sequence $\{x_n\}$ to an element of $\mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix}(\mathrm{W})$ will be proved. For each $z \in \mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix}(\mathrm{W_n})$ and $n \in \mathbb{N}$,

$$\langle x_n - x, J(x_n - z)\rangle \leq 0,$$

where J is the duality mapping of E. Indeed, for each $z \in \mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix}(\mathrm{W})$ and $x^* \in E^*$,

$$\langle T_\mu W_n z, x^*\rangle = \mu_t \langle T(t) W_n z, x^*\rangle = \mu \langle z, x^*\rangle = \langle z, x^*\rangle$$

and hence $z = T_\mu W_n z$ for each $z \in \mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix}(\mathrm{W})$. Therefore, from (1),

$$\begin{aligned}
\langle x_n - x, J(x_n - z)\rangle &= (n-1) \langle T_\mu W_n x_n - x_n, J(x_n - z)\rangle \\
&= (n-1)(\langle T_\mu W_n x_n - T_\mu W_n z, J(x_n - z)\rangle + \langle z - x_n, J(x_n - z)\rangle) \\
&\leqslant (n-1) \left(\|T_\mu W_n x_n - T_\mu W_n z\| \|x_n - z\| - \|x_n - z\|^2\right) \\
&\leqslant (n-1) \left(\|x_n - z\|^2 - \|x_n - z\|^2\right) \\
&= 0.
\end{aligned}$$

Furthermore, from (1),

$$\|x_n - T_\mu W_n x_n\| = \frac{1}{n}\|x - T_\mu W_n x_n\|$$

for each $n \in \mathbb{N}$, hence $\lim_{n\to\infty} \|x_n - T_\mu W_n x_n\| = 0$.

Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be subsequences of $\{x_n\}$ such that $\{x_{n_i}\}$ and $\{x_{n_j}\}$ converge strongly to y and z, respectively. It will be shown that $y = z$. The mapping $W : C \to C$, given by $Wx := \lim_n W_n x$ satisfies

$$\limsup_{n\to\infty} \|W_n y - Wy\| = 0. \tag{2}$$

Note that,

$$\begin{aligned}
\|y - T_\mu y\| &\leqslant \lim_{i\to\infty} \big(\|y - x_{n_i}\| + \|x_{n_i} - T_\mu W_{n_i} x_{n_i}\| + \|T_\mu W_{n_i} x_{n_i} - T_\mu W_{n_i} y\|\big) \\
&\leqslant \lim_{i\to\infty} \|W_{n_i} x_{n_i} - W_{n_i} y\| \\
&\leqslant \lim_{i\to\infty} \|x_{n_i} - y\| = 0,
\end{aligned}$$

for each $i \in \mathbb{N}$, therefore $y = T_\mu y$ and $y \in \mathrm{Fix}(\mathcal{S})$, hence, by our assumption, $T_i y \in \mathrm{Fix}(\mathcal{S})$ for all $i \in \mathbb{N}$ and then $W_n y \in \mathrm{Fix}(\mathcal{S})$. Hence, $T_\mu W_n y = W_n y$, therefore by Theorems 2.1 and 2.2 $T_\mu W y = W y$. Consider the set of the asymptotic center $A(x_{n_j})$ of $\{x_{n_j}\}$ with respect to H. Since $x_{n_j} \rightharpoonup y$, Lemma 2.1 implies that $A(x_{n_j}) = \{y\}$. It is concluded by the definition of $A(x_{n_j})$ that

$$\limsup_{j\to\infty} \|x_{n_j} - z\| \leqslant \limsup_{j\to\infty} \|x_{n_j} - T_t x_{n_j}\| \qquad (t \in S).$$

for all $z \in A(x_{n_j})$. Since $A(x_{n_j}) = \{y\}$, $x_{n_j} \to y$. Using (2),

$$\begin{aligned}
\limsup_{j\to\infty} \|x_{n_j} - Wy\| &\leqslant \limsup_{j\to\infty} \|x_{n_j} - T_\mu W_{n_j} x_{n_j}\| + \limsup_{j\to\infty} \|T_\mu W_{n_j} x_{n_j} - T_\mu W_{n_j} y\| \\
&\quad + \limsup_{j\to\infty} \|T_\mu W_{n_j} y - Wy\| \\
&\leqslant \limsup_{j\to\infty} \|x_{n_j} - T_\mu W_{n_j} x_{n_j}\| + \limsup_{j\to\infty} \|x_{n_j} - y\| \\
&\quad + \limsup_{j\to\infty} \|W_{n_j} y - Wy\| = 0.
\end{aligned}$$

This implies that $W(y) = y$. Hence, $y \in \mathrm{Fix(W)}$. Therefore, $y \in \mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix(W)}$. Because C is bounded, so there exists a positive number M such that $\|f(W_n z_n) - T_\mu W_n z_n\|^2 < M$. Furthermore,

$$\langle y - x, J(y - z)\rangle = \lim_{i\to\infty} \langle x_{n_i} - x, J(x_{n_i} - z)\rangle \leqslant 0.$$

Similarly, $\langle z - x, J(z - y)\rangle \leqslant 0$ and hence $y = z$. Thus, $\{x_n\}$ converges strongly to an element of $\mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix(W)}$. Let us define a mapping P from C into itself by $Px = \lim_{n\to\infty} x_n$. Then for each $z \in \mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix(W)}$,

$$\langle x - Px, J(z - Px)\rangle = \lim_{n\to\infty} \langle x_n - x, J(x_n - z)\rangle \leqslant 0. \tag{3}$$

Therefore, P is a sunny nonexpansive retraction from C onto $\mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix(W)}$.

Let Q be another sunny nonexpansive retraction from C onto $\mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix(W)}$. For each $x \in C$ and $z \in \mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix(W)}$, it is concluded that

$$\langle x - Qx, J(z - Qx)\rangle \leqslant 0. \tag{4}$$

Putting $z = Qx$ in (3) and $z = Px$ in (4), it is implied that

$$\langle x - Px, J(Qx - Px)\rangle \leqslant 0 \quad \textit{and} \quad \langle x - Qx, J(Px - Qx)\rangle \leqslant 0$$

and hence $\langle Qx - Px, J(Qx - Px)\rangle \leqslant 0$. Therefore, This implies that $Qx = Px$, so it completes the proof. □

Theorem 3.1. *Let S be a semigroup, and C a nonempty compact convex subset of a real strictly convex, reflexive and smooth Banach space E and, $\{T_i\}_{i\in\mathbb{N}}$ a family of nonexpansive self mappings on C. Suppose that $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as nonexpansive mapping from C into itself such that $\mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix}(\mathrm{W}) \neq \emptyset$ and $T_i(\mathrm{Fix}(\mathcal{S})) \subseteq \mathrm{Fix}(\mathcal{S})$ for each $i \in \mathbb{N}$. Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \mapsto \langle T_t x, x^*\rangle$ is an element of X for each $x \in C$ and $x^* \in E^*$. Let $\{\mu_n\}$ be a left regular sequence of means on X. Suppose that f is an α-contraction on C. Let ϵ_n be a sequence in $(0,1)$ such that $\lim_n \epsilon_n = 0$. Then there exists a unique sunny nonexpansive retraction P from C onto $\mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix}(\mathrm{W})$ and $x \in C$ such that the following sequence $\{z_n\}$ generated by*

$$z_n = \epsilon_n f(W_n z_n) + (1-\epsilon_n) T_{\mu_n} W_n z_n \quad (n \in \mathbb{N}), \tag{5}$$

strongly converges to Px which is the unique solution of the following variational inequality

$$\langle x - Px, J(z - Px)\rangle \leqslant 0 \qquad (z \in \mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix}(\mathrm{W})).$$

Proof. From Proposition 1.7.3 and Theorem 1.9.21 in [1], any compact subset C of a reflexive Banach space E is weakly compact and from Proposition 1.9.18 in [1], any closed convex subset of a weakly compact subset C of a Banach space E is itself weakly compact and by Proposition 1.9.13 in [1], any convex subset C of a normed space E is weakly closed if and only if C is closed. Therefore, weak closure of $\{T_t x : t \in S\}$ is weakly compact for each $x \in C$.

The proof will be presented in five steps.

Step 1. The existence of z_n which satisfies (5).
The mapping N_n given by

$$N_n x := \epsilon_n f(W_n x) + (1-\epsilon_n) T_{\mu_n} W_n x \qquad (x \in C)$$

is a contraction for every $n \in \mathbb{N}$. Because, $0 \leq \beta_n < 1$ where $\beta_n = (1 + \epsilon_n(\alpha - 1))$, for each $(n \in \mathbb{N})$. Then

$$\begin{aligned}\|N_n x - N_n y\| &\leqslant \epsilon_n \|f(W_n x) - f(W_n y)\| + (1-\epsilon_n)\|T_{\mu_n} W_n x - T_{\mu_n} W_n y\| \\ &\leqslant \epsilon_n \alpha \|W_n x - W_n y\| + (1-\epsilon_n)\|W_n x - W_n y\| \\ &\leqslant \epsilon_n \alpha \|x - y\| + (1-\epsilon_n)\|x - y\| \\ &\leqslant (1 - \epsilon_n(\alpha - 1))\|x - y\| = \beta_n \|x - y\|.\end{aligned}$$

Hence, from the Banach contraction principle [9], there exists a unique point $z_n \in C$ such that $N_n z_n = z_n$.

Step 2. $\lim_{n\to\infty} \|z_n - T_t z_n\| = 0$, for each $t \in S$.
Let $t \in S$ and $\epsilon > 0$. By Lemma 1 in [7], there exists $\delta > 0$ such that $\overline{\mathrm{co}} F_\delta(T_t) + 2B_\delta \subseteq F_\epsilon(T_t)$. Also, from Corollary 2.8 in [2], there exists a natural number N such that

$$\Big\| \frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y - T_t\Big(\frac{1}{N+1} \sum_{i=0}^{N} T_{t^i s} y\Big)\Big\| \leq \delta, \tag{6}$$

for each $s \in S$ and $y \in C$. Suppose that $p \in \mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix}(\mathrm{W})$ and M_0 be a positive number such that, $\sup_{y\in C} \|y\| \leq M_0$. Let $t \in S$, since $\{\mu_n\}$ is strongly left regular, there exists $N_0 \in \mathbb{N}$

such that $\|\mu_n - l^*_{t^i}\mu_n\| \le \frac{\delta}{(3M_0)}$ for all $n \ge N_0$ and $i = 1, 2, \cdots, N$. Therefore,

$$\begin{aligned}
&\sup_{y\in C}\left\|T_{\mu_n}y - \int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y\,\mathrm{d}\mu_n(s)\right\| \\
&= \sup_{y\in C}\sup_{\|x^*\|=1}\left|\langle T_{\mu_n}y, x^*\rangle - \left\langle \int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y\,\mathrm{d}\mu_n(s), x^*\right\rangle\right| \\
&= \sup_{y\in C}\sup_{\|x^*\|=1}\left|\frac{1}{N+1}\sum_{i=0}^{N}(\mu_n)_s\langle T_s y, x^*\rangle - \frac{1}{N+1}\sum_{i=0}^{N}(\mu_n)_s\langle T_{t^i s}y, x^*\rangle\right| \\
&\le \frac{1}{N+1}\sum_{i=0}^{N}\sup_{y\in C}\sup_{\|x^*\|=1}\left|(\mu_n)_s\langle T_s y, x^*\rangle - (l^*_{t^i}\mu_n)_s\langle T_s y, x^*\rangle\right| \\
&\le \max_{i=1,2,\cdots,N}\|\mu_n - l^*_{t^i}\mu_n\|(M_0 + 2\|p\|) \\
&\le \max_{i=1,2,\cdots,N}\|\mu_n - l^*_{t^i}\mu_n\|(3M_0) \\
&\le \delta \quad (\mathrm{n} \ge \mathrm{N}_0).
\end{aligned} \tag{7}$$

From Theorem 2.3, it is concluded that

$$\int \frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y\,\mathrm{d}\mu_n(s) \in \overline{\mathrm{co}}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i}(T_s y) : s \in S\right\}. \tag{8}$$

It follows from (6)-(8) that

$$\begin{aligned}
T_{\mu_n}y &\in \overline{\mathrm{co}}\left\{\frac{1}{N+1}\sum_{i=0}^{N} T_{t^i s}y : s \in S\right\} + B_\delta \\
&\subset \overline{\mathrm{co}}F_\delta(T_t) + 2B_\delta \subset F_\epsilon(T_t),
\end{aligned}$$

for each $y \in C$ and $n \ge N_0$. Hence, $\limsup_{n\to\infty}\sup_{y\in C}\|T_t(T_{\mu_n}y) - T_{\mu_n}y\| \le \epsilon$. Since $\epsilon > 0$ is arbitrary, it is implied that

$$\limsup_{n\to\infty}\sup_{y\in C}\|T_t(T_{\mu_n}y) - T_{\mu_n}y\| = 0. \tag{9}$$

Suppose that $t \in S$ and $\epsilon > 0$, then there exists $\delta > 0$, which satisfies (6). Put $L_0 = (1+\alpha)2M_0 + \|f(p) - p\|$. Therefore from the condition $\lim_n \epsilon_n = 0$ and from (9) there exists a natural number N_1 such that $T_{\mu_n}y \in F_\delta(T_t)$ for each $y \in C$ and $\epsilon_n < \frac{\delta}{2L_0}$ for each $n \ge N_1$. Since $p \in \mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix(W)}$, it is concluded that

$$\begin{aligned}
\epsilon_n\|f(W_n z_n) - T_{\mu_n}W_n z_n\| &\le \epsilon_n(\|f(W_n z_n) - f(W_n p)\| + \|f(W_n p) - p\| \\
&\quad + \|T_{\mu_n}W_n p - T_{\mu_n}W_n z_n\|) \\
&\le \epsilon_n\left(\alpha\|W_n z_n - W_n p\| + \|(f(W_n p) - p\| + \|W_n p - W_n z_n\|\right) \\
&\le \epsilon_n\left(\alpha\|z_n - p\| + \|f(p) - p\| + \|z_n - p\|\right) \\
&\le \epsilon_n\left((1+\alpha)\|z_n - p\| + \|f(p) - p\|\right) \\
&\le \epsilon_n\left((1+\alpha)2M_0 + \|f(p) - p\|\right) \\
&= \epsilon_n L_0 \le \frac{\delta}{2},
\end{aligned}$$

for each $n \geq N_1$. Note that

$$\begin{aligned} z_n &= \epsilon_n f(W_n z_n) + (1-\epsilon_n) T_{\mu_n} W_n z_n \\ &= T_{\mu_n} W_n z_n + \epsilon_n \left(f(W_n z_n) - T_{\mu_n} W_n z_n \right) \\ &\in F_\delta(T_t) + B_{\frac{\delta}{2}} \\ &\subseteq F_\delta(T_t) + 2B_\delta \\ &\subseteq F_\epsilon(T_t). \end{aligned}$$

for each $n \geq N_1$. Then

$$\|z_n - T_t z_n\| \leq \epsilon \quad (\mathrm{n} \geq \mathrm{N}_1).$$

Since $\epsilon > 0$ is arbitrary, it is implied that $\lim_{n\to\infty} \|z_n - T_t z_n\| = 0$.

Step 3. $\mathfrak{S}\{z_n\} \subset \mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix(W)}$, where $\mathfrak{S}\{z_n\}$ is the set of strongly limit points of $\{z_n\}$.
Suppose that $z \in \mathfrak{S}\{z_n\}$, and consider a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $z_{n_j} \to z$. Then,

$$\begin{aligned} \|T_t z - z\| \leq& \|T_t z - T_t z_{n_j}\| + \|T_t z_{n_j} - z_{n_j}\| + \|z_{n_j} - z\| \\ \leq& 2\|z_{n_j} - z\| + \|T_t z_{n_j} - z_{n_j}\|, \end{aligned}$$

then by Step 2,

$$\|T_t z - z\| \leq 2 \lim_j \|z_{n_j} - z\| + \lim_j \|T_t z_{n_j} - z_{n_j}\| = 0,$$

so $z \in \mathrm{Fix}(\mathcal{S})$.
Clearly, $\lim_j \|z_{n_j} - T_{\mu_{n_j}} W_{n_j} z_{n_j}\| = 0$. Indeed,

$$\begin{aligned} &\lim_j \|z_{n_j} - T_{\mu_{n_j}} W_{n_j} z_{n_j}\| \\ &= \lim_j \|\epsilon_{n_j} f(W_{n_j} z_{n_j}) + (1-\epsilon_{n_j}) T_{\mu_{n_j}} W_{n_j} z_{n_j} - T_{\mu_{n_j}} W_{n_j} z_{n_j}\| \\ &= \lim_j \epsilon_{n_j} \|f(W_{n_j} z_{n_j}) - T_{\mu_{n_j}} W_{n_j} z_{n_j}\| \\ &= 0, \end{aligned}$$

therefore,

$$\begin{aligned} \limsup_j \|z_{n_j} - T_{\mu_{n_j}} W z_{n_j}\| &\leqslant \limsup_j \|z_{n_j} - T_{\mu_{n_j}} W_{n_j} z_{n_j}\| \\ &\quad + \limsup_j \|T_{\mu_{n_j}} W_{n_j} z_{n_j} - T_{\mu_{n_j}} W z_{n_j}\| \\ &\leqslant \limsup_j \|W_{n_j} z_{n_j} - W z_{n_j}\| \\ &\leqslant \limsup_j (\|W_{n_j} z_{n_j} - W_{n_j} z\| + \|W_{n_j} z - W z\| + \|W z - W z_{n_j}\|) \\ &\leqslant 2 \limsup_j (\|z_{n_j} - z\| + \|W_{n_j} z - W z\|) = 0, \end{aligned}$$

so

$$\limsup_j \|z_{n_j} - T_{\mu_{n_j}} W z_{n_j}\| = 0. \tag{10}$$

Then from (10) and from the condition that $T_i(\mathrm{Fix}(\mathcal{S})) \subseteq \mathrm{Fix}(\mathcal{S})$, it is concluded that $W(\mathrm{Fix}(\mathcal{S})) \subseteq \mathrm{Fix}(\mathcal{S})$. Hence $T_t(Wz) = Wz$ and $T_{\mu_{n_j}}(Wz) = Wz$, then it is implied that

$$\begin{aligned}\limsup_j \|z_{n_j} - Wz\| &\leqslant \limsup_j \|z_{n_j} - T_{\mu_{n_j}} W z_{n_j}\| + \limsup_j \|T_{\mu_{n_j}} W z_{n_j} - T_{\mu_{n_j}} W z\| \\ &\quad + \limsup_j \|T_{\mu_{n_j}} W z - W z\| \\ &\leqslant \limsup_j \|z_{n_j} - z\| = 0,\end{aligned}$$

then $z \in \mathrm{Fix(W)}$. Hence, $\mathfrak{S}(z_n) \subset \mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix(W)}$.
Step 4. There exists a unique sunny nonexpansive retraction P of C onto $\mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix}(\mathrm{W_n})$ and $x \in C$ such that

$$\Gamma := \limsup_n \langle x - Px \, , \, J(z_n - Px)\rangle \leq 0. \tag{11}$$

By Lemma 3.1, there exists a unique sunny nonexpansive retraction P from C onto $\mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix(W)}$. The Banach Contraction Mapping Principle guarantees that fP has a unique fixed point $x \in C$. Next, it will be proved that

$$\Gamma = \limsup_n \langle x - Px \, , \, J(z_n - Px)\rangle \leq 0.$$

Since C is a compact subset of E, a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ can be found with the following properties:

(i) $\lim_j \langle x - Px \, , \, J(z_{n_j} - Px)\rangle = \Gamma$;

(ii) $\{z_{n_j}\}$ converges strongly to a point z;

from Step 3, $z \in \mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix(W)}$. Since E is smooth, it is implied that

$$\Gamma = \lim_j \langle x - Px \, , \, J(z_{n_j} - Px)\rangle = \langle x - Px \, , \, J(z - Px)\rangle \leq 0.$$

Also $fPx = x$, so $(f - I)Px = x - Px$. Now,

$$\begin{aligned}&\epsilon_n(\alpha - 1)\|z_n - Px\|^2 \\ &\geq \Big[\epsilon_n \alpha \big\|z_n - Px\big\| + (1 - \epsilon_n)\|z_n - Px\|\Big]^2 - \|z_n - Px\|^2 \\ &\geq \Big[\epsilon_n \big\|f(W_n z_n) - f(W_n Px)\big\| + (1 - \epsilon_n)\|T_{\mu_n} W_n z_n - Px\|\Big]^2 \\ &\qquad - \|z_n - Px\|^2 \\ &\geq 2\Big\langle \epsilon_n \Big(f(W_n z_n) - f(W_n Px)\Big) + (1 - \epsilon_n)(T_{\mu_n} W_n z_n - Px) \\ &\qquad - (z_n - Px) \, , \, J(z_n - Px)\Big\rangle \\ &= -2\epsilon_n \langle (f - I)Px \, , \, J(z_n - Px)\rangle \\ &= -2\epsilon_n \langle x - Px \, , \, J(z_n - Px)\rangle,\end{aligned}$$

for all $n \in \mathbb{N}$ (see page 99 in [9]), hence,

$$\|z_n - Px\|^2 \leq \frac{2}{1 - \alpha} \langle x - Px \, , \, J(z_n - Px)\rangle. \tag{12}$$

Step 5. $\{z_n\}$ strongly converges to Px.
Indeed, since $Px \in \mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix(W)}$, by applying (11), (12), it is deduced that

$$\limsup_n \|z_n - Px\|^2 \leq \frac{2}{1 - \alpha} \limsup_n \langle x - Px \, , \, J(z_n - Px)\rangle \leq 0.$$

That is $z_n \to Px$. □

4. Numerical example

Example 4.1. *Consider Theorem 3.1. Let* $S = \{0, 1, 2, 3, \cdots, \}$, $E = \mathbb{R}$ *and* $C = [0, 1]$. *Let* $\mu(g) = g(0)$ *for each* $g \in B(S)$. *If* $g = 1$ *then* $\mu(1) = 1(0) = 1$. *Also,*

$$1 = \mu(1) \leqslant \|\mu\| = \sup_{\|g\| \leqslant 1} \|\mu(g)\| = \sup_{\|g\| \leqslant 1} |g(0)| \leqslant 1$$
$$\Rightarrow \|\mu\| = 1 \Rightarrow \|\mu\| = \mu(1) = 1,$$

so μ *is a mean and obviously* μ *is invariant. Let* $T : C \longrightarrow C$ *be a nonexpansive mapping and let* $T^0 = I$ *and* $\mathcal{S} = \{T^k : k \in S\}$. *It is obvious that* $\mu(g) = \mu_k(g(k))$ *for each* $k \in S$ *and* $g \in B(S)$. *Consider* $g : k \longrightarrow \left\langle T^k x, y \right\rangle$ *for each* $x \in C$ *and* $y \in \mathbb{R}$. *Then, it is implied for each* $x, y \in C$ *that*

$$\mu(g) = \mu_k(g(k)) = \mu_k \left\langle T^k x, y \right\rangle = \langle T^0 x, y \rangle = \langle x, y \rangle,$$

also, $\langle T_\mu x, y \rangle = \mu_k \left\langle T^k x, y \right\rangle$ *then* $\langle T_\mu x, y \rangle = \langle x, y \rangle$ *so* $T_\mu x = x$ *hence* $T_\mu = I$. *Suppose that* $\{T_i : T_i = I,$ *for each* $i = 1, 2, 3, \ldots\}$ *be the family of nonexpansive mapping in Theorem 3.1. Let* $f(x) = \frac{1}{2}x$ *be our contraction mapping and* $P = 0$ *be the desired retraction from* $[0, 1]$ *onto* $\mathrm{Fix}(\mathcal{S}) \cap \mathrm{Fix(W)} = \{0\}$ *then using Table 1 and as in Figure 1, the sequence* $\{z_n\}$ *generated by* (5) *is converged to* $P0 = 0$.

Table 1. $f(x) = \frac{1}{2}x, \quad N = 20, \quad i = 1, \cdots, 20.$

n	$\lambda_{n,i}$	W_n	z_n
1	$\frac{1}{2}$	$0.5128\,I$	0
2	$\frac{1}{3}$	$0.6780\,I$	0
3	$\frac{1}{4}$	$0.7595\,I$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
8	$\frac{1}{9}$	$0.8939\,I$	0
9	$\frac{1}{10}$	$0.9045\,I$	0
10	$\frac{1}{11}$	$0.9132\,I$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
18	$\frac{1}{19}$	$0.9499\,I$	0
19	$\frac{1}{20}$	$0.9524\,I$	0
20	$\frac{1}{21}$	$0.9547\,I$	0

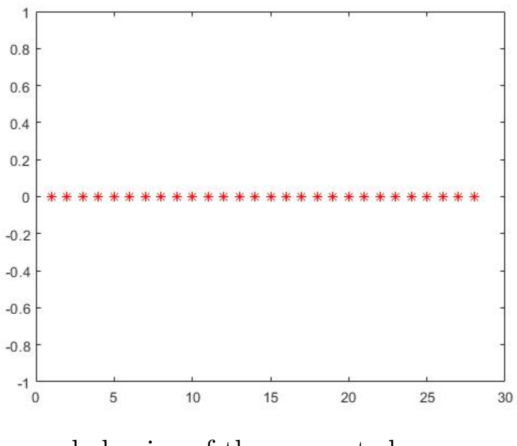

Figure 1. Convergence behavior of the generated sequences by Example (4.1).

5. Conclusion

Firstly, an algorithm in a Banach space is introduced. Then a nonexpansive retraction is found and the convergence of the proposed scheme to an element in the range of the retraction is proved.

6. Acknowledgements

The first author is grateful to the University of Lorestan for their support.

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